CUNTZ-PIMSNER ALGEBRAS FOR SUBPRODUCT SYSTEMS

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ABSTRACT. In this paper we generalize the notion of Cuntz-Pimsner algebras of C^*-correspondences to the setting of subproduct systems. The construction is justified in several ways, including the Morita equivalence of the operator algebras under suitable conditions, and examples are provided to illustrate its naturality. We also demonstrate why some features of the Cuntz-Pimsner algebras of C^*-correspondences fail to generalize to our setting, and discuss what we have instead.

CONTENTS

INTRODUCTION

The study of the Cuntz-Pimsner algebra of C^*-correspondences has its origins in the influential paper of Pimsner [21]. Initially defined merely for faithful C^*-correspondences, the Cuntz-Pimsner algebra was shown to be a quotient of the Toeplitz algebra with a special universal property. Katsura [11] provided a definition for arbitrary C^*-correspondences, promoting the Cuntz-Pimsner algebras even further. The construction is flexible enough to generalize, at the same time, the Cuntz-Krieger algebras, crossed products by Hilbert C^*-bimodules [1] (particularly, crossed products by partial automorphisms) and others. The Cuntz-Pimsner algebra has been highly popular in research ever since it was introduced. Many aspects of it have been comprehensively studied, for instance: K-theory [21, 12], Morita equivalence [18], exactness and nuclearity [12], ideal structure [13] and more, and it has served as a tool in various papers. A crucial step was made when it was established in [12] that the Cuntz-Pimsner algebras could be characterized using another universal property already known for Cuntz-Krieger algebras, namely the "gauge-invariant uniqueness theorem". This celebrated discovery (which, particularly, revealed the structure of the isomorphic representations of the algebra) is so powerful, that it allows an easy proof of many properties of the Cuntz-Pimsner algebras that have been proven earlier using more complicated means.

Subproduct systems were introduced in [23], where they were studied systematically from several aspects. In our recent paper [26] we continued these lines, focusing on the tensor and Toeplitz algebras associated to a subproduct system and their representations. Since these constructions seem to attract more and more attention recently, it seems that a generalization of the Cuntz-Pimsner algebra for *subproduct* systems may have a lot of potential for applications. Nevertheless, it is not clear *a priori* how the algebra should be defined so as to have it bear the desirable properties described above.

In this paper we suggest a possible way to define Cuntz-Pimsner algebras for subproduct systems, which is natural in view of the analysis in, e.g., [21, §3], [8, §4] and [24, §3]. This has already been done, in the specific context of a symbolic dynamical system called subshifts, by Matsumoto (see [15] and its follow-ups), and our definition reduces to his. Throughout the paper we give several justifications for the "correctness" of our approach. In §2 we explain why the universal characterizations of the Cuntz-Pimsner algebras for C^*-correspondences, particularly the gauge-invariant uniqueness, cannot be employed as-is to the subproduct systems case. After giving our definition of the algebra, we show that it generalizes the original construction of Pimsner in two different manners. The construction is demonstrated by examples in §3. A (partial) characterization of the algebra, in terms of essential representations of the ambient Toeplitz algebra, is then presented in §4. In §5 we define a notion of strong Morita equivalence for subproduct systems, and show that it implies the Morita equivalence of all associated operator algebras. This can be seen as another strength of the proposed definition. The last section is devoted to some open questions.

1. Preliminaries

We start with a brief summary of the definitions we need (see [26] and the references therein for a more thorough background), notation and general assumptions. The reader should be familiar with the basics of Hilbert C^*-modules found in [14, Ch. 1-4]. The notation $\langle\cdot,\cdot\rangle$ is reserved for the rigging in Hilbert modules.

Definition 1.1 ([17, Definition 2.1])**.** Let $\mathscr{M}, \mathscr{N}$ be C^*-algebras. A (right) Hilbert C^*-module E over $\mathscr{M}$ is called an $\mathscr{N} - \mathscr{M}$ (C^*-) *correspondence* if it is also equipped with a left $\mathscr{N}$-module structure, implemented by a $*$-homomorphism $\varphi : \mathscr{N} \to \mathcal{L}(E)$; that is, $a \cdot \zeta := \varphi(a)\zeta$ for $a \in \mathscr{N}$, $\zeta \in E$. We say that E is *faithful* if φ is faithful and *essential* if $\varphi(\mathscr{N})E$ is total in E. If $\mathscr{N} = \mathscr{M}$, we say that E is a (C^*-) correspondence over $\mathscr{M}$.

Example 1.2. Every Hilbert space is a C^*-correspondence over $\mathbb{C}$.

Example 1.3. If $\mathscr{M}$ is a C^*-algebra and α is an endomorphism of $\mathscr{M}$, we write ${}_\alpha\mathscr{M}$ for the C^*-correspondence that is equal to $\mathscr{M}$ as sets, with the obvious right $\mathscr{M}$-module structure, and left $\mathscr{M}$-action given by $\varphi(a)b := \alpha(a)b$ for $a, b \in \mathscr{M}$.

Example 1.4. Every quiver (directed graph) possesses an associated C^*-correspondence. See [21, p. 193, (2)] or [17, Example 2.9] for details.

The full Fock space of a C^*-correspondence E over $\mathscr{M}$ is the C^*-correspondence $\mathcal{F}_E := \bigoplus_{n\in\mathbb{Z}_+} E^{\otimes n}$ (also over $\mathscr{M}$), where $E^{\otimes 0}$ equals $\mathscr{M}$ by definition.

The original definition of subproduct systems ([23, Definitions 1.1, 6.2]) is for the context of von Neumann algebras. We require its adaptation to the C^*-setting of [26].

Definition 1.5. A family $X = (X(n))_{n\in\mathbb{Z}_+}$ of C^*-correspondences over a C^*-algebra $\mathscr{M}$ is called a (standard) *subproduct system* if $X(0) = \mathscr{M}$ and for all $n, m \in \mathbb{Z}_+$, $X(n+m)$ is an orthogonally-complementable sub-correspondence of $X(n) \otimes X(m)$. This implies, in particular, that $X(n)$ is essential for each $n \in \mathbb{N}$.

Example 1.6. If E is an essential C^*-correspondence over $\mathscr{M}$, the *product* system X_E, defined by $X_E(n) := E^{\otimes n}$ for each $n \in \mathbb{Z}_+$, is trivially a subproduct system.

Example 1.7 ([23, Example 1.3])**.** Fix a Hilbert space $\mathcal{H}$, and let $X(n) := \mathcal{H}^{\circledS n}$ (the n-fold symmetric tensor product of $\mathcal{H}$) for every n. The resulting family X satisfies the requirements of Definition 1.5. It is called the *symmetric subproduct system* over $\mathcal{H}$, and denoted by $\mathrm{SSP}_{\mathcal{H}}$. Specifically, we put $\mathrm{SSP}_d := \mathrm{SSP}_{\mathbb{C}^d}$ for $d \in \mathbb{N}$ and $\mathrm{SSP}_\infty := \mathrm{SSP}_{\ell_2(\mathbb{N})}$.

The reader is urged to consult [23] for many other interesting examples of subproduct systems.

Given a subproduct system X, we shall use the following notation throughout the paper. Set $E := X(1)$. The X-*Fock* space is the sub-correspondence

$$\mathcal{F}_X := \bigoplus_{n\in\mathbb{Z}_+} X(n)$$

of the full Fock space $\mathcal{F}_E$. For all $n \in \mathbb{Z}_+$ we have $X(n) \subseteq E^{\otimes n}$. Let $p_n \in \mathcal{L}(E^{\otimes n})$ stand for the (orthogonal) projection of $E^{\otimes n}$ onto $X(n)$, denote by $Q_n \in \mathcal{L}(\mathcal{F}_X)$ the projection of $\mathcal{F}_X$ onto the direct summand $X(n)$, and define $R_n := Q_0 + Q_1 + \ldots + Q_n$, $R'_n := \bigvee_{k \geq n} Q_k$.

The X-*shifts* are the operators $S_n(\zeta) \in \mathcal{L}(\mathcal{F}_X)$ ($n \in \mathbb{Z}_+$, $\zeta \in X(n)$) given by

$$S_n(\zeta)\eta := p_{n+m}(\zeta \otimes \eta)$$

for $m \in \mathbb{Z}_+$, $\eta \in X(m)$. We write $\varphi_\infty(\cdot)$ for $S_0(\cdot)$. In case the context is not clear, we will add the subproduct system letter as a superscript, e.g. p_n^X, Q_n^X, S_n^X, etc. A direct calculation shows that the adjoint $S_n^X(\zeta)^*$ is a restriction of the adjoint in the full Fock space, $S_n^{X_E}(\zeta)^*$, to $\mathcal{F}_X$. It satisfies $S_n^{X_E}(\zeta)^*(\eta_1 \otimes \eta_2) = \langle \zeta, \eta_1 \rangle \eta_2$ for each $\zeta, \eta_1 \in E^{\otimes n}$ and $\eta_2 \in E^{\otimes m}$.

The *tensor algebra* $\mathcal{T}_+(X)$ is the non-selfadjoint subalgebra of $\mathcal{L}(\mathcal{F}_X)$ generated by all X-shifts. The *Toeplitz algebra* $\mathcal{T}(X)$ is the C^*-subalgebra of $\mathcal{L}(\mathcal{F}_X)$ generated by the same operators. It admits a natural action of $\mathbb{T}$, called the *gauge action*, defined by $\alpha_\lambda(S_n(\zeta)) := \lambda^n S_n(\zeta)$ for all $\lambda \in \mathbb{T}$, $n \in \mathbb{Z}_+$ and $\zeta \in X(n)$. It is useful that the kth spectral subset of α ([7, Definition 2.1]) equals the closed linear span of all monomials in $\mathcal{T}(X)$ of degree k, denoted by $\mathcal{T}_k(X)$ (see [26], Definition 4.6 and the text surrounding equation (4.13)).

Definition 1.8 ([17, Definition 2.11])**.** Let E be a C^*-correspondence over $\mathscr{M}$ and let $\mathcal{H}$ be a Hilbert space. A pair (T, σ) is called a *covariant representation* of E on $\mathcal{H}$ if:

(1) σ is a C^*-representation of $\mathscr{M}$ on $\mathcal{H}$;
(2) T is a linear mapping from E to $B(\mathcal{H})$;
(3) and T is a bimodule map with respect to σ, that is, $T(\zeta a) = T(\zeta)\sigma(a)$ and $T(a\zeta) = \sigma(a)T(\zeta)$ for all $\zeta \in E$ and $a \in \mathscr{M}$.

When this holds, the formula $\widetilde{T}(\zeta \otimes h) := T(\zeta)h$ ($\zeta \in E$, $h \in \mathcal{H}$) defines a contraction $\widetilde{T} : E \otimes_\sigma \mathcal{H} \to \mathcal{H}$.

Definition 1.9 ([23, Definition 1.5], [26, Definitions 2.6, 2.20])**.** Let $X = (X(n))_{n \in \mathbb{Z}_+}$ be a subproduct system. A family $T = (T_n)_{n \in \mathbb{Z}_+}$ is called a *covariant representation* of X on $\mathcal{H}$ if:

(1) writing $\sigma := T_0$, the pair (T_n, σ) is a covariant representation of $X(n)$ on $\mathcal{H}$ (in the sense of Definition 1.8) for all $n \in \mathbb{N}$;
(2) and for every $n, m \in \mathbb{Z}_+$, $\zeta \in X(n)$ and $\eta \in X(m)$, we have

$$T_{n+m}(p_{n+m}(\zeta \otimes \eta)) = T_n(\zeta) T_m(\eta).$$

To such a family we associate the operators $\widetilde{T}_n : X(n) \otimes_\sigma \mathcal{H} \to \mathcal{H}$ as explained above. We say that T is *pure* if the sequence $\{\widetilde{T}_n \widetilde{T}_n^*\}_{n=1}^\infty$ converges to zero in the strong operator topology, and *fully coisometric* if $\widetilde{T}_1$ is coisometric.

If π is a representation of $\mathcal{T}_+(X)$ (or of $\mathcal{T}(X)$), setting $T_n(\zeta) := \pi(S_n(\zeta))$ yields a covariant representation of X. The opposite direction—namely, determining which covariant representations arise this way—was the main theme of [26].

Standing Hypothesis: throughout this paper, *all subproduct systems are assumed faithful* (X is faithful if $X(n)$ is faithful for all $n \in \mathbb{N}$), and *all representations are assumed nondegenerate*.

2. Construction of the algebra

Definition 2.1. For a (faithful) subproduct system X we denote by $\mathcal{J}$ the ideal $\varphi^{-1}(\mathcal{K}(E))$ of $\mathscr{M}$.

When X is a *product* system $X = X_E$ (with E faithful), $\mathcal{K}(\mathcal{F}_E \mathcal{J})$ is an ideal in $\mathcal{T}(E)$ (actually, it is equal to $\mathcal{T}(E) \cap \mathcal{K}(\mathcal{F}_E)$), and the Cuntz-Pimsner algebra $\mathcal{O}(E)$ is defined in [21] to be the quotient $\mathcal{T}(E)/\mathcal{K}(\mathcal{F}_E \mathcal{J})$.

Proposition 2.2. *Let X be a subproduct system. For all $a \in \mathcal{J}$ we have $\varphi_\infty(a)Q_0 \in \mathcal{T}(X)$. Moreover, the following subsets of $\mathcal{L}(\mathcal{F}_X)$ are equal:*

(1) $\mathcal{K}(\mathcal{F}_X \mathcal{J})$
(2) *the ideal of $\mathcal{T}(X)$ generated by $\varphi_\infty(\mathcal{J})Q_0$*
(3) *the ideal of $\mathcal{L}(\mathcal{F}_X)$ generated by $\varphi_\infty(\mathcal{J})Q_0$.*

As a result, $\mathcal{K}(\mathcal{F}_X \mathcal{J})$ is an ideal of $\mathcal{T}(X)$.

The proof is almost identical to that of the corresponding assertion for product systems (compare [17, Lemma 2.17]). Since the proposition is not essential for the rest of the paper, we omit the details.

Assume that E is a faithful C^*-correspondence. Katsura's gauge-invariant uniqueness theorem [13, Proposition 7.14][1] asserts that the ideal $\mathcal{K}(\mathcal{F}_E \mathcal{J})$ has the following property: it is the *largest* ideal of $\mathcal{T}(E)$ such that 1) it is gauge invariant, and 2) its intersection with $\varphi_\infty(\mathscr{M})$ is $\{0\}$. If X is a subproduct system, it would be plausible to define $\mathcal{O}(X)$ to be the quotient of $\mathcal{T}(X)$ by such an ideal. However, the following example demonstrates that such an ideal fails to exist even in simple cases.

[1] the term "gauge-invariant uniqueness theorem" refers sometimes to [12, Theorem 6.4]; but this is a consequence of [13, Proposition 7.14].

Example 2.3. Consider the symmetric subproduct system $X = \mathrm{SSP}_2$ (see Example 1.7). Suppose that there exists a largest ideal $\mathcal{I}$, which does not contain the unit $I \in \mathcal{T}(X)$, and which is gauge invariant. The ideal $\mathbb{K}$ fulfills these two conditions, so we must have $\mathbb{K} \subseteq \mathcal{I}$. Since $\mathcal{T}(X)/\mathbb{K}$ is canonically isomorphic to $C(\partial B_2)$ (see [2, Theorem 5.7]), $\mathcal{I}/\mathbb{K}$ can be identified with an ideal of $C(\partial B_2)$. There thus exists a nonempty compact set M such that this ideal equals $\mathcal{V}(M)$, the set of all elements of $C(\partial B_2)$ vanishing on M. Direct calculation shows that the gauge action $\tilde{\alpha}$ on $C(\partial B_2)$, induced by the gauge action on $\mathcal{T}(X)$, is given by $(\tilde{\alpha}_\lambda(f))(w) = f(\lambda w)$ (for all $\lambda \in \mathbb{T}$, $f \in C(\partial B_2)$ and $w \in \partial B_2$).

Suppose first that M is not contained in a set of the form

$$M_{a,b} := \{(z_1, z_2) \in \partial B_2 : |z_1| = a, |z_2| = b\}$$

(for fixed $a, b \geq 0$ with $|a|^2 + |b|^2 = 1$). Choose $w_1, w_2 \in M$ with the property that there are scalars $\alpha, \beta, \gamma \in \mathbb{C}$ such that the polynomial $f(w) := \alpha |z_1|^2 + \beta |z_2|^2 + \gamma$ ($w = (z_1, z_2)$) satisfies $f(w_1) \neq 0$, $f(w_2) = 0$. Write

$$f(S) := \alpha S_1(e_1)S_1(e_1)^* + \beta S_1(e_2)S_1(e_2)^* + \gamma I \in \mathcal{T}(X)$$

and consider the ideal $\mathcal{I}_1 := \langle \mathcal{I} \cup \{f(S)\}\rangle$ of $\mathcal{T}(X)$. Since $\mathcal{I}$ and $\{f(S)\}$ are gauge invariant, so is $\mathcal{I}_1$. Moreover, $\mathcal{I}_1/\mathbb{K}$ is isomorphic to the ideal $\langle \mathcal{V}(M) \cup \{f\}\rangle$ of $C(\partial B_2)$. We have $\langle \mathcal{V}(M) \cup \{f\}\rangle \supsetneqq \mathcal{V}(M)$ (thus $\mathcal{I}_1 \supsetneqq \mathcal{I}$) because $f(w_1) \neq 0$. In addition, $g(w_2) = 0$ for all $g \in \langle \mathcal{V}(M) \cup \{f\}\rangle$, hence $1 \notin \langle \mathcal{V}(M) \cup \{f\}\rangle$. In conclusion, we have constructed a gauge-invariant ideal of $\mathcal{T}(X)$, of which I is not an element, and which strictly contains $\mathcal{I}$. This is a contradiction.

