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Optimal Insurance Strategies in a Risk Process with Restrictions on Policyholder Risks

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Abstract—We consider the optimal choice problem by a risk-bearing function for an insurer to divide risks between him and his clients in a dynamic insurance model, the so-called Cramer–Lundberg risk process. In this setting, we take into account restrictions imposed on policyholder risks, either on the mean value or a constraint with probability one. We solve the optimal control problem on an infinite time interval for the optimality criterion of the stationary coefficient of variation. We show that in the model with a restriction on average risk the stop-loss insurance strategy will be most profitable. For a probability one restriction, the optimal insurance is a combination of a stop-loss strategy and a deductible. We show that these results extend to a number of problems with other optimality criteria, e.g., the problems of maximizing unit utility and minimizing the probability of deviating from the mean value.

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1. INTRODUCTION

The classical Cramer–Lundberg risk process that describes capital dynamics in an insurance company has the form

$$X_t = x_0 + ct - \sum_{i=1}^{N_t} Y_i, \tag{1}$$

where $x_0 > 0$ is the initial capital of the company, $\{N_t\}$ is the Poisson process of the claims with parameter λ that defines the number of filed claims, i.e., the number of insurance events, on the interval $[0, t]$; $\{Y_i\}$ are independent identically distributed insurance payments (or risks) with distribution function $F(x)$ and a finite second moment $EY^2 < \infty$, which do not depend on $\{N_t\}$. The speed c of accumulating insurance premium is determined by the mean value principle (see, e.g., [1,2]). That is, the total premium on $[0, t]$ equals $(1+\alpha)EX_t$, where the total loss is $X_t = \sum_{i=1}^{N(t)} Y_i$, and $\alpha > 0$ is a predefined load coefficient that shows the mark-up, in percents, over the average risk EX_t. Since $EN_t = \lambda t$, the speed of premium accumulation is $c = (1 + \alpha)\lambda EY$.

Studying the optimization problem of risk-bearing in insurance began with the work [3], where, for a static model, it was shown that the optimal risk-bearing from the viewpoint of expected policyholder (client) utility is a deductible. In [4], a description of Pareto optimal risk-bearing functions was presented, and, in particular, it was shown that the solution of the risk-bearing problem leads to a stop-loss risk-bearing if the risk-bearing function is chosen by the insurance company (the insurer). The situation when an insurer has the choice of insuring and reinsuring each individual risk is considered in [5].

The optimal control problem in dynamic insurance models based on the Cramer–Lundberg risk process has been studied in [6–11]. In [6], the problem of minimizing the ruin probability for an insurer by selecting reinsurance in the class of stop-loss risk-bearing functions applied to each

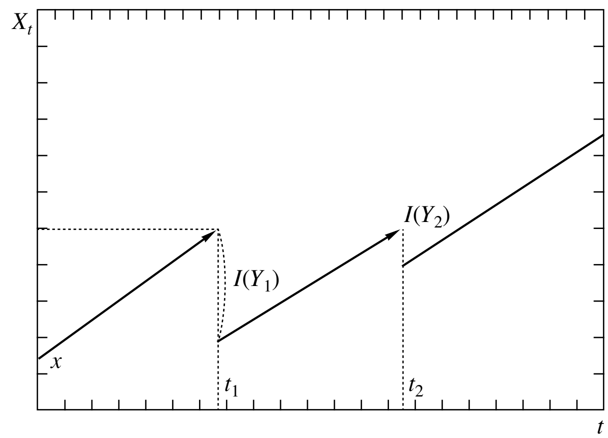

Fig. 1. Trajectory of the controlled risk process.

policyholder's risk has been studied. The work [7] is devoted to the same problem, but in the case when the insurer has the option of proportional reinsuring and investing in a risky asset. In a close problem setting, but without reinsurance control, the problem of minimizing ruin probability has been studied in [8], where the asymptotics of the target functional has been determined. The risk process arising as a diffuse approximation of the Cramer–Lundberg process was studied in [9], where optimal investing and proportional reinsurance strategies were found for the cases of both unlimited investing budget (for the case of using short sales) and if a constraint is present. The problem with another optimality criteria, namely the expected consuming utility of a policyholder, was solved in [10], where optimal investing and insuring strategies were found.