Assume that M is contained in $M_{a,b}$ for some a, b. Since $M_{a,b}$ fulfills the desirable conditions, we must have $M = M_{a,b}$, and by symmetry, necessarily $a = b = \frac{1}{\sqrt{2}}$. Now let $f(w) := |z_1 - z_2|^2$, $w_1 := (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and $w_2 := (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, and continue as above to get a contradiction.

The structure of the Toeplitz algebra of a general subproduct system X is much more complicated than that of a product system, and defining the Cuntz-Pimsner algebra of X to be $\mathcal{T}(X)/\mathcal{K}(\mathcal{F}_X\mathcal{J})$ is useless, as could be seen by examples (cf. Remark 3.7 below). We are thus looking for an alternative.

Lemma 2.4. *Let $\mathcal{G}_1 \trianglelefteq \mathcal{T}(X)$ be gauge invariant.*

(1) *The set $\bigcup_{k\in\mathbb{Z}}\big(\mathcal{G}_1 \cap \mathcal{T}_k(X)\big)$ is total in $\mathcal{G}_1$.*
(2) *If $\mathcal{G}_2 \trianglelefteq \mathcal{T}(X)$ and $(\mathcal{G}_1 \cap \mathcal{T}_0(X))_+ \subseteq \mathcal{G}_2$, then $\mathcal{G}_1 \subseteq \mathcal{G}_2$.*

Proof. We use the routine methods. Let $\{k_n\}_{n=1}^\infty$ denote Fejér's kernel. For $S \in \mathcal{T}(X)$, write $\sigma_n(S) := \frac{1}{2\pi}\int_{t=0}^{2\pi} \alpha_\lambda(S) k_n(\lambda)\,\mathrm{d}t \in \mathcal{T}(X)$ ($\lambda := e^{it}$). If S is a monomial of degree m, then $\alpha_\lambda(S) = \lambda^m S$, and so $\sigma_n(S) = \left(\frac{1}{2\pi}\int_{t=0}^{2\pi} \lambda^m k_n(\lambda)\,\mathrm{d}t\right) S \xrightarrow[n\to\infty]{} S$. Hence $\sigma_n(S) \to S$ for every (finite) "polynomial" in $\mathcal{T}(X)$. The estimate $\|\sigma_n(S)\| \le \|S\|$ then yields that actually $\sigma_n(S) \to S$ for all $S \in \mathcal{T}(X)$.

Pick $S \in \mathcal{G}_1$. For every $k \in \mathbb{Z}$, $\Phi_k(S) := \int_{t=0}^{2\pi} \alpha_\lambda(S)\lambda^{-k}\,\mathrm{d}t$ belongs to $\mathcal{G}_1 \cap \mathcal{T}_k(X)$. Since $\sigma_n(S)$ is a linear combination of $\{\Phi_k(S)\}_{k=-n}^{n}$, (1) is established. Moreover, $\Phi_k(S)^*\Phi_k(S) \in (\mathcal{G}_1 \cap \mathcal{T}_0(X))_+$, and under the assumptions of (2) this implies that $\Phi_k(S)^*\Phi_k(S) \in \mathcal{G}_2$, and consequently $\Phi_k(S) \in \mathcal{G}_2$ (by [5, Theorem I.5.3]). Therefore $\sigma_n(S) \in \mathcal{G}_2$ for all n. As a result, $S \in \mathcal{G}_2$. □

Theorem 2.5. *Let X be a subproduct system. Define an ideal $\mathcal{I}' \trianglelefteq \mathcal{T}(X)$ by*

$$\mathcal{I}' := \left\langle S \in \mathcal{T}_0(X) : \lim_{n\to\infty} \|SQ_n\| = 0 \right\rangle$$

and a subset $\mathcal{I}'' \subseteq \mathcal{T}(X)$ by

$$\mathcal{I}'' := \left\{ S \in \mathcal{T}(X) : \lim_{n\to\infty} \|SQ_n\| = 0 \right\}.$$

Then $\mathcal{I}' = \mathcal{I}''$, and it is gauge invariant. In particular, $\mathcal{I}'' \trianglelefteq \mathcal{T}(X)$.

In comparison with the product system case, $\mathcal{I}'$ and $\mathcal{I}''$ parallel $\langle \varphi_\infty(\mathcal{J})Q_0\rangle$ and $\mathcal{T}(E) \cap \mathcal{K}(\mathcal{F}_E)$ (both are equal to $\mathcal{K}(\mathcal{F}_E\mathcal{J})$), respectively.

Proof. The set $\mathcal{I}''$ is gauge invariant as α_λ is unitarily implemented by $W_\lambda \in \mathcal{L}(\mathcal{F}_X)$ given by $\bigoplus_{n\in\mathbb{Z}_+} \zeta_n \mapsto \bigoplus_{n\in\mathbb{Z}_+} \lambda^n\zeta_n$, which commutes with Q_n for all n.

Let us prove that $\mathcal{I}''$ is an ideal. It is clearly a linear subspace and a left ideal. It is also norm-closed. Indeed, let $S \in \overline{\mathcal{I}''}$ and $\varepsilon > 0$ be given, and choose $T \in \mathcal{I}''$ with $\|S - T\| < \varepsilon$. In particular, $\|SQ_n - TQ_n\| < \varepsilon$ for all n. Since n_0 can be produced so as to have $\|TQ_n\| < \varepsilon$ for all $n \ge n_0$, we have $\|SQ_n\| < 2\varepsilon$ for all $n \ge n_0$, thus $S \in \mathcal{I}''$. We next verify that $\mathcal{I}''$ is a right ideal. If $S \in \mathcal{I}''$ and $T \in \mathcal{T}(X)$, we may assume, having proved that $\mathcal{I}''$ is a closed subspace, that T is a monomial, say of degree $k \in \mathbb{Z}$. Then T maps $X(n)$ to $X(n+k)$ when $n+k \ge 0$, and consequently $\|STQ_n\| = \|SQ_{n+k}TQ_n\| \le \|SQ_{n+k}\| \cdot \|T\| \to 0$ as $n \to \infty$.

In conclusion, $\mathcal{I}'' \trianglelefteq \mathcal{T}(X)$, and evidently $\mathcal{I}' \subseteq \mathcal{I}''$. For the converse, we may use Lemma 2.4 with $\mathcal{G}_1 = \mathcal{I}''$ and $\mathcal{G}_2 = \mathcal{I}'$, as it is clear that $\mathcal{I}'' \cap \mathcal{T}_0(X) = \mathcal{I}'$. □

Definition 2.6. Let X be a subproduct system. Denote by $\mathcal{I}$ the gauge-invariant ideal $\mathcal{I}' = \mathcal{I}''$ of $\mathcal{T}(X)$. The *Cuntz-Pimsner algebra* of X is defined as $\mathcal{O}(X) := \mathcal{T}(X)/\mathcal{I}$.

Corollary 2.7. *For every $S \in \mathcal{I}$, $\|SR_n'\| \xrightarrow[n\to\infty]{} 0$.*

Proof. If $S \in \mathcal{I} \cap \mathcal{T}_k(X)$ for some $k \in \mathbb{Z}$ then $\|SR_n'\| = \sup_{m\ge n} \|SQ_m\| \xrightarrow[n\to\infty]{} 0$. The proof is complete using (1) of Lemma 2.4 and an approximation argument. □

From Proposition 2.2 we clearly obtain $\mathcal{K}(\mathcal{F}_X\mathcal{J}) \subseteq \mathcal{T}(X) \cap \mathcal{K}(\mathcal{F}_X) \subseteq \mathcal{I}$. For product systems the converse also holds, so our definition generalizes indeed that of Pimsner ([21]).

Proposition 2.8. *If X is a (faithful)* product *system X_E, then $\mathcal{I} = \mathcal{K}(\mathcal{F}_E\mathcal{J})$, that is, $\mathcal{O}(X_E) = \mathcal{O}(E)$.*

Proof. On one hand, $\mathcal{K}(\mathcal{F}_E\mathcal{J}) \subseteq \mathcal{I}$. On the other hand, $\mathcal{I}$ is gauge invariant, and if $0 \neq a \in \mathscr{M}$ then $\|\varphi_\infty(a)Q_n\| = \|a\|$ for all n, and so $\mathcal{I} \cap \varphi_\infty(\mathscr{M}) = \{0\}$. By Katsura's gauge-invariant uniqueness theorem we have $\mathcal{I} \subseteq \mathcal{K}(\mathcal{F}_E\mathcal{J})$. □

Let π be a representation of $\mathcal{T}(X)$ on a Hilbert space $\mathcal{H}$. Since $\mathcal{I} \trianglelefteq \mathcal{T}(X)$, one can decompose π as $\pi_{\mathcal{I}} \oplus \pi_{\mathcal{T}(X)/\mathcal{I}}$, where $\pi_{\mathcal{I}}$ represents $\mathcal{T}(X)$ on the invariant subspace $\overline{\operatorname{span}}\, \pi(\mathcal{I})\mathcal{H}$ and $\pi_{\mathcal{T}(X)/\mathcal{I}}$ on its orthogonal complement. Generally, if $\mathcal{H}' \subseteq \mathcal{H}$ is an invariant subspace for π, then the subrepresentation π' on $\mathcal{H}'$ has covariant representation T', with $T'_n : X(n) \otimes \mathcal{H}' \to \mathcal{H}'$ satisfying $\widetilde{T}'_n(\zeta \otimes h) = \widetilde{T}_n(\zeta \otimes h)$ and $\widetilde{T}'^*_n h = \widetilde{T}_n h$ for all $n \in \mathbb{Z}_+$, $\zeta \in X(n)$ and $h \in \mathcal{H}'$. Consequently $\widetilde{T}'_n \widetilde{T}'^*_n = (\widetilde{T}_n \widetilde{T}^*_n)_{|\mathcal{H}'}$.

Proposition 2.9. *Let X be a subproduct system whose fibers are Hilbert spaces (not necessarily finite dimensional) and π a C^*-representation of $\mathcal{T}(X)$ on a Hilbert space $\mathcal{H}$. Then the representation $\pi_{\mathcal{I}}$ of $\mathcal{T}(X)$ is pure.*

Proof. Denote by $T = (T_n)_{n\in\mathbb{Z}_+}, C = (C_n)_{n\in\mathbb{Z}_+}$ the covariant representations of $\pi, \pi_{\mathcal{I}}$, respectively. To verify that $\pi_{\mathcal{I}}$ is pure, it is enough to establish that $(\widetilde{C}_n \widetilde{C}^*_n x, x) \to 0$ as $n \to \infty$ for each $x = \pi(S)h$ where $S \in \mathcal{I}$ and $0 \neq h \in \mathcal{H}$.

Let x be as above. Given $\varepsilon > 0$, fix n with $\|S^* R'_n\| \leq \varepsilon / \|h\|$ (see Corollary 2.7). Let $(e_\kappa)_{\kappa\in K}$ be an orthonormal base for $X(n)$. Then from [26, Lemma 3.5],

$$
\begin{aligned}
(\widetilde{C}_n \widetilde{C}^*_n x, x) &= (\widetilde{T}_n \widetilde{T}^*_n x, x) = \sum_\kappa \big(\pi(S_n(e_\kappa)S_n(e_\kappa)^*)x, x\big) \\
&= \sum_\kappa \big(\pi(S^* S_n(e_\kappa)S_n(e_\kappa)^* S)h, h\big) = \sum_\kappa \big(\pi(S^* R'_n S_n(e_\kappa)S_n(e_\kappa)^* R'_n S)h, h\big).
\end{aligned}
$$

For every finite subset $F \subseteq K$ we have $\sum_{\kappa\in F} S_n(e_\kappa)S_n(e_\kappa)^* \leq I$. Consequently,

$$
\begin{aligned}
\big|(\widetilde{C}_n \widetilde{C}^*_n x, x)\big| &= \lim_{\substack{F\subseteq K \\ F \text{ is finite}}} \Big|\big(\pi(\sum_{\kappa\in F} S^* R'_n S_n(e_\kappa)S_n(e_\kappa)^* R'_n S)h, h\big)\Big| \\
&\leq \lim_{\substack{F\subseteq K \\ F \text{ is finite}}} \left\| \sum_{\kappa\in F} S^* R'_n S_n(e_\kappa)S_n(e_\kappa)^* R'_n S \right\| \cdot \|h\|^2 \\
&\leq \|S^* R'_n\|^2 \cdot \|h\|^2 \leq \varepsilon^2.
\end{aligned}
$$

This completes the proof. □

Before giving examples, we show in another way why the definition of $\mathcal{O}(X)$ as $\mathcal{T}(X)/\mathcal{I}$ makes sense. For $n \in \mathbb{Z}_+$, consider $\mathcal{L}(\oplus_{k=0}^n X(k))$ as a subspace of $\mathcal{L}(\mathcal{F}_X)$, and let $\mathcal{B} := \bigcup_{n=0}^\infty \mathcal{L}(\oplus_{k=0}^n X(k))$. Then $\mathcal{B}$ is a $*$-algebra, whose closure $\overline{\mathcal{B}}$ is a C^*-subalgebra of $\mathcal{L}(\mathcal{F}_X)$. Since, additionally, the inclusion $\overline{\mathcal{B}} \subseteq \mathcal{L}(\mathcal{F}_X)$ is nondegenerate, we may consider the multiplier algebra $M(\overline{\mathcal{B}})$ as a C^*-subalgebra of $\mathcal{L}(\mathcal{F}_X)$ in the usual manner. It is straightforward to check that $\mathcal{T}(X) \subseteq M(\overline{\mathcal{B}})$ as the set of monomials is total in $\mathcal{T}(X)$. Denote by q the quotient map $M(\overline{\mathcal{B}}) \to M(\overline{\mathcal{B}})/\overline{\mathcal{B}}$. Recall that in case X is the product system $X = X_E$ (we are assuming that E is faithful), Pimsner proved in [21] that $\mathcal{O}(E) \cong q(\mathcal{T}(E))$. (As a matter of fact, this was the *original* definition of $\mathcal{O}(E)$).

Proposition 2.10. *The ideal* $\ker q_{|\mathcal{T}(X)} = \overline{\mathcal{B}} \cap \mathcal{T}(X)$ *of* $\mathcal{T}(X)$ *is equal to* $\mathcal{I}$. *Equivalently,* $\mathcal{O}(X) \cong q(\mathcal{T}(X))$.

Proof. If $S \in \mathcal{I} \cap \mathcal{T}_0(X)$ then for every $\varepsilon > 0$ there exists some $n_0 \in \mathbb{N}$ such that, upon defining $T := SR_{n_0}$, we have $T \in \mathcal{B}$ and $\|S - T\| \leq \varepsilon$. Therefore $\mathcal{I} \cap \mathcal{T}_0(X) \subseteq \overline{\mathcal{B}}$, thus $\mathcal{I} \subseteq \overline{\mathcal{B}} \cap \mathcal{T}(X)$ (because $\overline{\mathcal{B}} \cap \mathcal{T}(X) = \ker q_{|\mathcal{T}(X)} \trianglelefteq \mathcal{T}(X)$). The converse holds similarly: if $S \in \overline{\mathcal{B}} \cap \mathcal{T}(X)$, then for all $\varepsilon > 0$ there is an operator $T \in \mathcal{B}$ such that $\|S - T\| \leq \varepsilon$, and if $n_0 \in \mathbb{N}$ is such that $T = TR_{n_0}$, then $\|S(I - R_{n_0})\| \leq 2\varepsilon$, so that $\|SQ_n\| \leq 2\varepsilon$ for $n > n_0$. □

3. Examples

The next theorem demonstrates certain circumstances under which the ideal $\mathcal{I}$ may be expressed somewhat more explicitly.

Theorem 3.1. *Let X be a subproduct system.*

(1) *If $Q_n \in \mathcal{T}(X)$ for all $n \in \mathbb{Z}_+$, then $\mathcal{I} = \langle Q_n : n \in \mathbb{Z}_+ \rangle$.*
(2) *If, additionally, $I \in \mathcal{T}(X)$ and π is a representation of $\mathcal{T}(X)$ whose associated covariant representation T satisfies $\widetilde{T}_n \widetilde{T}_n^* = \pi(R_n')$ for all $n \in \mathbb{N}$, then π admits a Wold decomposition—that is, it is unitarily equivalent to the direct sum of an induced representation and a fully-coisometric representation.*

Proof. (1) We clearly have $\langle Q_n : n \in \mathbb{Z}_+ \rangle \subseteq \mathcal{I}$. Conversely, suppose that $S \in \mathcal{I}$. From Corollary 2.7 one has $\|S - SR_m\| = \left\|SR_{m+1}'\right\| \to 0$. Since $R_m \in \langle Q_n : n \in \mathbb{Z}_+ \rangle$ for all m, we get $S \in \langle Q_n : n \in \mathbb{Z}_+ \rangle$.

(2) If π is such a representation of $\mathcal{T}(X)$ on $\mathcal{H}$, consider its decomposition with respect to the ideal $\mathcal{I}$, $\pi = \pi_{\mathcal{I}} \oplus \pi_{\mathcal{T}(X)/\mathcal{I}}$, as explained above, and write $\mathcal{H}' := \overline{\operatorname{span}}\, \pi(\mathcal{I})\mathcal{H}$. Since $\pi_{\mathcal{T}(X)/\mathcal{I}}$ factors through $\mathcal{T}(X)/\mathcal{I}$ by construction, $I - R_1' = Q_0 \in \mathcal{I}$ and $\widetilde{T}_1 \widetilde{T}_1^* = \pi(R_1')$, the representation $\pi_{\mathcal{T}(X)/\mathcal{I}}$ is fully coisometric. Denote by

$C = (C_n)_{n\in\mathbb{Z}_+}$ the covariant representation of $\pi_{\mathcal{I}}$. Then $\widetilde{C}_n\widetilde{C}_n^* = \pi(R'_n)_{|\mathcal{H}'}$, and in the terminology of [26, Definition 2.8] we have that $\Delta_*(C) = \pi(Q_0)_{|\mathcal{H}'}$ and

$$\begin{aligned}\Delta_*(C)C_n(\zeta)^*C_n(\zeta)\Delta_*(C) &= \pi(Q_0S_n(\zeta)^*S_n(\zeta)Q_0)_{|\mathcal{H}'}\\ &= \pi(\varphi_\infty(\langle\zeta,\zeta\rangle)Q_0)_{|\mathcal{H}'} = C_0(\langle\zeta,\zeta\rangle)\Delta_*(C).\end{aligned}$$

In other words, C is relatively isometric ([26, Definition 3.3]). To verify that it is pure, it is enough to establish that $\widetilde{C}_n\widetilde{C}_n^*x \to 0$ as $n\to\infty$ for vectors x of the form $\pi(S)h$, $S\in\mathcal{I}$ and $h\in\mathcal{H}$. Indeed, $\widetilde{C}_n\widetilde{C}_n^*x = \pi(R'_nS)h \to 0$ from Corollary 2.7.

In conclusion, all conditions of [26, Theorem 3.8] are satisfied, so that C (equivalently, $\pi_{\mathcal{I}}$) is induced. □

Corollary 3.2. *If X is a subproduct system of finite dimensional Hilbert spaces, then $\mathcal{I} = \mathbb{K}$ (the compacts over the separable Hilbert space $\mathcal{F}_X$), and X fulfills the requirements of Theorem 3.1 for every representation.*

Proof. From [23, Proposition 8.1] it follows that $\mathbb{K} \subseteq \mathcal{T}(X)$, and it is easily seen that $\langle Q_n : n\in\mathbb{Z}_+\rangle = \mathbb{K}$. The second assertion is a consequence of [26, Lemma 3.5]. □

Example 3.3. Take $d\in\mathbb{N}$. By [2, Theorem 5.7] we get the expected result $\mathcal{O}(\mathrm{SSP}_d) \cong C(\partial B_d)$ (see Example 1.7).

Example 3.4. Let Λ be a *subshift* in the sense of [15]. Then the C^*-algebra $\mathcal{O}_\Lambda$ associated with Λ is equal to $\mathcal{O}(X_\Lambda)$, where X_Λ is the subproduct system associated with Λ as in [23, §12] (see Definition 12.1 and Remark 12.2 there).

The conditions of Theorem 3.1 also hold for subproduct systems whose fibers are *not* Hilbert spaces. Example 3.8 below is an illustration of this.

Subproduct systems whose fibers are *infinite* dimensional Hilbert spaces, which do not satisfy the conditions of Theorem 3.1, are also of interest. We next consider the subproduct system SSP_∞. The following lemma is required to express $\mathcal{I}$ concretely.

Lemma 3.5. *Let $\mathscr{A}$ be a unital Abelian C^*-algebra. Suppose that there exists a bounded linear mapping $A : \ell_2(\mathbb{N}) \to \mathscr{A}$ such that:*

(1) $\|A(e_n)\| = 1$ *for each* $n\in\mathbb{N}$
(2) $\mathscr{A}$ *is generated by* $\{I, A(e_1), A(e_2), \ldots\}$
(3) *the inequality* $A(e_1)^*A(e_1) + \ldots + A(e_n)^*A(e_n) \le I$ *holds for all* $n\in\mathbb{N}$
(4) *for every unitary* $U\in B(\ell_2(\mathbb{N}))$*, the mapping* $A(x)\mapsto A(Ux)$ *extends to an automorphism* α_U *of* $\mathscr{A}$*.*

Then the structure space of $\mathscr{A}$ can be naturally identified with the unit ball

$$B := \left\{ (z_n)_{n\in\mathbb{N}} \in \overline{\mathbb{D}}^{\mathbb{N}} : \sum_{n=1}^{\infty} |z_n|^2 \leq 1 \right\}$$

of ℓ_2 endowed with the Tychonoff topology.

Proof. Denote the structure space of $\mathscr{A}$ by M. From assumptions (2) and (3) it follows (see [6, Theorem IX.2.11], for example) that the map $\rho \mapsto (\rho(A(e_n)))_{n\in\mathbb{N}}$ is a (topological) embedding of M into B. We should prove that it is surjective.

By (1) there is a pure state (one-dimensional representation) ρ_1 of $\mathscr{A}$ with $|\lambda| = 1$ where $\lambda := \rho_1(A(e_1))$. We must therefore have $\rho_1(A(e_k)) = 0$ for all $k \geq 2$. Write $B' := \{z \in B : \|z\|_2 = 1\}$. Given $z = (z_n)_{n\in\mathbb{N}} \in B'$, let $U \in B(\ell_2(\mathbb{N}))$ be a unitary with $(Ue_n, e_1) = \overline{\lambda} z_n$ for all $n \in \mathbb{N}$. The pure state $\rho_1 \circ \alpha_U$ (see (4)) satisfies

$$(\rho_1 \circ \alpha_U)(A(e_n)) = \rho_1(A(Ue_n)) = \sum_{k=1}^{\infty} (Ue_n, e_k)\rho_1(A(e_k)) = z_n \qquad (\forall n \in \mathbb{N}),$$

and consequently z belongs to M. Since B' is dense in B and M is closed, we have $B = M$, as desired. ☐

Example 3.6. Consider the subproduct system $X := \mathrm{SSP}_\infty$. Its Cuntz-Pimsner algebra is the commutative counterpart of $\mathcal{O}_\infty$. Let $\mathcal{I}_1$ denote the ideal in $\mathcal{T}(X)$ generated by the commutators $[S_1(e_n), S_1(e_m)^*]$, $n, m \in \mathbb{N}$. As in [2, Proposition 5.3], one sees that every such commutator is in $\mathcal{I}$, so that $\mathcal{I}_1 \subseteq \mathcal{I}$. The quotient $\mathcal{T}(X)/\mathcal{I}_1$ is a unital Abelian C^*-algebra.

We would like to apply Lemma 3.5 to $\mathcal{T}(X)/\mathcal{I}_1$ and $\mathcal{T}(X)/\mathcal{I}$ with $A(x)$ being defined as $S_1(x) + \mathcal{I}_1$ and $S_1(x) + \mathcal{I}$, respectively. It follows from the definition of $\mathcal{I}$ that for all $n \in \mathbb{N}$ and $T \in \mathcal{I}$,

$$\|S_1(e_n) + T\| \geq \lim_{m\to\infty} \|(S_1(e_n) + T)Q_m\| = \lim_{m\to\infty} \|S_1(e_n)Q_m\| = 1$$

(because $S_1(e_n)(e_n^{\otimes m}) = e_n^{\otimes(m+1)}$ for all m). Therefore $\|S_1(e_n) + \mathcal{I}\| = 1$, and thus $\|S_1(e_n) + \mathcal{I}_1\| = 1$, proving (1). Assumptions (2) and (3) clearly hold in both cases.

To establish (4), let $U \in B(\ell_2(\mathbb{N}))$ be unitary. Define a unitary $W \in B(\mathcal{F}_X)$ to be the restriction to $\mathcal{F}_X$ of the unitary $\bigoplus_{n\in\mathbb{Z}_+} U^{\otimes n}$ over the full Fock space $\mathcal{F}_E$. The automorphism α_U of $\mathcal{T}(X)$ mapping $S_1(x)$ to $S_1(Ux)$ ($x \in E$) is implemented by W. Direct calculation shows that $\alpha_U([S_1(e_n), S_1(e_m)^*]) \in \mathcal{I}_1$ for all $n, m \in \mathbb{N}$, thus $\alpha_U(\mathcal{I}_1) = \mathcal{I}_1$. Furthermore, $\alpha_U(\mathcal{I}) = \mathcal{I}$ as W commutes with Q_n for all n.

In conclusion, it follows from Lemma 3.5 that $\mathcal{I}_1 = \mathcal{I}$, and that we have the exact sequence

$$0 \to \mathcal{I} \to \mathcal{T}(X) \to C(B) \to 0$$

(compare [2, Theorem 5.7]). Consequently, $\mathcal{O}(X) \cong C(B)$.

For a given point $(z_n)_{n\in\mathbb{N}} = z \in B$, the corresponding representation $\rho_z : \mathcal{T}(X) \to \mathbb{C}$ (pulled back from $\mathcal{T}(X)/\mathcal{I}$) satisfies $\rho_z(S_1(e_n)) = z_n$ for all $n \in \mathbb{N}$. Therefore, denoting by T the suitable covariant representation of $\mathcal{T}(X)$, we obtain $\widetilde{T}_1\widetilde{T}_1^* = \sum_{n=1}^{\infty} |z_n|^2 = \|z\|_2^2$ (for general $m \in \mathbb{N}$, $\widetilde{T}_m\widetilde{T}_m^* = \sum_{\alpha\in\mathbb{N}^m} |z_{a_1}|^2 \cdots |z_{\alpha_n}|^2 = \|z\|_2^{2m}$). Hence, if $z \neq 0$, then $\widetilde{T}_1 : E \to \mathbb{C}$ is a partial isometry if and only if $z \in B'$ (see the notation of the proof of Lemma 3.5), if and only if T is fully coisometric. In particular, there is an abundance of representations of $\mathcal{T}(X)$ whose associated covariant representations are not a partial isometry. In the pathological case $z = 0$ we have $\rho_z(S_1(e_n)) = 0$ for all n and $\widetilde{T}_1 = 0$. The covariant representation T is trivially pure, but it is by no means relatively isometric (see [26, §3]). Particularly, T extends to a C^*-representation although the conditions of [26, Theorem 3.8] are not satisfied.

Remark 3.7. The last example shows very clearly that for subproduct systems, defining the Cuntz-Pimsner algebra as $\mathcal{T}(X)/\mathcal{K}(\mathcal{F}_X\mathcal{J})$ is counter-intuitive, since $\mathcal{J}$ of SSP_∞ is $\{0\}$. This stands in stark contrast to the Cuntz algebra $\mathcal{O}_\infty$, which equals its corresponding Toeplitz algebra $\mathcal{T}\mathcal{O}_\infty$.

Example 3.8 (The subproduct system of a "positive" matrix). For a unital C^*-algebra $\mathscr{M}$ and a completely positive map P over $\mathscr{M}$, the C^*-correspondence $\mathscr{M} \otimes_P \mathscr{M}$ over $\mathscr{M}$ (see [20, §5]) is constructed from the algebraic $\mathbb{C}$-balanced tensor product $\mathscr{M} \otimes_{\mathrm{alg}} \mathscr{M}$ by giving it the standard left and right actions and the rigging

$$\langle a \otimes_P b, c \otimes_P d\rangle = b^* P(a^* c) d.$$

A sufficient condition for $\mathscr{M} \otimes_P \mathscr{M}$ to be faithful is that P be faithful.

Let P_1, P_2 be two completely positive maps over $\mathscr{M}$. Then P_2P_1 is also a completely positive map over $\mathscr{M}$, and there is a correspondence isometry

$$V_{P_1,P_2} : \mathscr{M} \otimes_{P_2P_1} \mathscr{M} \to (\mathscr{M} \otimes_{P_1} \mathscr{M}) \otimes (\mathscr{M} \otimes_{P_2} \mathscr{M})$$

defined by $a \otimes_{P_2P_1} b \mapsto (a \otimes_{P_1} I_{\mathscr{M}}) \otimes (I_{\mathscr{M}} \otimes_{P_2} b)$, $a, b \in \mathscr{M}$.

Henceforth we take $\mathscr{M} := \mathbb{C}^d$, $d \in \mathbb{N}$. In this case, a linear map $P : \mathscr{M} \to \mathscr{M}$ can be identified with a matrix $P = (P_{ij}) \in M_d(\mathbb{C})$. The map P is completely positive if and only if it is positive (as $\mathscr{M}$ is commutative), which is equivalent to that $P_{ij} \geq 0$ for all i, j. We also assume that P is faithful, equivalently: every column of P has at least one entry with value strictly greater than zero.

Let $e_1, \dots, e_d$ be the standard basis of $\mathbb{C}^d$. Write e_{ij} for (the equivalence class of) the element $e_i \otimes e_j$ of $\mathscr{M} \otimes_P \mathscr{M}$. Notice that

$$\langle e_{ij}, e_{kl} \rangle_{\mathscr{M} \otimes_P \mathscr{M}} = e_j^* P(e_i^* e_k) e_l = \begin{cases} P_{ji} e_j & \text{if } (i,j) = (k,l) \\ 0 & \text{else.} \end{cases} \tag{3.1}$$

In particular, $e_{ij} \neq 0$ in $\mathscr{M} \otimes_P \mathscr{M}$ if and only if $P_{ji} > 0$.

Let now G_P stand for the quiver with vertices $1, 2, \dots, d$, and with an edge going from j to i (denoted by g_{ij}) if and only if $P_{ji} > 0$. This quiver is the *support* of P. Write f_i for the element of $C(G_P^{(0)}) \cong \mathscr{M}$ mapping i to 1 and all other vertices to 0, and f_{ij} for the element of $C(G_P^{(1)})$ mapping g_{ij} to 1 and all other edges to 0 (if $P_{ji} > 0$; otherwise, set $f_{ij} := 0$). Then the C^*-correspondence $C(G_P^{(1)})$ of G_P (see Example 1.4) is naturally isomorphic to $\mathscr{M} \otimes_P \mathscr{M}$ via $\Psi_P : \mathscr{M} \otimes_P \mathscr{M} \to C(G_P^{(1)})$ defined by $e_{ij} \mapsto \sqrt{P_{ji}} f_{ij}$.

For all $n \in \mathbb{N}$, the map P^n is (completely) positive over $\mathscr{M}$ and faithful. Let $\left(P^n_{ij}\right) \in M_d(\mathbb{C})$ be its representing matrix, and denote by $X(n)$ the C^*-correspondence $\mathscr{M} \otimes_{P^n} \mathscr{M}$. Write also $X(0) := \mathscr{M}$. Fix $n, m \in \mathbb{N}$. Regarding e_{ij}, e_{kl} as elements of $X(n)$, $X(m)$, respectively, one sees from (3.1) that

$$\begin{aligned} \langle e_{ij} \otimes e_{kl}, e_{ij} \otimes e_{kl} \rangle_{X(n) \otimes X(m)} &= \left\langle e_{kl}, \langle e_{ij}, e_{ij} \rangle_{X(n)} \cdot e_{kl} \right\rangle_{X(m)} \\ &= P^n_{ji} \langle e_{kl}, e_j \cdot e_{kl} \rangle_{X(m)} = P^m_{lk} P^n_{ji} e_l \delta_{j,k}. \end{aligned}$$

In particular, $e_{ij} \otimes e_{kl} \neq 0$ in $X(n) \otimes X(m)$ if and only if $j = k$ and $P^n_{ji}, P^m_{lk} > 0$.

As seen above, we may regard $X(n+m)$ as a sub-correspondence of $X(n) \otimes X(m)$ via the embedding $V_{n,m} := V_{P^n, P^m}$. Now $V_{n,m}$ is adjointable, and its adjoint $V^*_{n,m} : X(n) \otimes X(m) \to X(n+m)$ is given by

$$V^*_{n,m}(e_{ij} \otimes e_{kl}) = \begin{cases} (P^{n+m}_{li})^{-1} P^m_{lj} P^n_{ji} e_{il} & \text{if } j = k \text{ and } P^{n+m}_{li} > 0 \\ 0 & \text{else.} \end{cases}$$

Indeed, for elements of the form $e_{ij} \in X(n)$, $e_{kl} \in X(m)$ and $e_{pq} \in X(n+m)$ we have by (3.1)

$$\begin{aligned} \langle V_{n,m} e_{pq}, e_{ij} \otimes e_{kl} \rangle_{X(n) \otimes X(m)} &= \langle (e_p \otimes_{P^n} I_{\mathscr{M}}) \otimes (I_{\mathscr{M}} \otimes_{P^m} e_q), e_{ij} \otimes e_{kl} \rangle_{X(n) \otimes X(m)} \\ &= \sum_{t=1}^d \langle e_{pt} \otimes e_{tq}, e_{ij} \otimes e_{kl} \rangle_{X(n) \otimes X(m)} = \sum_{t=1}^d \left\langle e_{tq}, \langle e_{pt}, e_{ij} \rangle_{X(n)} \cdot e_{kl} \right\rangle_{X(m)} \\ &= P^n_{ji} \delta_{p,i} \langle e_{jq}, e_j \cdot e_{kl} \rangle_{X(m)} = \delta_{p,i} \delta_{q,l} \delta_{j,k} P^m_{lj} P^n_{ji} e_l. \end{aligned}$$

It is easy to check that $(V_{n,m} \otimes I_{X(k)})V_{n+m,k} = (I_{X(n)} \otimes V_{m,k})V_{n,m+k}$ for all n, m, k, making $X = (X(n))_{n\in\mathbb{Z}_+}$ a subproduct system.