The main feature distinguishing this work from previous studies are bounds on the risk remaining with the client after insurance. These (upper) bounds are natural from the policyholder's point of view; he would like to avoid "large" (in some sense) losses. In this work, we consider both a constraint on average risk and a "hard" constraint with probability one. Managing the risk process consists of selecting an insurance strategy (we do not consider the possibilities of reinsurance and investment management). While the basic instrument for finding an optimal strategy in the above-mentioned works has been the Hamilton–Jacobi–Bellman equation, which can very seldom be solved analytically and usually requires numerical methods, the present paper reduces the optimal control problem to the static case and obtain an analytic form for the optimal strategy. A similar problem setting was considered in [11], where simultaneous optimization of insurance and reinsurance during the risk process was allowed. However, in that paper average maximal insurer losses are being minimized, while the present work uses other criteria, e.g., the stationary variation criterion. Besides, significantly, in [11], a constraint with probability one was not considered.

Let us proceed to the formal description of the model in question, i.e., the controlled risk process. Suppose that at the payment moment $t = t_i$ ($i \geq 0$, $t_0 = 0$) the insurer makes a decision, i.e., chooses a risk-bearing function $I_t(\cdot)$, which means that now $I_t(Y_{i+1})$ is the share of the next payment reimbursed to the client, and the premium accumulation speed becomes equal $c_t = (1+\alpha)\lambda E\, I_t(Y)$ until the next payment. The controlled risk process is, then,

$$X_t = x_0 + \int\limits_0^t c_s ds - \sum_{j=1}^{N_t} I_{t_{j-1}}(Y_j), \qquad (2)$$

where admissible strategies $I = \{I_t\}$ are nonanticipating controls measurable with respect to the natural filtration $\{\mathcal{F}_t\}$ and satisfying standard inequalities $0 \leq I_t(x) \leq x$ on $[0, \infty)$ (in other words, reimbursement is always nonnegative and does not exceed the client's damage). The trajectory of the X_t process is shown on Fig. 1.

The process operation interval is supposed to be infinite, and after determining the target functional $J[I]$ that characterizes the process operation quality this problem, without additional constraints, takes the form

$$\min\,(\max)\, J[I], \tag{3}$$

where the minimum (maximum) is taken over the set of admissible insurance strategies $I = \{I_t\}$ defined above.

In Section 2, we consider the stationary coefficient of variation as the minimization criterion $J[I]$ and study both cases of the constraints on policyholder risk: a bound on the mean value and a bound with probability one. Optimal insurance risk-bearing functions are substantially different in these two cases; in the former case stop-loss insurance is optimal, while in the latter case the so-called SD-insurance, which is a combination of the stop-loss strategy and the deductible, becomes optimal. We present optimality equations that define parameters of these insurance strategies for both constraint types. To construct optimal strategies, we have first reduced the original problem to a static optimization problem and, second, used the methods of moments theory to the latter, in particular, the Neyman–Pearson lemma. In Section 3, this method is used to solve the problem with another criterion, insurer utility (a linear function of the expectation and variance) per unit of time. The form of optimal insurance risk-bearing functions is the same, only the optimality equations for parameters of these risk-bearing functions have changed. In Section 4, as the minimization criterion we have considered the probability of a time normalized risk process to deviate from the mean value. In the first case of a bound on the average risk stop loss insurance remains the solution, while in the case of a bound with probability one the solution will be a degenerate SD insurance, namely a deductible. At the end of this work, we show a numerical example that illustrates our results for the case of an exponential distribution of the payments.

2. MINIMIZATION PROBLEM FOR THE STATIONARY COEFFICIENT OF VARIATION

Suppose that the functional minimized in problem (3) has the form of an upper limit

$$J[I] = \overline{\lim_{t\to\infty}} D\, X_t / E\, X_t. \tag{4}$$

Similar to a well-known term in risk theory, we call this functional the "stationary coefficient of variation," despite the fact that usually this term means a ratio of the standard deviation, not the variance, to the expectation. In our case, even for an unmanaged risk process (1) (considering the fact that $E\sum_{i=1}^{N_t} Y_i = \lambda t E\, Y$ and $D\sum_{i=1}^{N_t} Y_i = \lambda t E\, Y^2$) the process

$$\lim_{t\to\infty} \frac{\sqrt{D\, X_t}}{E\, X_t} = \lim_{t\to\infty} \frac{\sqrt{\lambda t E\, Y^2}}{x_0 + \alpha\lambda t E\, Y}$$

is always zero, so as the criterion we select the ratio of the variance of X_t to the average value. In insurance (see, e.g., [2]), the variation coefficient is often used as an estimate of a company's financial stability: the less it is, the better the insurance portfolio is balanced.