For $n \in \mathbb{N}$, let $Y(n)$ denote the C^*-correspondence of G_{P^n}. Then $Y = (Y(n))_{n\in\mathbb{Z}_+}$ is a subproduct system with respect to the embeddings

$$W_{n,m} := (\Psi_{P^n} \otimes \Psi_{P^m})\, V_{n,m}\Psi^{-1}_{P^{n+m}} : Y(n+m) \to Y(n) \otimes Y(m),$$

which satisfy

$$W_{n,m} f_{ij} = \frac{1}{\sqrt{P^{n+m}_{ji}}} \sum_{t=1}^{d} (\Psi_{P^n} \otimes \Psi_{P^m})\, e_{it} \otimes e_{tj} = \sum_{t=1}^{d} \sqrt{\frac{P^m_{jt} P^n_{ti}}{P^{n+m}_{ji}}} f_{it} \otimes f_{tj}$$

and

$$W^*_{n,m}\,(f_{ik} \otimes f_{kl}) = \begin{cases} \sqrt{\frac{P^m_{lk} P^n_{ki}}{P^{n+m}_{li}}} f_{il} & \text{if } P^{n+m}_{li} > 0 \\ 0 & \text{else.} \end{cases}$$

Abbreviate $S^Y_n(\zeta)$ by $S_n(\zeta)$. Let $n, m \in \mathbb{N}$. Regard f_{ij}, f_{kl} as elements of $Y(n)$, $Y(m)$, respectively. Then

$$S_n(f_{ij}) f_{kl} = W^*_{n,m}(f_{ij} \otimes f_{kl}) = \begin{cases} \sqrt{\frac{P^m_{lk} P^n_{ki}}{P^{n+m}_{li}}} \delta_{j,k} f_{il} & P^{n+m}_{li} > 0 \\ 0 & \text{else.} \end{cases} \in Y(n+m)$$

Regarding f_{ij}, f_{kl} as elements of $Y(n)$, $Y(n+m)$, respectively, we obtain

$$S^*_n(f_{ij}) f_{kl} = \sum_{t=1}^{d} \sqrt{\frac{P^m_{lt} P^n_{tk}}{P^{n+m}_{lk}}} \langle f_{ij}, f_{kt}\rangle_{Y(n)} \cdot f_{tl} = \sqrt{\frac{P^m_{lj} P^n_{jk}}{P^{n+m}_{lk}}} \delta_{i,k} f_{jl} \in Y(m)$$

while if $f_{ij}, f_{kl} \in Y(n)$ then

$$S^*_n(f_{ij}) f_{kl} = \langle f_{ij}, f_{kl}\rangle_{Y(n)} = \delta_{(i,j),(k,l)} f_l.$$

Given $n, m \in \mathbb{N}$, consider f_{kl} as an element of $Y(n+m)$ (assuming $P^{n+m}_{lk} > 0$). Then

$$\sum_{s,t=1}^{d} S_n(f_{ts}) S_n(f_{ts})^* f_{kl} = \sum_{t=1}^{d} \delta_{t,k} \sum_{s=1}^{d} \sqrt{\frac{P^m_{ls} P^n_{sk}}{P^{n+m}_{lk}}} S_n(f_{ks}) f_{sl} = \sum_{s=1}^{d} \frac{P^m_{ls} P^n_{sk}}{P^{n+m}_{lk}} f_{kl} = f_{kl}. \tag{3.2}$$

Similarly, it is interesting to note that for all $n \in \mathbb{N}$ the left multiplication in $Y(n)$ is implemented by compacts: $\varphi(f_t) = \sum_{s=1}^{d} f_{ts} \otimes f^*_{ts}$.

Proposition 3.9. *Let $P \in M_d(\mathbb{C})$ be as in the last example, and Y be the associated subproduct system. Let π be a representation of Y on a Hilbert space $\mathcal{H}$, with T the* *associated covariant representation. Then* $\{Q_n : n \in \mathbb{Z}_+\} \subseteq \mathcal{T}(Y)$ *and for all* n*,*

$$\widetilde{T}_n^* h = \sum_{i,j=1}^{d} f_{ij} \otimes T_n(f_{ij})^* h \qquad (\forall h \in \mathcal{H}) \tag{3.3}$$

and

$$\widetilde{T}_n \widetilde{T}_n^* = \pi(R'_n).$$

Hence, Y *fulfills the requirements of Theorem 3.1 for every representation.*

Proof. Equation (3.3) is checked by a simple calculation, because

$$\langle f_{ij}, f_{kl} \rangle_{Y(n)} = \begin{cases} \delta_{(i,j),(k,l)} f_j & P^n_{ji} > 0, \text{ equivalently: } f_{ij} \neq 0 \\ 0 & \text{else.} \end{cases}$$

The other assertions follow from (3.2). We omit the details. □

4. Essential and fully-coisometric representations

As further justification for the definition of $\mathcal{O}(X)$ as $\mathcal{T}(X)/\mathcal{I}$, we sought a suitable "universality" property of $\mathcal{I}$, which could replace the gauge-invariant uniqueness theorem (cf. Example 2.3). More specifically, our goal was to express $\mathcal{I}$ as the intersection of a certain set of ideals, as in the next proposition. Unfortunately, we could generally establish only half of this characterization in Theorem 4.3. Nevertheless, we exemplify many subproduct systems for which $\mathcal{I}$ has this property, the most non-standard of which is the infinite-dimensional symmetric subproduct system SSP_∞ (Example 3.6).

Proposition 4.1. *Let* E *be a faithful and essential* C^**-correspondence. Then the intersection of the kernels of all fully-coisometric* C^**-representations of* $\mathcal{T}(E)$ *is* $\mathcal{K}(\mathcal{F}_E\mathcal{J})$*.*

Proof. In case E is *full*, this result is a reformulation of [9, Theorem 1.2].

For the general case, denote the above-mentioned intersection by $\mathcal{P}$. Let π be a fully-coisometric C^*-representation of $\mathcal{T}(E)$. Then $\mathcal{K}(\mathcal{F}_E\mathcal{J}) \subseteq \ker\pi$ by [17, Lemma 5.5]. Moreover, if $\lambda \in \mathbb{T}$, then the C^*-representation $\pi \circ \alpha_\lambda$ of $\mathcal{T}(E)$ is also fully coisometric. As a result, $\mathcal{P}$ is gauge invariant. By [25, Theorem 8.3], there exists a fully-coisometric C^*-representation π of $\mathcal{T}(E)$ such that $\pi \circ \varphi_\infty$ is faithful. Consequently, $\mathcal{P} \cap \varphi_\infty(\mathscr{M}) = \{0\}$. Katsura's gauge-invariant uniqueness theorem therefore implies that $\mathcal{P} = \mathcal{K}(\mathcal{F}_E\mathcal{J})$, as desired. □

Definition 4.2. Let X be a subproduct system. A C^*-representation π of $\mathcal{T}(X)$ on $\mathcal{H}$ is said to be *essential* if the associated covariant representation T satisfies that $\operatorname{Im}\widetilde{T}_n$ is dense in $\mathcal{H}$ (equivalently: $\bigcup_{\zeta \in X(n)} \operatorname{Im} T_n(\zeta)$ is total in $\mathcal{H}$) for all n.

This requirement is weaker than π being fully coisometric, and it is often strictly weaker; see Example 3.6 (also compare [26, Remark 4.2]). Nevertheless, in some special cases, π is essential if and only if it is fully coisometric. This happens, in particular, when the operators $\widetilde{T}_n$ are automatically partial isometries. For instance:

(1) if X is a *product* system, because then the operators $\widetilde{T}_n$ are isometries;
(2) if π satisfies the conditions of Theorem 3.1, (2); for example, if the fibers of X are finite-dimensional Hilbert spaces, or if X is the subproduct system over $\mathbb{C}^d$ constructed in Example 3.8 (by Proposition 3.9).

Theorem 4.3. *If X is a subproduct system and π is an essential C^*-representation of $\mathcal{T}(X)$, then $\mathcal{I} \subseteq \ker \pi$.*

Proof. Suppose that π represents $\mathcal{T}(X)$ on the Hilbert space $\mathcal{H}$. Let $S \in \mathcal{I}$. We have to show that $S \in \ker \pi$. Fix $x \in \mathcal{H}$ and $\varepsilon > 0$, and choose n such that $\|SR'_n\| \leq \varepsilon$ (see Corollary 2.7). By assumption, the set $\operatorname{span}\{\pi(S_n(\zeta))y : \zeta \in X(n), y \in \mathcal{H}\}$ is dense in $\mathcal{H}$, so there exist $t \in \mathbb{N}$, $\zeta_1, \ldots, \zeta_t \in X(n)$ and $y_1, \ldots y_t \in \mathcal{H}$ so that

$$\|x - z\|_{\mathcal{H}} \leq \varepsilon \text{ for } z := \pi(S_n(\zeta_1))y_1 + \ldots + \pi(S_n(\zeta_t))y_t.$$

Then

$$\begin{aligned}\|\pi(S)z\|_{\mathcal{H}}^2 &= \sum_{i,j=1}^{t} \big(\pi(S_n(\zeta_j)^* S^* S S_n(\zeta_i))y_i, y_j\big)_{\mathcal{H}} \\ &= \left(\pi^{(t)}\left(\big(S_n(\zeta_i)^* S^* S S_n(\zeta_j)\big)_{i,j=1}^t\right)(y_k)_{k=1}^t, (y_\ell)_{\ell=1}^t\right)_{\mathcal{H}\otimes\mathbb{C}^t}.\end{aligned}$$

For all $\zeta \in X(n)$ we have $SS_n(\zeta) = SR'_n S_n(\zeta)$ and $0 \leq R'_n S^* S R'_n \leq \varepsilon^2 I_{\mathcal{F}_X}$. Hence, using the (positive) matrix inequality

$$\big(S_n(\zeta_i)^* R'_n S^* S R'_n S_n(\zeta_j)\big)_{i,j=1}^t \leq \varepsilon^2 \cdot \big(S_n(\zeta_i)^* S_n(\zeta_j)\big)_{i,j=1}^t,$$

we see that

$$\|\pi(S)z\|_{\mathcal{H}}^2 \leq \varepsilon^2 \left(\pi^{(t)}\left(\big(S_n(\zeta_i)^* S_n(\zeta_j)\big)_{i,j=1}^t\right)(y_k)_{k=1}^t, (y_\ell)_{\ell=1}^t\right)_{\mathcal{H}\otimes\mathbb{C}^t} = \varepsilon^2 \|z\|_{\mathcal{H}}^2.$$

Finally we have

$$\|\pi(S)x\| \leq \|\pi(S)(x-z)\| + \|\pi(S)z\| \leq \varepsilon(\|S\| + \|z\|) \leq \varepsilon(\|S\| + \|x\| + \varepsilon),$$

so $\pi(S) = 0$ indeed. ☐

Definition 4.4. A subproduct system X is called *tame* if $\mathcal{I} = \bigcap \ker \pi$, when π ranges over all fully-coisometric C^*-representations of $\mathcal{T}(X)$.

Example 4.5. If X fulfills the requirements of Theorem 3.1 for all C^*-representations of $\mathcal{T}(X)$, then every such representation that factors through $\mathcal{T}(X)/\mathcal{I}$ is fully coisometric. Hence, by Theorem 4.3, X is tame. This class of subproduct systems is wide—see Corollary 3.2 and Proposition 3.9.

Example 4.6 (cont. of Example 3.6)**.** The subproduct system $X := \mathrm{SSP}_\infty$ is tame, for if $S \in \mathcal{T}(X)\backslash\mathcal{I}$ and $f \neq 0$ is the corresponding element of $C(B)$, there exists $z \in B'$ with $f(z) \neq 0$ (as B' is dense), and ρ_z gives rise to a fully-coisometric C^*-representation π of $\mathcal{T}(X)$ such that $\pi(S) \neq 0$.

Conjecture 4.7. *All subproduct systems satisfying some mild hypotheses are tame, at least if the adjective "fully-coisometric" is replaced by "essential" in Definition 4.4.*

We conclude this section by giving a rough structure theory for the representations of $\mathcal{T}(\mathrm{SSP}_\infty)$.

Example 4.8 (cont. of Example 3.6)**.** For $X := \mathrm{SSP}_\infty$, let π be a C^*-representation of $\mathcal{T}(X)$ on $\mathcal{H}$. As in Proposition 2.9 and the preceding paragraph, decompose π as $\pi_{\mathcal{I}} \oplus \pi_{\mathcal{T}(X)/\mathcal{I}}$, and let T be the covariant representation of $\pi_{\mathcal{T}(X)/\mathcal{I}}$. We already know that $\pi_{\mathcal{I}}$ is pure (whether more could be said is an open question). Since $\pi_{\mathcal{T}(X)/\mathcal{I}}$ factors through $\mathcal{T}(X)/\mathcal{I} \cong C(B)$, we consider $\pi_{\mathcal{T}(X)/\mathcal{I}}$ as a C^*-representation of $C(B)$. Write $\mathscr{K}$ for the closure of $\operatorname{Im} \widetilde{T}\widetilde{T}^*$ (equivalently, of $\operatorname{Im} \widetilde{T}$). Then $\mathscr{K}$ is the closed span of the union of the images of $T(e_n)$, $n \in \mathbb{N}$, which, by virtue of normality, contains the images of $T(e_n)^*$, $n \in \mathbb{N}$. Thus, $\mathscr{K}$ is invariant for $\pi_{\mathcal{T}(X)/\mathcal{I}}$. Decompose $\pi_{\mathcal{T}(X)/\mathcal{I}}$ as $\pi' \oplus \pi''$ with respect to $\mathscr{K}$ and $\mathscr{K}^\perp$. By construction, the C^*-representation π' is essential and π'' satisfies $\pi''(S_n(\zeta)) = 0$ for all $n \in \mathbb{N}$ and $\zeta \in X(n)$.

5. Morita equivalence

In this section we generalize ideas of [18] to develop a notion of Morita equivalence for subproduct systems. It is proved in Theorems 5.9, 5.11 and 5.15 that if two subproduct systems are equivalent in this sense, then so are their tensor, Toeplitz and Cuntz-Pimsner algebras. In particular, the last theorem is proved by showing that the Rieffel correspondence associated with the equivalence of the Toeplitz algebras carries the ideal $\mathcal{I}$ of the first to that of the second. This is yet another evidence of the naturality of the definition of the Cuntz-Pimsner algebra for subproduct systems as the quotient by $\mathcal{I}$. The results of this section should also be compared to those of [1].

5.1. **Strong Morita equivalence of subproduct systems.** Our standard reference for Morita equivalence is [22]. We assume that the reader has basic familiarity with [18, §1-2], part of which is summarized here for the sake of convenience.

Let $\mathscr{A}, \mathscr{B}$ be C^*-algebras, and suppose that they are Morita equivalent via an imprimitivity bimodule M. We denote by $\widetilde{\mathsf{M}}$ the opposite (dual) bimodule, and recall that the maps $m_{\mathscr{A}} : \mathsf{M} \otimes_{\mathscr{B}} \widetilde{\mathsf{M}} \to \mathscr{A}$, $m_{\mathscr{B}} : \widetilde{\mathsf{M}} \otimes_{\mathscr{A}} \mathsf{M} \to \mathscr{B}$ given by $x \otimes \widetilde{y} \mapsto {}_{\mathscr{A}}\langle x, y\rangle$ and $\widetilde{x} \otimes y \mapsto \langle x, y\rangle_{\mathscr{B}}$, respectively, are correspondence isomorphisms.

Definition 5.1 ([18, Definition 2.1])**.** Let E, F be C^*-correspondences over $\mathscr{A}, \mathscr{B}$, respectively. If the C^*-algebras $\mathscr{A}, \mathscr{B}$ are Morita equivalent via an imprimitivity bimodule M, and if there exists a correspondence isomorphism from $\mathsf{M} \otimes_{\mathscr{B}} F$ onto $E \otimes_{\mathscr{A}} \mathsf{M}$, we say that E and F are strongly Morita equivalent, and write $E \overset{\mathrm{SME}}{\sim}_{\mathsf{M}} F$.

Example 5.2. If $\mathscr{A}$ and $\mathscr{B}$ are Morita equivalent C^*-algebras, then they are (strongly) Morita equivalent as C^*-correspondences.

When the conditions of Definition 5.1 hold, the isomorphism $W : \mathsf{M} \otimes_{\mathscr{B}} F \to E \otimes_{\mathscr{A}} \mathsf{M}$ of Definition 5.1 induces an isomorphism $\widetilde{W}$ from $\widetilde{\mathsf{M}} \otimes_{\mathscr{A}} E$ onto $F \otimes_{\mathscr{B}} \widetilde{\mathsf{M}}$. Additionally, $E^{\otimes n} \overset{\mathrm{SME}}{\sim}_{\mathsf{M}} F^{\otimes n}$ for each $n \in \mathbb{N}$, with correspondence isomorphisms $W_n : \mathsf{M} \otimes_{\mathscr{B}} F^{\otimes n} \to E^{\otimes n} \otimes_{\mathscr{A}} \mathsf{M}$ satisfying $W_1 = W$ and

$$W_{n+m} = (I_{E^{\otimes n}} \otimes W_m)(W_n \otimes I_{F^{\otimes m}}). \tag{5.1}$$

Letting W_0 denote the natural isomorphism from $\mathsf{M} \otimes_{\mathscr{B}} \mathscr{B}$ onto $\mathscr{A} \otimes_{\mathscr{A}} \mathsf{M}$, this last equation actually holds for all $n, m \in \mathbb{Z}_+$.

In the sequel , when $\mathscr{A}$ and $\mathscr{B}$ are Morita equivalent via M, we let L be the "linking C^*-algebra" of $\mathscr{A}$ and $\mathscr{B}$ ([4]), namely

$$L := \begin{pmatrix} \mathscr{B} & \widetilde{\mathsf{M}} \\ \mathsf{M} & \mathscr{A} \end{pmatrix},$$

and for E, F as above, we write Z for the Hilbert L-module

$$Z := \begin{pmatrix} F & F \otimes_{\mathscr{B}} \widetilde{\mathsf{M}} \\ E \otimes_{\mathscr{A}} \mathsf{M} & E \end{pmatrix}$$

(see [18, p. 121]).

Proposition 5.3 ([18, Proposition 2.6])**.** *If $E \overset{\mathrm{SME}}{\sim}_{\mathsf{M}} F$ then there is a left action of L on Z, $\varphi_Z : L \to \mathcal{L}(Z)$, making Z an L-correspondence, satisfying $\overline{\operatorname{span}}\,(\varphi_Z(L)\left(\begin{smallmatrix} F & 0 \\ 0 & E \end{smallmatrix}\right)) = Z$ and $\varphi_Z\left(\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix}\right)\left(\begin{smallmatrix} \eta & 0 \\ 0 & \zeta \end{smallmatrix}\right) = \left(\begin{smallmatrix} b\eta & 0 \\ 0 & a\zeta \end{smallmatrix}\right)$ for $a \in \mathscr{A}$, $b \in \mathscr{B}$, $\zeta \in E$ and $\eta \in F$. Particularly, Z is essential.*

The complete definition of φ_Z is given in [18, p. 125].

The following notion of Morita equivalence of subproduct systems is natural in light of Definition 5.1, as well as [25, Definition 5.10].

Definition 5.4. Let X, Y be subproduct systems over $\mathscr{A}, \mathscr{B}$ respectively, and write $E := X(1), F := Y(1)$. We say that X and Y are *strongly Morita equivalent* with respect to M and denote $X \overset{\mathrm{SME}}{\sim}_{\mathsf{M}} Y$ if $E \overset{\mathrm{SME}}{\sim}_{\mathsf{M}} F$ in the sense of Definition 5.1, with implementing correspondence isomorphism $W : \mathsf{M} \otimes_{\mathscr{B}} F \to E \otimes_{\mathscr{A}} \mathsf{M}$ that satisfies

$$W_n(\mathsf{M} \otimes_{\mathscr{B}} Y(n)) = X(n) \otimes_{\mathscr{A}} \mathsf{M}, \tag{5.2}$$

or, equivalently,

$$W_n(I_{\mathsf{M}} \otimes p_n^Y) = (p_n^X \otimes I_{\mathsf{M}})W_n, \tag{5.3}$$

for all $n \in \mathbb{N}$. In particular, this implies that $X(n) \overset{\mathrm{SME}}{\sim}_{\mathsf{M}} Y(n)$ (with W_n implementing the equivalence). Depending upon the context, we will regard W_n as a mapping either from $\mathsf{M} \otimes_{\mathscr{B}} F^{\otimes n}$ to $E^{\otimes n} \otimes_{\mathscr{A}} \mathsf{M}$ or from $\mathsf{M} \otimes_{\mathscr{B}} Y(n)$ to $X(n) \otimes_{\mathscr{A}} \mathsf{M}$. The relation $\overset{\mathrm{SME}}{\sim}$ is certainly an equivalence relation.

Remark 5.5. If X is a subproduct system over $\mathscr{A}$, F is an essential C^*-correspondence over $\mathscr{B}$ and $E := X(1) \overset{\mathrm{SME}}{\sim}_{\mathsf{M}} F$, then the implementing isomorphism W can be used to canonically induce a subproduct system Y over $\mathscr{B}$ with $Y(1) = F$ such that $X \overset{\mathrm{SME}}{\sim}_{\mathsf{M}} Y$. Indeed, let

$$Y(n) := (m_{\mathscr{B}} \otimes I_{F^{\otimes n}})\big(\widetilde{\mathsf{M}} \otimes_{\mathscr{A}} W_n^{-1}(X(n) \otimes \mathsf{M})\big)$$

for every $n \in \mathbb{N}$. Then $Y(n)$ is an orthogonally-complementable sub-correspondence of $F^{\otimes n}$, $Y(n+m) \subseteq Y(n) \otimes Y(m)$ for all n, m, and (5.2) holds. The details are left to the reader.

In what follows we assume that the conditions of Definition 5.4 are satisfied unless stated otherwise. For $n \in \mathbb{N}$, denote by $Z(n)$ the L-correspondence associated with the equivalence $X(n) \overset{\mathrm{SME}}{\sim}_{\mathsf{M}} Y(n)$ on account of Proposition 5.3,

$$Z(n) = \begin{pmatrix} Y(n) & Y(n) \otimes_{\mathscr{B}} \widetilde{\mathsf{M}} \\ X(n) \otimes_{\mathscr{A}} \mathsf{M} & X(n) \end{pmatrix},$$

and let $Z(0) := L$ (this makes sense as $\mathscr{A} \otimes_{\mathscr{A}} \mathsf{M} \cong \mathsf{M}$ and $\mathscr{B} \otimes_{\mathscr{B}} \widetilde{\mathsf{M}} \cong \widetilde{\mathsf{M}}$). We will require the subspace $C(n) := \begin{pmatrix} Y(n) & 0 \\ 0 & X(n) \end{pmatrix}$ of $Z(n)$. If $n, m \in \mathbb{N}$, then from [18, Lemmas 2.7, 2.8] we have $X(n) \otimes X(m) \overset{\mathrm{SME}}{\sim}_{\mathsf{M}} Y(n) \otimes Y(m)$, with associated L-correspondence

$$Z_{n,m} := \begin{pmatrix} Y(n) \otimes Y(m) & Y(n) \otimes Y(m) \otimes_{\mathscr{B}} \widetilde{\mathsf{M}} \\ X(n) \otimes X(m) \otimes_{\mathscr{A}} \mathsf{M} & X(n) \otimes X(m) \end{pmatrix};$$

furthermore, there is a natural L-correspondence isomorphism

$$\Psi_{n,m} : Z_{n,m} \to Z(n) \otimes_L Z(m),$$

which restricts to the map from $\begin{pmatrix} Y(n)\otimes Y(m) & 0 \\ 0 & X(n)\otimes X(m) \end{pmatrix}$ onto $C(n) \otimes_L C(m)$ given by $\begin{pmatrix} \eta_1\otimes\eta_2 & 0 \\ 0 & \zeta_1\otimes\zeta_2 \end{pmatrix} \mapsto \begin{pmatrix} \eta_1 & 0 \\ 0 & \zeta_1 \end{pmatrix} \otimes \begin{pmatrix} \eta_2 & 0 \\ 0 & \zeta_2 \end{pmatrix}$.

Lemma 5.6. *Suppose that X, Y are subproduct systems over $\mathscr{A}, \mathscr{B}$ respectively with $X \overset{\text{SME}}{\sim}_{\mathsf{M}} Y$. Using the above-mentioned notation, the family $Z := (Z(n))_{n\in\mathbb{Z}_+}$ of essential L-correspondences is a subproduct system: $Z(n+m)$ embeds in $Z(n) \otimes_L Z(m) \cong Z_{n,m}$ as an orthogonally-complementable sub-correspondence in a canonical fashion, and the maps $(\Psi_{n,m} \otimes I_{Z(k)})\Psi_{n+m,k}$ and $(I_{Z(n)} \otimes \Psi_{m,k})\Psi_{n,m+k}$ agree on $Z(n+m+k)$, for all $n, m, k \in \mathbb{N}$.*

Proof. First, since X, Y are subproduct systems, $Z(n+m) \subseteq Z_{n,m}$ as sets. We have to check that the L-correspondence structure of $Z(n+m)$ (associated with $X(n+m) \overset{\text{SME}}{\sim}_{\mathsf{M}} Y(n+m)$) agrees with that of $Z_{n,m}$ (associated with $X(n) \otimes X(m) \overset{\text{SME}}{\sim}_{\mathsf{M}} Y(n) \otimes Y(m)$ as above). To this end, we use the three formulas in the top of [18, p. 125]. Given $a \in \mathscr{A}$, $b \in \mathscr{B}$, $x, y, z, v \in \mathsf{M}$, $\zeta_1, \zeta_2 \in X(n+m)$ and $\eta_1, \eta_2 \in Y(n+m)$, we compute:

$$\varphi_{Z(n+m)} \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} \eta_1 & \eta_2 \otimes \widetilde{z} \\ \zeta_1 \otimes v & \zeta_2 \end{pmatrix} = \begin{pmatrix} b\eta_1 & b\eta_2 \otimes \widetilde{z} \\ a\zeta_1 \otimes v & a\zeta_2 \end{pmatrix} = \varphi_{Z_{n.m}} \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} \eta_1 & \eta_2 \otimes \widetilde{z} \\ \zeta_1 \otimes v & \zeta_2 \end{pmatrix},$$

$$\varphi_{Z(n+m)} \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \begin{pmatrix} \eta_1 & \eta_2 \otimes \widetilde{z} \\ \zeta_1 \otimes v & \zeta_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ W_{n+m}(x \otimes \eta_1) & (I_{X(n+m)} \otimes m_{\mathscr{A}})(W_{n+m} \otimes I_{\widetilde{\mathsf{M}}})(x \otimes \eta_2 \otimes \widetilde{z}) \end{pmatrix},$$

$$\varphi_{Z(n+m)} \begin{pmatrix} 0 & \widetilde{y} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 & \eta_2 \otimes \widetilde{z} \\ \zeta_1 \otimes v & \zeta_2 \end{pmatrix} = \begin{pmatrix} (m_{\mathscr{B}} \otimes I_{Y(n+m)})(I_{\widetilde{\mathsf{M}}} \otimes W_{n+m}^{-1})(\widetilde{y} \otimes \zeta_1 \otimes v) & \widetilde{W}_{n+m}(\widetilde{y} \otimes \zeta_2) \\ 0 & 0 \end{pmatrix}.$$

The operator

$$W_{n,m} := (I_{X(n)} \otimes W_m)(W_n \otimes I_{Y(m)}) : \mathsf{M} \otimes Y(n) \otimes Y(m) \to X(n) \otimes X(m) \otimes \mathsf{M}$$

implementing the equivalence $X(n) \otimes X(m) \overset{\text{SME}}{\sim}_{\mathsf{M}} Y(n) \otimes Y(m)$ extends W_{n+m} by (5.1), and so $\widetilde{W}_{n,m}$ extends $\widetilde{W}_{n+m}$. Thus

$$\varphi_{Z(n+m)} \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \begin{pmatrix} \eta_1 & \eta_2 \otimes \widetilde{z} \\ \zeta_1 \otimes v & \zeta_2 \end{pmatrix} = \varphi_{Z_{n,m}} \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix} \begin{pmatrix} \eta_1 & \eta_2 \otimes \widetilde{z} \\ \zeta_1 \otimes v & \zeta_2 \end{pmatrix}$$

and

$$\varphi_{Z(n+m)} \begin{pmatrix} 0 & \widetilde{y} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 & \eta_2 \otimes \widetilde{z} \\ \zeta_1 \otimes v & \zeta_2 \end{pmatrix} = \varphi_{Z_{n,m}} \begin{pmatrix} 0 & \widetilde{y} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 & \eta_2 \otimes \widetilde{z} \\ \zeta_1 \otimes v & \zeta_2 \end{pmatrix}.$$

It is easy to show that $Z(n+m)$ is orthogonally complementable in $Z_{n,m}$: the linear mapping $p^Z_{n,m} := \left(\begin{smallmatrix} p^Y_{n+m} & p^Y_{n+m}\otimes I_{\widetilde{\mathsf{M}}} \\ p^X_{n+m}\otimes I_{\mathsf{M}} & p^X_{n+m}\end{smallmatrix}\right)$ from $Z_{n,m}$ to itself is an (orthogonal) projection in $\mathcal{L}(Z_{n,m})$ (for a direct calculation shows that it is a right L-module map), whose range is $Z(n+m)$. In conclusion, $(\Psi_{n,m})_{|Z(n+m)}$ is an isometric, adjointable L-correspondence mapping from $Z(n+m)$ to $Z_{n,m}$.

For the second part of the assertion, it follows from the construction of Ψ that $(\Psi_{n,m}\otimes I_{Z(k)})\Psi_{n+m,k}$ and $(I_{Z(n)}\otimes\Psi_{m,k})\Psi_{n,m+k}$ agree on $C(n+m+k)$. Since all the maps involved are (continuous) L-correspondence maps, and $\varphi_{Z(n+m+k)}(L)C(n+m+k)$ is total in $Z(n+m+k)$ by Proposition 5.3, we infer that the desired equality holds. □

Corollary 5.7. *Under the conditions of the last lemma, $\mathcal{F}_X \overset{\mathrm{SME}}{\sim}_{\mathsf{M}} \mathcal{F}_Y$ and the associated L-correspondence is $\mathcal{F}_Z$.*

5.2. **Equivalence of the operator algebras.** Let us see how the shift operator of the subproduct system Z, denoted by S^Z, acts. Let $n,m\in\mathbb{Z}_+$, $\zeta_1,\zeta_2\in X(n)$, $\eta_1,\eta_2\in Y(n)$, $\varrho_1,\varrho_2\in X(m)$, $\xi_1,\xi_2\in Y(m)$ and $u,v,w,z\in\mathsf{M}$ be given. By [18, Lemma 2.9] and similar computations that are left to the reader,

$$
\begin{aligned}
&S^Z_n\begin{pmatrix}\eta_1 & \eta_2\otimes\widetilde{w}\\ \zeta_1\otimes v & \zeta_2\end{pmatrix}\begin{pmatrix}\xi_1 & \xi_2\otimes\widetilde{z}\\ \varrho_1\otimes u & \varrho_2\end{pmatrix}\\
&= p^Z_{n,m}\Psi^{-1}_{n,m}\left[\begin{pmatrix}\eta_1 & \eta_2\otimes\widetilde{w}\\ \zeta_1\otimes v & \zeta_2\end{pmatrix}\otimes_L\begin{pmatrix}\xi_1 & \xi_2\otimes\widetilde{z}\\ \varrho_1\otimes u & \varrho_2\end{pmatrix}\right]\\
&= p^Z_{n,m}\begin{pmatrix}\eta_1\otimes\xi_1 + c(\eta_2\otimes\widetilde{w}, W_m^{-1}(\varrho_1\otimes u)) & \eta_1\otimes\xi_2\otimes\widetilde{z}+\eta_2\otimes\widetilde{W_m}(\widetilde{w}\otimes\varrho_2)\\ \zeta_1\otimes W_m(v\otimes\xi_1)+\zeta_2\otimes\varrho_1\otimes u & \widetilde{c}(\zeta_1\otimes v,\widetilde{W_m}^{-1}(\xi_2\otimes\widetilde{z}))+\zeta_2\otimes\varrho_2\end{pmatrix}
\end{aligned}
\tag{5.4}
$$

where $c: Y(n)\otimes\widetilde{\mathsf{M}}\times\mathsf{M}\otimes Y(m)\to Y(n)\otimes Y(m)$ and $\widetilde{c}: X(n)\otimes\mathsf{M}\times\widetilde{\mathsf{M}}\otimes X(m)\to X(n)\otimes X(m)$ are given by $(\eta\otimes\widetilde{x}, y\otimes\xi)\mapsto\eta\otimes\langle x,y\rangle_{\mathscr{B}}\xi$ and $(\zeta\otimes x,\widetilde{y}\otimes\rho)\mapsto\zeta\otimes_{\mathscr{A}}\langle x,y\rangle\rho$, respectively.