2.1. The Case of Bounded Average Risk

It is easy to see that the functional (4) reaches the minimal zero value in case if all risk-bearing functions are zero, $I_t(x) \equiv 0$, which means that the company refuses to insure at all. In this case, $X_t = x_0$ and $D\,X_t/E\,X_t = 0$. To avoid this situation, in which the clients' desire to insure their risks is completely ignored, we introduce an additional constraint: we consider admissible only those insurance risk-bearing functions I which satisfy an upper bound on the average risk remaining with the client (policyholder) after insurance, $E\{Y - I_t(Y)\} \le C$. Here t is the (fixed) payment moment, $C \in [0, E\,Y)$ is the constant defined by the policyholder that defines the maximal average risk he agrees to have left. An equivalent constraint on the insurer's risk can be written as $E\,I_t(Y) \ge M$, where $M = E\,Y - C$. Our subject in this subsection is the problem of minimizing the functional defined in (4) with additional constraints

$$J[I] \to \min, \quad E\,I_t(Y) \ge M. \tag{5}$$

Theorem 1. *In problem* (5), *the minimum is achieved on the stationary strategy $I^*(\cdot)$ independent on the current process state. The optimal risk-bearing function is the stop-loss insurance $I^*(x) = x \wedge k^*$ (here and below $x \wedge y$ denotes* $\min\{x, y\}$), *where parameter k^* is the only root of the equation $\int\limits_0^k \bar{F}(x)dx = M$, where $\bar{F}(x) = 1 - F(x)$.*

Proof. According to definition (2), the risk process is a controlled Markov uniform process with infinite horizon (see, e.g., [12]). We define the Bellman function $V(t, x) = \inf_I J[I]$ for a process on the interval $[t, \infty)$ with initial state $X_t = x$. Due to the specifics of the considered $J[I]$ criterion, we have $V(t, x) = V(0, x)$ and $V(t, x) = V(t, 0)$. Therefore, since the class of Markov strategies is sufficient for solving the optimization problem, we get that the minimum in problem (5) can be searched for in the class of constant strategies $I_t(\cdot) = I(\cdot)$, i.e., strategies independent of both the decision making moment t and the current state x.

In the class of these strategies, by the Wald's identity we get

$$E\,X_t = x_0 + \alpha\lambda t E\,I(Y) \quad \text{and} \quad D\,X_t = \lambda t E\,I^2(Y), \tag{6}$$

therefore, (4) implies that the original problem (5) takes the form

$$J[I] \equiv E\,I^2(Y)/(\alpha E\,I(Y)) \to \min, \quad E\,I(Y) \ge M. \tag{7}$$

The set of admissible risk-bearing functions I here is the set of Borel functions satisfying inequalities $0 \le I(x) \le x$, $E\,I(Y) \ge M$.

For a solution (7), we use the approach employed in [11] for a similar problem: we first show that the solution of an auxiliary problem

$$\inf_{I:\,E\,I(Y)=m} J[I] \tag{8}$$

with parameter $m \in [M, E\,Y]$ is a stop-loss insurance $x \wedge k_m$, and then find the value of k_m for the problem (7). Since $J[I]$ increases in $E\,I^2(Y)$ for a fixed $E\,I(Y)$, minimization in (8) reduces to minimizing $E\,I^2(Y)$. Since $E\,I^2(Y)$ is a convex function of I, an admissible risk-bearing function I_m^* minimizes $E\,I^2(Y)$ if and only if the derivative

$$\frac{d}{d\rho}\,E\left\{\rho I_m^*(Y) + (1-\rho)I(Y)\right\}^2\Big|_{\rho=1} = 2\int\limits_0^T I_m^*(x)[I_m^*(x) - I(x)]dF(x) \le 0$$

for any admissible I. This is equivalent to the fact that I_m^* is a solution of the problem

$$\min_I \int\limits_0^\infty I_m^*(x)I(x)dF(x) \quad \text{with constraint} \quad \int\limits_0^\infty I(x)dF(x) = m.$$

By the Neyman–Pearson lemma (see, e.g., [13]), an admissible risk-bearing function I^*_m is optimal in this problem if and only if there exists a constant k such that

$$I^*_m(x) = \begin{cases} 0, & \text{if } I^*_m(x) - k > 0 \\ x, & \text{if } I^*_m(x) - k < 0 \end{cases} \qquad \text{up to a set of zero } F\text{-measure.}$$

It is easy to see that the only function satisfying this condition is $I^*_m(x) = x \wedge k$, where $k = k_m$ is defined by $E\,[Y \wedge k] = m$, i.e., $\int\limits_0^k \bar{F}(x)dx = m$.