The Fock space $\mathcal{F}_Z = \begin{pmatrix}\mathcal{F}_Y & \mathcal{F}_Y\otimes_{\mathscr{B}}\widetilde{\mathsf{M}}\\ \mathcal{F}_X\otimes_{\mathscr{A}}\mathsf{M} & \mathcal{F}_X\end{pmatrix}$ has the following two closed linear subspaces:

$$
\mathcal{F}'_Z := \begin{pmatrix}\mathcal{F}_Y & 0\\ \mathcal{F}_X\otimes_{\mathscr{A}}\mathsf{M} & 0\end{pmatrix},\ \mathcal{F}''_Z := \begin{pmatrix}0 & \mathcal{F}_Y\otimes_{\mathscr{B}}\widetilde{\mathsf{M}}\\ 0 & \mathcal{F}_X\end{pmatrix}
$$

(which are left, but not right, L-submodules of $\mathcal{F}_Z$). From (5.4) it is apparent that both subspaces are invariant under the tensor algebra $\mathcal{T}_+(Z)$. As for the adjoints, suppose that $n, m \in \mathbb{N}$, $z \in Z(n)$, $c \in C(n+m)$ and $l \in L$. Approximate c by a sum of the form $\sum_i \begin{pmatrix} \eta_i^1 \otimes \eta_i^2 & 0 \\ 0 & \zeta_i^1 \otimes \zeta_i^2 \end{pmatrix}$. Then by the construction of $\Psi_{n,m}$, $S_n^Z(z)^*(\varphi_{Z(n+m)}(l)c)$ can be approximated by

$$\sum_i \varphi_{Z(m)} \left(\left\langle z, \varphi_{Z(n)}(l) \begin{pmatrix} \eta_i^1 & 0 \\ 0 & \zeta_i^1 \end{pmatrix} \right\rangle \right) \begin{pmatrix} \eta_i^2 & 0 \\ 0 & \zeta_i^2 \end{pmatrix}.$$

We therefore deduce from Proposition 5.3 that $\mathcal{F}_Z'$ and $\mathcal{F}_Z''$ are also invariant under $\mathcal{T}_+(Z)^*$. Consequently, they reduce the Toeplitz algebra $\mathcal{T}(Z)$. For convenience, we occasionally drop the zero columns from $\mathcal{F}_Z'$ and $\mathcal{F}_Z''$.

Lemma 5.8. *The restriction mappings $T \mapsto T_{|\mathcal{F}_Z'}$ and $T \mapsto T_{|\mathcal{F}_Z''}$, from $\mathcal{T}(Z)$ to linear operators over $\mathcal{F}_Z'$ and $\mathcal{F}_Z''$, respectively, are injective.*

Proof. Let $T \in \mathcal{T}(Z)$ be given, and suppose that $T_{|\mathcal{F}_Z'} = 0$. Fix $\zeta \in \mathcal{F}_X$, $\eta \in \mathcal{F}_Y$ and $m, z, w \in \mathsf{M}$. Write $\alpha := \begin{pmatrix} \eta & 0 \\ 0 & 0 \end{pmatrix}, \beta := \begin{pmatrix} 0 & 0 \\ \zeta \otimes z & 0 \end{pmatrix} \in \mathcal{F}_Z'$ and $l_1 := \begin{pmatrix} 0 & \widetilde{m} \\ 0 & 0 \end{pmatrix}, l_2 := \begin{pmatrix} 0 & \widetilde{w} \\ 0 & 0 \end{pmatrix} \in L$. Then

$$T \begin{pmatrix} 0 & \eta \otimes \widetilde{m} \\ 0 & 0 \end{pmatrix} = T(\alpha \cdot l_1) = T(\alpha) \cdot l_1 = T_{|\mathcal{F}_Z'}(\alpha) \cdot l_1 = 0$$

and

$$T \begin{pmatrix} 0 & 0 \\ 0 & \zeta \cdot {}_{\mathscr{A}}\langle z, w \rangle \end{pmatrix} = T(\beta \cdot l_2) = T(\beta) \cdot l_2 = T_{|\mathcal{F}_Z'}(\beta) \cdot l_2 = 0.$$

Hence $T \begin{pmatrix} 0 & \eta \otimes \widetilde{m} \\ 0 & \zeta \langle z, w \rangle \end{pmatrix} = 0$. Since the closed span of vectors of the form $\begin{pmatrix} 0 & \eta \otimes \widetilde{m} \\ 0 & \zeta \langle z, w \rangle \end{pmatrix}$ is dense in $\mathcal{F}_Z''$, we infer that $T_{|\mathcal{F}_Z''} = 0$, and all in all, $T = 0$. The proof of $T_{|\mathcal{F}_Z''} = 0 \Rightarrow T = 0$ is similar. □

Endow $\mathcal{F}_Z'$ with a right $\mathscr{B}$-module structure in the obvious manner (although as a subspace of $\mathcal{F}_Z$ it is *not* a right L-submodule). This makes $\mathcal{F}_Z'$ a Hilbert C^*-module, whose $\mathscr{B}$-valued rigging corresponds naturally to the L-valued rigging of $\mathcal{F}_Z'$ as a subset of $\mathcal{F}_Z$. If $T \in \mathcal{T}(Z)$, it is easy to see that $T_{|\mathcal{F}_Z'}$ is a module map, so it belongs to $\mathcal{L}(\mathcal{F}_Z')$. From Lemma 5.8 it follows that the C^*-algebras homomorphism $\mathcal{T}(Z) \to \mathcal{L}(\mathcal{F}_Z')$ given by $T \mapsto T_{|\mathcal{F}_Z'}$ is injective, so that we can identify $\mathcal{T}(Z)$ with its image under this map.

Denote by $\mathbf{p}$ and $\mathbf{q}$ the projections of $\mathcal{F}_Z'$ onto $\begin{pmatrix} \mathcal{F}_Y \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \mathcal{F}_X \otimes_{\mathscr{A}} \mathsf{M} \end{pmatrix}$, respectively.

Theorem 5.9. *Suppose that X, Y are subproduct systems over $\mathscr{A}, \mathscr{B}$ respectively with $X \overset{\mathrm{SME}}{\sim}_{\mathsf{M}} Y$. Identify $\mathcal{T}(Z)$ with the subalgebra of $\mathcal{L}(\mathcal{F}_Z')$ as above. Then:*

(1) $\mathbf{p}\mathcal{T}_+(Z)\mathbf{p} \cong \mathcal{T}_+(Y)$ *and* $\mathbf{q}\mathcal{T}_+(Z)\mathbf{q} \cong \mathcal{T}_+(X)$.

(2) *The (non-selfadjoint) operator algebras $\mathcal{T}_+(X)$ and $\mathcal{T}_+(Y)$ are strongly Morita equivalent in the sense of [3].*

Lemma 5.10. *If G is a Hilbert C^*-module over $\mathscr{A}$ and M is an $\mathscr{A}$-$\mathscr{B}$ imprimitivity bimodule, then the map $T \mapsto T \otimes I_{\mathsf{M}}$ is an isomorphism from $\mathcal{L}(G)$ onto $\mathcal{L}(G \otimes \mathsf{M})$.*

The proof is exactly as that of [18, Lemma 2.12]. The details are omitted.

Proof of Theorem 5.9. (1) Fix $n, m \in \mathbb{Z}_+$, $\zeta_1, \zeta_2 \in X(n)$, $\eta_1, \eta_2 \in Y(n)$ and $v, w \in \mathsf{M}$. Writing

$$\alpha := \begin{pmatrix} \eta_1 & \eta_2 \otimes \widetilde{w} \\ \zeta_1 \otimes v & \zeta_2 \end{pmatrix} \in Z(n) \tag{5.5}$$

(remember: $Z(0) = L$) we have from (5.4) that for $\nu \in Y(m)$,

$$\begin{aligned} S_n^Z(\alpha) \begin{pmatrix} \nu \\ 0 \end{pmatrix} &= p_{n,m}^Z \begin{pmatrix} \eta_1 \otimes \nu & 0 \\ \zeta_1 \otimes W_m(v \otimes \nu) & 0 \end{pmatrix} \\ &= \begin{pmatrix} p_{n+m}^Y(\eta_1 \otimes \nu) & 0 \\ (p_{n+m}^X \otimes I_{\mathsf{M}})\,(\zeta_1 \otimes W_m(v \otimes \nu)) & 0 \end{pmatrix} = \begin{pmatrix} S_n^Y(\eta_1)\nu & 0 \\ (S_n^X(\zeta_1) \otimes I_{\mathsf{M}}) W_m(v \otimes \nu) & 0 \end{pmatrix}, \end{aligned} \tag{5.6}$$

so that

$$\mathbf{p} S_n^Z(\alpha) \mathbf{p} \begin{pmatrix} \nu \\ 0 \end{pmatrix} = \begin{pmatrix} S_n^Y(\eta_1)\nu \\ 0 \end{pmatrix}.$$

Hence $\mathbf{p}\mathcal{T}_+(Z)\mathbf{p}$ is (unitarily equivalent, and hence) completely isometrically isomorphic to $\mathcal{T}_+(Y)$. Similarly, for $\mu \in X(m)$ and $z \in \mathsf{M}$,

$$\begin{aligned} S_n^Z(\alpha) \begin{pmatrix} 0 \\ \mu \otimes z \end{pmatrix} &= p_{n,m}^Z \begin{pmatrix} c(\eta_2 \otimes \widetilde{w}, W_m^{-1}(\mu \otimes z)) & 0 \\ \zeta_2 \otimes \mu \otimes z & 0 \end{pmatrix} \\ &= \begin{pmatrix} p_{n+m}^Y\big(c(\eta_2 \otimes \widetilde{w}, W_m^{-1}(\mu \otimes z))\big) & 0 \\ (p_{n+m}^X \otimes I_{\mathsf{M}})(\zeta_2 \otimes \mu \otimes z) & 0 \end{pmatrix} \\ &= \begin{pmatrix} S_n^Y(\eta_2)(m_{\mathscr{B}} \otimes I_{Y(m)})(\widetilde{w} \otimes W_m^{-1}(\mu \otimes z)) & 0 \\ (S_n^X(\zeta_2) \otimes I_{\mathsf{M}})(\mu \otimes z) & 0 \end{pmatrix}, \end{aligned} \tag{5.7}$$

thus

$$\mathbf{q} S_n^Z(\alpha) \mathbf{q} \begin{pmatrix} 0 \\ \mu \otimes z \end{pmatrix} = \begin{pmatrix} 0 \\ (S_n^X(\zeta_2) \otimes I_{\mathsf{M}})(\mu \otimes z) \end{pmatrix}.$$

The map $T \mapsto T \otimes I_{\mathsf{M}}$ from $\mathcal{L}(\mathcal{F}_X)$ to $\mathcal{L}(\mathcal{F}_X \otimes_{\mathscr{B}} \mathsf{M})$ is a C^*-isomorphism by Lemma 5.10, and therefore $\mathbf{q}\mathcal{T}_+(Z)\mathbf{q}$ is completely isometrically isomorphic to $\mathcal{T}_+(X)$.

(2) We follow the proof of [18, Theorem 3.2, (3)] to show that

$$(\mathbf{p}\mathcal{T}_+(Z)\mathbf{p}, \mathbf{q}\mathcal{T}_+(Z)\mathbf{q}, \mathbf{p}\mathcal{T}_+(Z)\mathbf{q}, \mathbf{q}\mathcal{T}_+(Z)\mathbf{p})$$

is a Morita context with the actions $(\mathbf{p}S_1\mathbf{q},\mathbf{q}S_2\mathbf{p}) := \mathbf{p}S_1\mathbf{q}S_2\mathbf{p}$ and $[\mathbf{q}S_1\mathbf{p},\mathbf{p}S_2\mathbf{q}] := \mathbf{q}S_1\mathbf{p}S_2\mathbf{q}$ $(S_1, S_2 \in \mathcal{T}_+(Z))$. The foregoing implies that $\mathbf{p}, \mathbf{q}$ belong to the multiplier algebra $M(\varphi_\infty(L))$ and that $\mathbf{p}\varphi_\infty(L)\mathbf{p}$ and $\mathbf{q}\varphi_\infty(L)\mathbf{q}$ are naturally isomorphic to $\varphi_\infty(\mathscr{B})$ and $\varphi_\infty(\mathscr{A}) \otimes I_{\mathsf{M}}$, respectively. If $l := \left(\begin{smallmatrix} b & \widetilde{y} \\ x & a \end{smallmatrix}\right) \in L$, then $\mathbf{p}\varphi_\infty(l)\mathbf{q}\varphi_\infty(l)\mathbf{p}$ and $\mathbf{q}\varphi_\infty(l)\mathbf{p}\varphi_\infty(l)\mathbf{q}$ "equal" $\varphi_\infty(\langle y, x\rangle_{\mathscr{B}})$ and $\varphi_\infty({}_{\mathscr{A}}\langle x, y\rangle) \otimes I_{\mathsf{M}}$, respectively. Consequently, the C^*-algebras $\varphi_\infty(\mathscr{B})$ and $\varphi_\infty(\mathscr{A}) \otimes I_{\mathsf{M}}$ are strongly Morita equivalent through the imprimitivity bimodule $\mathbf{p}\varphi_\infty(L)\mathbf{q}$. From [3, Theorem 6.1] this implies that $(\varphi_\infty(\mathscr{B}), \varphi_\infty(\mathscr{A}) \otimes I_{\mathsf{M}}, \mathbf{p}\varphi_\infty(L)\mathbf{q}, \mathbf{q}\varphi_\infty(L)\mathbf{p})$ is a Morita context. We omit the rest of the details, which are identical to those of [18]. ☐

A straightforward computation using (5.6) and (5.7) shows that $S_n^Z(\alpha)\mathbf{p}, \mathbf{p}S_n^Z(\alpha) \in \mathcal{T}(Z)$ for all $n \in \mathbb{Z}_+$ and $\alpha \in Z(n)$. Hence $\mathbf{p}, \mathbf{q} \in M(\mathcal{T}(Z))$ $(\subseteq \mathcal{L}(\mathcal{F}'_Z))$.

Theorem 5.11. *Suppose that X, Y are subproduct systems over $\mathscr{A}, \mathscr{B}$ respectively with $X \overset{\mathrm{SME}}{\sim}_{\mathsf{M}} Y$. Identify $\mathcal{T}(Z)$ with the subalgebra of $\mathcal{L}(\mathcal{F}'_Z)$ as above. Then:*

(1) *$\mathbf{p}\mathcal{T}(Z)\mathbf{p} \cong \mathcal{T}(Y)$ and $\mathbf{q}\mathcal{T}(Z)\mathbf{q} \cong \mathcal{T}(X)$.*
(2) *$\mathcal{T}(X) \overset{\mathrm{SME}}{\sim} \mathcal{T}(Y)$.*

We shall require three technical lemmas.

Lemma 5.12.

(1) *Let $n, k \in \mathbb{N}$. For all $w, w' \in \mathsf{M}$, $\eta \in Y(n)$ and $\epsilon \in Y(k)$, the operators in $\mathcal{L}(\mathcal{F}_X \otimes \mathsf{M})$, defined on $X(m) \otimes \mathsf{M}$, $m \in \mathbb{Z}_+$, by the formulas*

$$W_{n+m}\left(w' \otimes \left[S_n^Y(\eta)(m_{\mathscr{B}} \otimes I_{Y(m)})(\widetilde{w} \otimes W_m^{-1}(\cdot)\right]\right) \tag{5.8}$$

and

$$W_{m-k}\left(w' \otimes \left[S_k^Y(\epsilon)^*(m_{\mathscr{B}} \otimes I_{Y(m)})(\widetilde{w} \otimes W_m^{-1}(\cdot)\right]\right) \tag{5.9}$$

(if $m \geq k$, otherwise 0) can be written as $S_n^X(\zeta) \otimes I_{\mathsf{M}}$ and $S_k^X(\theta)^* \otimes I_{\mathsf{M}}$, respectively, for suitable $\zeta \in X(n)$ and $\theta \in X(k)$.

(2) *Let $n, k \in \mathbb{N}$. For all $v, v' \in \mathsf{M}$, $\zeta \in X(n)$ and $\xi \in X(k)$, the operators in $\mathcal{L}(\mathcal{F}_Y)$, defined on $Y(m)$, $m \in \mathbb{Z}_+$, by the formulas*

$$(m_{\mathscr{B}} \otimes I_{Y(n+m)})\left(\widetilde{v'} \otimes \left[W_{n+m}^{-1}(S_n^X(\zeta) \otimes I_{\mathsf{M}})W_m(v \otimes \cdot)\right]\right) \tag{5.10}$$

and

$$(m_{\mathscr{B}} \otimes I_{Y(m-k)})\left(\widetilde{v'} \otimes \left[W_{m-k}^{-1}(S_k^X(\xi)^* \otimes I_{\mathsf{M}})W_m(v \otimes \cdot)\right]\right)$$

can be written as $S_n^Y(\eta)$ and $S_k^Y(\varrho)^*$, respectively, for suitable $\eta \in Y(n)$ and $\varrho \in Y(k)$.

Proof. The proofs of (1) and (2) are similar, so we give details only for the former. To prove the first part, fix $m \in \mathbb{Z}_+$, $\mu \in X(m)$ and $z \in \mathsf{M}$. Approximate $W_m^{-1}(\mu \otimes z)$

as the finite sum $\sum_i z_i \otimes \rho_i$ ($z_i \in \mathsf{M}$ and $\rho_i \in Y(m)$ for all i). Then

$$w' \otimes \left[S_n^Y(\eta)(m_{\mathscr{B}} \otimes I_{Y(m)})(\widetilde{w} \otimes W_m^{-1}(\mu \otimes z)\right]$$

can be approximated by

$$(I_{\mathsf{M}} \otimes p_{n+m}^Y)\big(\sum_i w' \otimes \eta \otimes \langle w, z_i\rangle_{\mathscr{B}}\rho_i\big).$$

Approximate $W_n(w' \otimes \eta)$ as the finite sum $\sum_j \xi_j \otimes x_j$ ($\xi_j \in X(n)$, $x_j \in \mathsf{M}$ for all j). Using (5.1) and (5.3),

$$\begin{aligned}
(5.8) &\sim \sum_i (p_{n+m}^X \otimes I_{\mathsf{M}})(I_{X(n)} \otimes W_m)\big(W_n(w' \otimes \eta) \otimes \langle w, z_i\rangle_{\mathscr{B}}\rho_i\big) \\
&\sim \sum_i \sum_j (p_{n+m}^X \otimes I_{\mathsf{M}})\big(\xi_j \otimes W_m(x_j \otimes \langle w, z_i\rangle_{\mathscr{B}}\rho_i)\big) \\
&= \sum_j (p_{n+m}^X \otimes I_{\mathsf{M}})\big(\xi_j \otimes {}_{\mathscr{A}}\langle x_j, w\rangle \cdot W_m(\sum_i z_i \otimes \rho_i)\big) \\
&\sim \sum_j (p_{n+m}^X \otimes I_{\mathsf{M}})\big(\xi_j {}_{\mathscr{A}}\langle x_j, w\rangle \otimes \mu \otimes z\big) \\
&= \big(S_n^X(\sum_j \xi_j {}_{\mathscr{A}}\langle x_j, w\rangle) \otimes I_{\mathsf{M}}\big)(\mu \otimes z).
\end{aligned}$$

The assertion is therefore true for $\zeta := (I_{X(n)} \otimes m_{\mathscr{A}})(W_n(w' \otimes \eta) \otimes \widetilde{w})$.

For the second part, fix $m \geq k$, $\mu \in X(m)$ and $z \in \mathsf{M}$. Approximate $W_m^{-1}(\mu \otimes z)$ as the finite sum $\sum_i z_i \otimes \rho_i^{(1)} \otimes \rho_i^{(2)}$ ($z_i \in \mathsf{M}$, $\rho_i^{(1)} \in Y(k)$ and $\rho_i^{(2)} \in Y(m-k)$ for all i), and $W_k(w \otimes \epsilon)$ as the finite sum $\sum_j \xi_j \otimes x_j$ ($\xi_j \in X(k)$, $x_j \in \mathsf{M}$ for all j). Then since W_k is unitary,

$$\begin{aligned}
(5.9) &\sim \sum_i W_{m-k}\left(w' \otimes \left\langle \epsilon, \langle w, z_i\rangle_{\mathscr{B}}\rho_i^{(1)}\right\rangle \rho_i^{(2)}\right) \\
&= \sum_i W_{m-k}\left(w' \otimes \left\langle w \otimes \epsilon, z_i \otimes \rho_i^{(1)}\right\rangle \rho_i^{(2)}\right) \\
&= \sum_i W_{m-k}\left(w' \otimes \left\langle W_k(w \otimes \epsilon), W_k(z_i \otimes \rho_i^{(1)})\right\rangle \rho_i^{(2)}\right) \\
&\sim \sum_i \sum_j W_{m-k}\left(w' \left\langle \xi_j \otimes x_j, W_k(z_i \otimes \rho_i^{(1)})\right\rangle \otimes \rho_i^{(2)}\right).
\end{aligned}$$

It is easy to show that $w' \langle \xi \otimes x, \Theta \rangle = (S^X_k(\xi \cdot {}_{\mathscr{A}}\langle x, w'\rangle)^* \otimes I_{\mathsf{M}})\Theta$ for all $\xi \in X(k)$ and $\Theta \in X(k) \otimes \mathsf{M}$. Thus, from (5.1),

$$\text{(5.9)} \sim \sum_i W_{m-k} \left(\left[\big(S^X_k(\sum_j \xi_j \cdot {}_{\mathscr{A}}\langle x_j, w'\rangle)^* \otimes I_{\mathsf{M}}\big) W_k(z_i \otimes \rho^{(1)}_i) \right] \otimes \rho^{(2)}_i \right)$$
$$= \left(S^X_k(\sum_j \xi_j \cdot {}_{\mathscr{A}}\langle x_j, w'\rangle)^* \otimes I_{\mathsf{M}} \right) \sum_i (I_{X(k)} \otimes W_{m-k}) \left(W_k(z_i \otimes \rho^{(1)}_i) \otimes \rho^{(2)}_i \right)$$
$$\sim \left(S^X_k(\sum_j \xi_j \cdot {}_{\mathscr{A}}\langle x_j, w'\rangle)^* \otimes I_{\mathsf{M}} \right) (\mu \otimes z).$$

Hence $\theta := (I_{X(k)} \otimes m_{\mathscr{A}})(W_k(w \otimes \epsilon) \otimes \widetilde{w'})$ fits. □

Lemma 5.13. *Let $n \in \mathbb{Z}_+$. Then*

(1) *for $\kappa \in Y(n)$ and $w \in \mathsf{M}$, the operator in $\mathcal{L}(\mathcal{F}_X \otimes \mathsf{M}, \mathcal{F}_Y)$ defined by*

$$X(m) \otimes \mathsf{M} \ni \mu \otimes z \mapsto S^Y_n(\kappa)(m_{\mathscr{B}} \otimes I_{Y(m)})(\widetilde{w} \otimes W^{-1}_m(\mu \otimes z)); \tag{5.11}$$

(2) *and for $\varsigma \in X(n)$ and $v \in \mathsf{M}$, the operator in $\mathcal{L}(\mathcal{F}_Y, \mathcal{F}_X \otimes \mathsf{M})$ defined by*

$$Y(m) \ni \nu \mapsto (S^X_n(\varsigma) \otimes I_{\mathsf{M}}) W_m(v \otimes \nu); \tag{5.12}$$

belong to the closed linear span of operators of the form

$$X(m) \otimes \mathsf{M} \ni \mu \otimes z \mapsto (m_{\mathscr{B}} \otimes I_{Y(n+m)}) \left(\widetilde{x} \otimes W^{-1}_{n+m} \left[(S^X_n(\zeta) \otimes I_{\mathsf{M}})(\mu \otimes z) \right] \right) \tag{5.13}$$

and

$$Y(m) \ni \nu \mapsto W_{n+m}(y \otimes S^Y_n(\rho)\nu), \tag{5.14}$$

respectively, where $\zeta \in X(n)$, $\rho \in Y(n)$ and $x, y \in \mathsf{M}$.

Similar assertions are valid when $S_n(\cdot)$ is replaced by its adjoint.

Proof. Write **I** and **II** for the operators given by (5.11) and (5.12), respectively. In order to make the operator **I** have the form of (5.8), we ought to "wrap" it with $W_{n+m}(w' \otimes \cdot)$ for some $w' \in \mathsf{M}$. As $Y(n)$ is essential, $S^Y_n(\kappa) = \varphi_\infty(b) S^Y_n(\kappa')$ for suitable b, κ'. Since M is full as a right $\mathscr{B}$-module, $\varphi_\infty(b) \in \mathcal{L}(\mathcal{F}_Y)$ belongs to the closed linear span of operators of the form

$$Y(p) \ni \tau \mapsto (m_{\mathscr{B}} \otimes I_{Y(p)})\big(\widetilde{x} \otimes W^{-1}_p(W_p(w' \otimes \tau))\big), \qquad p \in \mathbb{Z}_+,$$

where $x, w' \in \mathsf{M}$. Using the first part of Lemma 5.12, (1), one deduces that the operator $\mathbf{I} \in \mathcal{L}(\mathcal{F}_X \otimes \mathsf{M}, \mathcal{F}_Y)$ belongs to the closed linear span of operators of the form (5.13), as stated.

To convert **II** to the form of (5.10), we employ the fullness of M as a left $\mathscr{A}$-module to conclude that for $a \in \mathscr{A}$, $\varphi_\infty(a) \otimes I_{\mathsf{M}} \in \mathcal{L}(\mathcal{F}_X \otimes \mathsf{M})$ belongs to the closed linear span of operators of the form $\varphi_\infty({}_{\mathscr{A}}\langle y, v'\rangle) \otimes I_{\mathsf{M}}$. Notice also that

$$\big(\varphi_\infty({}_{\mathscr{A}}\langle y, v'\rangle) \otimes I_{\mathsf{M}}\big)\Theta = W_p\left[y \otimes \Big((m_{\mathscr{B}} \otimes I_{Y(p)})(\widetilde{v'} \otimes W_p^{-1}\Theta)\Big)\right] \tag{5.15}$$

for each $\Theta \in X(p) \otimes \mathsf{M}$. As above, using the first part of Lemma 5.12, (2) and that $X(n)$ is essential, the operator $\mathbf{II} \in \mathcal{L}(\mathcal{F}_Y, \mathcal{F}_X \otimes \mathsf{M})$ belongs to the closed linear span of operators of the form (5.14).

The proof for $S_n(\cdot)^*$ goes along the lines of the preceding one, using the second parts of Lemma 5.12, (1) and (2). □

Lemma 5.14. *Let $k \in \mathbb{N}$, $\zeta_1, \zeta_2 \in X(k)$, $\eta_1, \eta_2 \in Y(k)$ and $v, w \in \mathsf{M}$. Write $\beta := \begin{pmatrix} \eta_1 & \eta_2\otimes\widetilde{w} \\ \zeta_1\otimes v & \zeta_2 \end{pmatrix} \in Z(k)$. Then for $m \geq k$, $S_k^Z(\beta)^*$ maps $\begin{pmatrix} \nu \\ \mu\otimes z \end{pmatrix} \in Z(m)$ to*

$$\begin{pmatrix} S_k^Y(\eta_1)^*\nu \\ (S_k^X(\zeta_2)^* \otimes I_{\mathsf{M}})(\mu \otimes z) \end{pmatrix} + \begin{pmatrix} (m_{\mathscr{B}} \otimes I_{Y(m-k)})\left(\widetilde{v} \otimes W_{m-k}^{-1}\left[(S_k^X(\zeta_1)^* \otimes I_{\mathsf{M}})(\mu \otimes z)\right]\right) \\ W_{m-k}(w \otimes S_k^Y(\eta_2)^*\nu) \end{pmatrix}.$$

Proof. Write $S := S_k^{X_L}(\beta)$ for the shift in the full Fock space, and remember that $S_k^Z(\beta)^* = (S^*)_{|\mathcal{F}_Z}$. Fix $m \geq k$. If $\nu_1 \in Y(k)$ and $\nu_2 \in Y(m-k)$, then for $\nu = \nu_1 \otimes \nu_2$,

$$\begin{aligned}
S^*\begin{pmatrix} \nu \\ 0 \end{pmatrix} &= \varphi_{Z(m-k)}\left(\left\langle \begin{pmatrix} \eta_1 & \eta_2 \otimes \widetilde{w} \\ \zeta_1 \otimes v & \zeta_2 \end{pmatrix}, \begin{pmatrix} \nu_1 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle\right) \begin{pmatrix} \nu_2 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \varphi_{Z(m-k)}\begin{pmatrix} \langle \eta_1, \nu_1\rangle & 0 \\ w\,\langle \eta_2, \nu_1\rangle & 0 \end{pmatrix} \begin{pmatrix} \nu_2 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \langle \eta_1, \nu_1\rangle\,\nu_2 \\ W_{m-k}(w\,\langle \eta_2, \nu_1\rangle \otimes \nu_2) \end{pmatrix} = \begin{pmatrix} S_k^Y(\eta_1)^*\nu \\ W_{m-k}(w \otimes S_k^Y(\eta_2)^*\nu) \end{pmatrix}.
\end{aligned}$$

(see [18, p. 125]). If $\mu_1 \in X(k)$, $\mu_2 \in X(m-k)$ and $z \in \mathsf{M}$, then $\Psi_{k,m-k}\begin{pmatrix} 0 & 0 \\ \mu_1\otimes\mu_2\otimes z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \mu_1 \end{pmatrix} \otimes_L \begin{pmatrix} 0 & 0 \\ \mu_2\otimes z & 0 \end{pmatrix}$ by [18, Lemma 2.9], and we obtain for $\mu = \mu_1 \otimes \mu_2$:

$$\begin{aligned}
S^*\begin{pmatrix} 0 \\ \mu \otimes z \end{pmatrix} &= \varphi_{Z(m-k)}\left(\left\langle \begin{pmatrix} \eta_1 & \eta_2 \otimes \widetilde{w} \\ \zeta_1 \otimes v & \zeta_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \mu_1 \end{pmatrix} \right\rangle\right) \begin{pmatrix} 0 & 0 \\ \mu_2 \otimes z & 0 \end{pmatrix} \\
&= \varphi_{Z(m-k)}\begin{pmatrix} 0 & \widetilde{v}\,\langle \zeta_1, \mu_1\rangle \\ 0 & \langle \zeta_2, \mu_1\rangle \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \mu_2 \otimes z & 0 \end{pmatrix} \\
&= \begin{pmatrix} (m_{\mathscr{B}} \otimes I_{Y(m-k)})\left(\widetilde{v} \otimes W_{m-k}^{-1}\left[(S_k^X(\zeta_1)^* \otimes I_{\mathsf{M}})(\mu \otimes z)\right]\right) \\ (S_k^X(\zeta_2)^* \otimes I_{\mathsf{M}})(\mu \otimes z) \end{pmatrix}.
\end{aligned}$$

□

Proof of Theorem 5.11. (1) We first claim that every monomial $S \in \mathcal{T}(Z)$ (say, of degree t) belongs to the closed linear span of operators of the form

$$Z(m) \ni \begin{pmatrix} \nu \\ \mu \otimes z \end{pmatrix} \mapsto \begin{pmatrix} T_1^Y \nu \\ (T_1^X \otimes I_{\mathsf{M}})(\mu \otimes z) \end{pmatrix} + \begin{pmatrix} (m_{\mathscr{B}} \otimes I_{Y(t+m)}) \left(\widetilde{x'} \otimes W_{t+m}^{-1} \left[(T_2^X \otimes I_{\mathsf{M}})(\mu \otimes z)\right]\right) \\ W_{t+m}(y' \otimes T_2^Y \nu) \end{pmatrix}$$

for some monomials $T_i^X \in \mathcal{T}(X)$, $T_i^Y \in \mathcal{T}(Y)$ ($i = 1, 2$) of degree t and $x', y' \in \mathsf{M}$. Indeed, suppose that S is as above. Given $n \in \mathbb{Z}_+$ and $\alpha \in Z(n)$, on account of (5.6), (5.7) and Lemma 5.13 we may assume that $S_n^Z(\alpha)$ maps $\left(\begin{smallmatrix} \nu \\ \mu\otimes z \end{smallmatrix}\right) \in Z(m)$ to

$$\begin{pmatrix} S_n^Y(\eta)\nu \\ (S_n^X(\xi) \otimes I_{\mathsf{M}})(\mu \otimes z) \end{pmatrix} + \begin{pmatrix} (m_{\mathscr{B}} \otimes I_{Y(n+m)}) \left(\widetilde{x} \otimes W_{n+m}^{-1} \left[(S_n^X(\zeta) \otimes I_{\mathsf{M}})(\mu \otimes z)\right]\right) \\ W_{n+m}(y \otimes S_n^Y(\rho)\nu) \end{pmatrix}$$

for some $\zeta, \xi \in X(n)$, $\eta, \rho \in Y(n)$ and $x, y \in \mathsf{M}$. Consequently, $S_n^Z(\alpha)S$ maps $\left(\begin{smallmatrix} \nu \\ \mu\otimes z \end{smallmatrix}\right) \in Z(m)$ to

$$\begin{pmatrix} S_n^Y(\eta) \left\{T_1^Y \nu + (m_{\mathscr{B}} \otimes I_{Y(t+m)}) \left(\widetilde{x'} \otimes W_{t+m}^{-1} \left[(T_2^X \otimes I_{\mathsf{M}})(\mu \otimes z)\right]\right)\right\} \\ (S_n^X(\xi) \otimes I_{\mathsf{M}}) \left\{(T_1^X \otimes I_{\mathsf{M}})(\mu \otimes z) + W_{t+m}(y' \otimes T_2^Y \nu)\right\} \end{pmatrix}$$
$$+ \begin{pmatrix} (m_{\mathscr{B}} \otimes I_{Y(n+t+m)}) \left(\widetilde{x} \otimes W_{n+t+m}^{-1} \left[(S_n^X(\zeta) \otimes I_{\mathsf{M}}) \left\{(T_1^X \otimes I_{\mathsf{M}})(\mu \otimes z) + W_{t+m}(y' \otimes T_2^Y \nu)\right\}\right]\right) \\ W_{n+t+m}(y \otimes S_n^Y(\rho) \left\{T_1^Y \nu + (m_{\mathscr{B}} \otimes I_{Y(t+m)}) \left(\widetilde{x'} \otimes W_{t+m}^{-1} \left[(T_2^X \otimes I_{\mathsf{M}})(\mu \otimes z)\right]\right)\right\}) \end{pmatrix}.$$

Utilizing Lemma 5.13 once again as well as (5.15) on this last expression yields the desired form.

We now do the same computation for the adjoints. Using Lemma 5.14 and its notation, $S_k^Z(\beta)^* S$ maps $\left(\begin{smallmatrix} \nu \\ \mu\otimes z \end{smallmatrix}\right)$ (when $m \geq k$) to

$$\begin{pmatrix} S_k^Y(\eta_1)^* \left\{T_1^Y \nu + (m_{\mathscr{B}} \otimes I_{Y(t+m)}) \left(\widetilde{x'} \otimes W_{t+m}^{-1} \left[(T_2^X \otimes I_{\mathsf{M}})(\mu \otimes z)\right]\right)\right\} \\ (S_k^X(\zeta_2)^* \otimes I_{\mathsf{M}}) \left\{(T_1^X \otimes I_{\mathsf{M}})(\mu \otimes z) + W_{t+m}(y' \otimes T_2^Y \nu)\right\} \end{pmatrix}$$
$$+ \begin{pmatrix} (m_{\mathscr{B}} \otimes I_{Y(t+m-k)}) \left(\widetilde{v} \otimes W_{t+m-k}^{-1} \left[(S_k^X(\zeta_1)^* \otimes I_{\mathsf{M}}) \left\{(T_1^X \otimes I_{\mathsf{M}})(\mu \otimes z) + W_{t+m}(y' \otimes T_2^Y \nu)\right\}\right]\right) \\ W_{t+m-k}(w \otimes S_k^Y(\eta_2)^* \left\{T_1^Y \nu + (m_{\mathscr{B}} \otimes I_{Y(t+m)}) \left(\widetilde{x'} \otimes W_{t+m}^{-1} \left[(T_2^X \otimes I_{\mathsf{M}})(\mu \otimes z)\right]\right)\right\}) \end{pmatrix}.$$

The claim is established by appealing to Lemma 5.13 and (5.15) once more.

The rest of the proof is now simple. It follows from the claim that for every $S \in \mathcal{T}(Z)$ correspond $S^X \in \mathcal{T}(X)$ and $S^Y \in \mathcal{T}(Y)$ so that $\mathbf{p}S\mathbf{p}$ maps $\left(\begin{smallmatrix} \nu \\ 0 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} S^Y\nu \\ 0 \end{smallmatrix}\right)$ and $\mathbf{q}S\mathbf{q}$ maps $\left(\begin{smallmatrix} 0 \\ \mu\otimes z \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} 0 \\ (S^X \otimes I_{\mathsf{M}})(\mu\otimes z) \end{smallmatrix}\right)$. As a result, $\mathbf{p}\mathcal{T}(Z)\mathbf{p}$ and $\mathbf{q}\mathcal{T}(Z)\mathbf{q}$ are unitarily equivalent to subalgebras of $\mathcal{T}(Y)$ and $\mathcal{T}(X) \otimes I_{\mathsf{M}}$, respectively. The converse "inclusion" is also true. For instance, if $n_1, \ldots, n_t, m_1, \ldots, m_t \in \mathbb{Z}_+$ and $\eta_i \in Y(n_i)$, $\omega_i \in Y(m_i)$ for all $1 \leq i \leq t$, set $\alpha_i := \left(\begin{smallmatrix} \eta_i & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\beta_i := \left(\begin{smallmatrix} \omega_i & 0 \\ 0 & 0 \end{smallmatrix}\right)$; then

$\mathbf{p}\big(\prod_{i=1}^{t} S_{n_i}^{Z}(\alpha_i)^* S_{m_i}^{Z}(\beta_i)\big)\mathbf{p}$ "equals" $\prod_{i=1}^{t} S_{n_i}^{Y}(\eta_i)^* S_{m_i}^{Y}(\omega_i)$. This completes the proof by Lemma 5.10.

(2) We will show that $\mathbf{p}\mathcal{T}(Z)\mathbf{q}$ is a $\mathbf{p}\mathcal{T}(Z)\mathbf{p}$-$\mathbf{q}\mathcal{T}(Z)\mathbf{q}$ imprimitivity bimodule, which, by the foregoing, is all we need. To this end, we merely have to verify that $\mathbf{p}$ and $\mathbf{q}$ are full. But we saw in the proof of Theorem 5.9, (2), that $\overline{\operatorname{span}}\,\mathbf{p}\varphi_\infty(L)\mathbf{q}\varphi_\infty(L)\mathbf{p}$ and $\overline{\operatorname{span}}\,\mathbf{q}\varphi_\infty(L)\mathbf{p}\varphi_\infty(L)\mathbf{q}$ "contain" $\varphi_\infty(\mathscr{B})$ and $\varphi_\infty(\mathscr{A})\otimes I_{\mathsf{M}}$, respectively; and the latter sets contain approximate identities for $\mathcal{T}(Y)$ and $\mathcal{T}(X)\otimes I_{\mathsf{M}}$, respectively. Thus $\overline{\operatorname{span}}\,\mathbf{p}\mathcal{T}(Z)\mathbf{q}\mathcal{T}(Z)\mathbf{p} = \mathbf{p}\mathcal{T}(Z)\mathbf{p}$ and $\overline{\operatorname{span}}\,\mathbf{q}\mathcal{T}(Z)\mathbf{p}\mathcal{T}(Z)\mathbf{q} = \mathbf{q}\mathcal{T}(Z)\mathbf{q}$. This completes the proof. □

Theorem 5.15. *Suppose that X, Y are subproduct systems over $\mathscr{A}, \mathscr{B}$ respectively with $X \overset{\mathrm{SME}}{\sim}_{\mathsf{M}} Y$. Then $\mathcal{O}(X) \overset{\mathrm{SME}}{\sim} \mathcal{O}(Y)$. More specifically, if we identify $\mathcal{T}(Z)$ with the subalgebra of $\mathcal{L}(\mathcal{F}_Z')$, $\mathbf{p}\mathcal{T}(Z)\mathbf{p}$ with $\mathcal{T}(Y)$ and $\mathbf{q}\mathcal{T}(Z)\mathbf{q}$ with $\mathcal{T}(X)$, and treat $\mathbf{p}\mathcal{T}(Z)\mathbf{q}$ as a $\mathbf{p}\mathcal{T}(Z)\mathbf{p}$-$\mathbf{q}\mathcal{T}(Z)\mathbf{q}$ imprimitivity bimodule, then the image of $\mathcal{I}_Y$ under the Rieffel correspondence ([22, Theorem 3.22]) is $\mathcal{I}_X$.*

Proof. The Morita equivalence of the Cuntz-Pimsner algebras follows from the succeeding assertion by [22, Proposition 3.25]. Recall that $\mathbf{p}\mathcal{T}(Z)\mathbf{p}$ and $\mathbf{q}\mathcal{T}(Z)\mathbf{q}$ are *naturally* unitarily equivalent to $\mathcal{T}(Y)$ and $\mathcal{T}(X)\otimes I_{\mathsf{M}}$, respectively. We have to check that

$$\mathbf{q}\mathcal{T}(Z)\mathbf{p}\cdot\mathcal{I}_Y\cdot\mathbf{p}\mathcal{T}(Z)\mathbf{q} = \mathcal{I}_X\otimes I_{\mathsf{M}}$$

(see [22, Proposition 3.24]). Since the Rieffel correspondence is a lattice isomorphism, it is sufficient to prove that $\mathbf{q}\mathcal{T}(Z)\mathbf{p}\cdot\mathcal{I}_Y\cdot\mathbf{p}\mathcal{T}(Z)\mathbf{q}\subseteq\mathcal{I}_X\otimes I_{\mathsf{M}}$ and that $\mathbf{p}\mathcal{T}(Z)\mathbf{q}\cdot(\mathcal{I}_X\otimes I_{\mathsf{M}})\cdot\mathbf{q}\mathcal{T}(Z)\mathbf{p}\subseteq\mathcal{I}_Y$. The two inclusions are proved similarly, so we show only the first.

Let $T_1, T_2\in\mathcal{T}(Z)$ and $S\in\mathcal{I}_Y$. Assume that T_2 is a monomial of degree $m\in\mathbb{Z}$. For large enough n, the range of $(\mathbf{p}T_2\mathbf{q})(Q_n^X\otimes I_{\mathsf{M}})$ is contained in $\binom{Y(n+m)}{0}$, and so

$$(\mathbf{q}T_1\mathbf{p}\cdot S\cdot\mathbf{p}T_2\mathbf{q})(Q_n^X\otimes I_{\mathsf{M}}) = (\mathbf{q}T_1\mathbf{p}\cdot SQ_{n+m}^Y\cdot\mathbf{p}T_2\mathbf{q})(Q_n^X\otimes I_{\mathsf{M}}),$$

and the norm of this operator is dominated by

$$\|T_1\|\,\big\|SQ_{n+m}^Y\big\|\,\|T_2\| \xrightarrow[n\to\infty]{} 0.$$

This proves (by Lemma 5.10) that $\mathbf{q}T_1\mathbf{p}\cdot S\cdot\mathbf{p}T_2\mathbf{q}\in\mathcal{I}_X\otimes I_{\mathsf{M}}$. Since the closed span of monomials of arbitrary degree in $\mathcal{T}(Z)$ is $\mathcal{T}(Z)$, we have the desired inclusion. □

Remark 5.16. The opposite direction, namely determining whether the Morita equivalence of the operator algebras implies the strong Morita equivalence of the subproduct systems, is very delicate. This is evident from the analysis of this question in the *product* system case (see [18]). We did not attempt to tackle this problem in the present paper.

5.3. **Examples.** See [18] for general examples of strong Morita equivalence of C^*-correspondences.

Example 5.17 (cf. [19])**.** Take $\mathscr{A} := \mathbb{K}$ and $\mathscr{B} := \mathbb{C}$, and let M stand for the standard $\mathbb{K}$-$\mathbb{C}$ imprimitivity bimodule, namely the Hilbert space $\mathcal{H} := \ell_2$. Fix $d \in \mathbb{N}$. For a Cuntz d-tuple of isometries $V_1, \dots, V_d$ over $\mathcal{H}$ write α for the endomorphism of $\mathbb{K}$ given by $\alpha(T) := \sum_{i=1}^d V_i T V_i^*$. Then ${}_\alpha\mathbb{K} \overset{\text{SME}}{\sim}_{\mathsf{M}} \mathbb{C}^d$ (see Example 1.3): indeed, $W : {}_\alpha\mathbb{K} \otimes_{\mathbb{K}} \mathcal{H} \to \mathcal{H} \otimes_{\mathbb{C}} \mathbb{C}^d$ given by

$$W(T \otimes h) := \sum_{i=1}^d V_i^* T h \otimes e_i \qquad (\forall T \in \mathbb{K}, h \in \mathcal{H}).$$

is a correspondence isomorphism. Now $W_n : ({}_\alpha\mathbb{K})^{\otimes n} \otimes_{\mathbb{K}} \mathcal{H} \to \mathcal{H} \otimes_{\mathbb{C}} (\mathbb{C}^d)^{\otimes n}$ satisfies

$$W_n(T_1 \otimes \cdots \otimes T_n \otimes h) = \sum_{i_1, i_2, \dots, i_n = 1}^d V_{i_1}^* T_1 V_{i_2}^* T_2 \cdots V_{i_n}^* T_n h \otimes e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}$$

for all $T_1, \dots, T_n \in \mathbb{K}$, $h \in \mathcal{H}$. Upon the identification $({}_\alpha\mathbb{K})^{\otimes n} \cong {}_{\alpha^n}\mathbb{K}$ (which holds since $V_1, \dots, V_d$ is "Cuntz"), given concretely by

$$T_1 \otimes \cdots \otimes T_n \mapsto \sum_{i_1, i_2, \dots, i_{n-1} = 1}^d V_{i_{n-1}} \cdots V_{i_1} T_1 V_{i_1}^* T_2 V_{i_2}^* \cdots V_{i_{n-1}}^* T_n,$$

we now get

$$W_n(T \otimes h) = \sum_{i_1, i_2, \dots, i_n = 1}^d V_{i_1}^* \cdots V_{i_n}^* T h \otimes e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} \qquad (\forall T \in \mathbb{K}, h \in \mathcal{H}).$$

By Remark 5.5, defining $Y(n)$ to be the sub-correspondence of ${}_{\alpha^n}\mathbb{K}$ consisting of all elements T such that $V_{i_1}^* \cdots V_{i_n}^* T = V_{i_{\sigma(1)}}^* \cdots V_{i_{\sigma(n)}}^* T$ for every $1 \le i_1, \dots, i_n \le d$ and $\sigma \in S_n$ gives a subproduct system Y over $\mathbb{K}$ with $Y(1) = {}_\alpha\mathbb{K}$ such that $Y \overset{\text{SME}}{\sim} \text{SSP}_d$. Theorems 5.9, 5.11 and 5.15 now assert that $\mathcal{T}_+(Y)$, $\mathcal{T}(Y)$ and $\mathcal{O}(Y)$ are strongly Morita equivalent to $\mathcal{T}_+(\text{SSP}_d)$, $\mathcal{T}(\text{SSP}_d)$ and $\mathcal{O}(\text{SSP}_d) \cong C(\partial B_d)$, respectively (each in the appropriate sense).

More generally, all subproduct systems whose fibers are finite-dimensional Hilbert spaces "come from polynomial identities" involving finitely many variables, and vice versa (see [23, Proposition 7.2]). Hence, the construction of the last paragraph can be adapted to every such subproduct system.

Example 5.18. The preceding example is valid when $B(\mathcal{H})$ replaces $\mathbb{K}$.

Example 5.19. Take $\mathscr{A}$, $\mathscr{B}$, M and $\mathcal{H}$ as in the last example. Fix a Cuntz sequence $(V_i)_{i \in \mathbb{N}}$ of isometries over $\mathcal{H}$ with $\sum_{i=1}^\infty V_i V_i^* = I$, and let α be the endomorphism of $B(\mathcal{H})$ given by $\alpha(T) := \sum_{i=1}^\infty V_i T V_i^*$ (all sums are in the strong operator topology).

Then ${}_{\alpha}B(\mathcal{H}) \overset{\mathrm{SME}}{\sim}_{\mathsf{M}} \mathcal{H}$ via $W : {}_{\alpha}B(\mathcal{H}) \otimes_{B(\mathcal{H})} \mathcal{H} \to \mathcal{H} \otimes_{\mathbb{C}} \mathcal{H}$ given by

$$W(T \otimes h) := \sum_{i=1}^{\infty} V_i^* T h \otimes e_i \qquad (\forall T \in B(\mathcal{H}), h \in \mathcal{H}).$$

Following the lines of Example 5.17 yields a concrete construction of a subproduct system $Y = (Y(n))_{n\in\mathbb{Z}_+}$ over $B(\mathcal{H})$ such that $Y(n)$ is a sub-correspondence of ${}_{\alpha^n}B(\mathcal{H})$ for all n and $Y \overset{\mathrm{SME}}{\sim} \mathrm{SSP}_{\infty}$.

6. Open questions

In this section we state a few open questions and possible future research directions. As usual, X denotes an arbitrary (faithful) subproduct system.

(1) **Other characterizations of $\mathcal{O}(X)$.** Is there a "strong" universality characterization of $\mathcal{O}(X)$ in the spirit of the gauge-invariant uniqueness theorem (whether or not based on Conjecture 4.7)?

(2) **Non-faithful subproduct systems.** How should the Cuntz-Pimsner algebra be defined for non-faithful subproduct systems? Considering the case $X(n) = \{0\}$ for $n \geq n_0$ makes it apparent that there is no obvious answer. Especially, it is not clear whether adapting Theorem 4.3 to this setting is feasible.

(3) **Semi-split exact sequences and K-theory.** Let E be a C^*-correspondence. An "extension of scalars" method is employed in [21, §2] to produce a C^*-correspondence E_∞ such that $\mathcal{T}(E) \hookrightarrow \mathcal{T}(E_\infty)$ and $\mathcal{O}(E) \cong \mathcal{O}(E_\infty)$ canonically, and which admits a semi-split exact sequence involving $\mathcal{T}(E_\infty)$ and $\mathcal{O}(E_\infty)$ (this is useful for obtaining a KK-theoretic six-term exact sequence for $\mathcal{O}(E)$). Could a similar technique be utilized for subproduct systems? What could be said about the K-theory of $\mathcal{O}(X)$ and $\mathcal{T}(X)$ in other methods (cf. [12, §8] and [16])?

(4) **The C^*-envelope of the tensor algebra $\mathcal{T}_+(X)$.** Denote by $C^*_{\mathrm{env}}(\mathcal{A})$ the C^*-envelope of an operator algebra $\mathcal{A}$. For every C^*-correspondence E we have $C^*_{\mathrm{env}}(\mathcal{T}_+(E)) \cong \mathcal{O}(E)$ by [10, Theorem 3.7]. In sharp contrast, from [2, Theorem 8.15] we obtain $C^*_{\mathrm{env}}(\mathcal{T}_+(\mathrm{SSP}_d)) \cong \mathcal{T}(\mathrm{SSP}_d)$. A general statement about the relation between $C^*_{\mathrm{env}}(\mathcal{T}_+(X))$, $\mathcal{T}(X)$ and $\mathcal{O}(X)$ would be very desirable.

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