Returning to problem (7), we note that

$$\inf_{I:\, EI(Y) \ge M} J[I] = \inf_{m \in [M, EY]} \ \min_{I:\, EI(Y)=m} J[I],$$

and the internal minimum in the right-hand side is reached, as we have shown, on $I^*_m(x) = x \wedge k_m$. Then the left-hand side problem is equivalent to minimizing over all risk-bearing functions $I(x) = x \wedge k$, $k \ge k_M$. After substituting a stop-loss risk-bearing function in the expression for $J[I]$ in (7) we get $J[I] = \int\limits_0^k 2x\bar{F}(x)dx \Big/ \Big\{\alpha \int\limits_0^k \bar{F}(x)dx\Big\}$. By differentiating, it is easy to show that this function increases in k, so the optimal risk-bearing function in (7) is $I^*(x) = x \wedge k^*$ with $k^* = k_M$.

The optimal risk-bearing function form $I^*(x) = x \wedge k^*$ found in Theorem 1 is a well-known in insurance practice contract with an insurance sum: if the client's damages do not exceed k^*, they are reimbursed completely, and if $Y > k^*$ then only the insurance sum k^* is paid.

2.2. The Case of a Bound with Probability One

Suppose that we consider admissible only those risk-bearing functions which satisfy an upper bound on the client's risk satisfied with probability one: $Y - I_t(Y) \le q$, where q is a constant given by the policyholder. Any contract admissible from the policyholder's point of view should leave him risk (i.e., potential damage) at most q. An equivalent restriction on the insurer's risk can be written as $I_t(x) \ge (x - q)_+$, $x \in [0, \infty)$, where $(x)_+$ denotes $\max\{0, x\}$.

Theorem 2. *The minimum in the problem*

$$J[I] \to \min, \quad I_t(x) \ge (x - q)_+, \tag{9}$$

is achieved on a stationary strategy $I^*(\cdot)$ independent of the current process state. The optimal risk-bearing function is a combination of stop-loss insurance and deductible, $I^*(x) = (x \wedge k^*) \vee (x - q)$, where $x \vee y$ denotes $\max\{x, y\}$. *The parameter k^* is the only root of the equation*

$$\int\limits_0^k (k - x)\bar{F}(x)dx + \int\limits_k^\infty (k - x)\bar{F}(x + q)dx = 0. \tag{10}$$

Proof. By repeating the arguments of the first part of the proof of Theorem 1, we get that it suffices to consider stationary strategies independent of the current process state. Therefore, the initial problem reduces to an analogue of problem (7):

$$J[I] \equiv E\,I^2(Y)/(\alpha E\,I(Y)) \to \min, \quad (x - q)_+ \le I(x) \le x. \tag{11}$$

Just like in Theorem 1, to solve (11) we consider an auxiliary problem with an additional constraint

$$\inf_{I:\, E\,I(Y)=m} E\,I^2(Y).$$

Applying the Neyman–Pearson lemma gives us necessary and sufficient optimality conditions in this problem: there exists a constant k such that

$$I_m^*(x) = \begin{cases} (x-q)_+ & \text{if } I_m^*(x) - k > 0 \\ x & \text{if } I_m^*(x) - k < 0. \end{cases} \tag{12}$$

We first assume that $k < I_m^*(0)\,(=0)$. Then (see (12)) $I_m^*(x) \equiv (x-q)_+$. If $I_m^*(0) - k = 0$ (i.e., $k = 0$) then $I_m^*(x)$ cannot become positive up until point q, since for such x (12) implies the equality $I_m^*(x) = 0$, which yields a contradiction. For $x > q$, the value of $I_m^*(x)$ coincides with the lower bound $x - q$ and, consequently, $I_m^*(x) \equiv (x-q)_+$. Suppose that $k > I_m^*(0)$. Then from (12) we have $I_m^*(x) = x$ up until the point $x = k$ of $I_m^*(x) - k$ touching the X axis. As x increases from k upwards, the value $I_m^*(x) - k$ remains zero, since both increase and decrease of this function would contradict (12). At the point $x = k + q$, it reaches the lower bound and, due to (12), remains equal to $x - q$. As a result, we get that the optimal risk-bearing function has to look like $I_k(x) = (x \wedge k) \vee (x - q)$, where $k \geq 0$.

Now, to solve problem (11) with no additional constraints it remains to find $\min\limits_{k\geq 0} J[I_k]$, where

$$J[I_k] = \frac{E\,I_k^2(Y)}{\alpha E\,I_k(Y)} = \frac{2\left(\int\limits_0^k x\bar{F}(x)dx + \int\limits_k^\infty x\bar{F}(x+q)dx\right)}{\alpha\left(\int\limits_0^k \bar{F}(x)dx + \int\limits_k^\infty \bar{F}(x+q)dx\right)}. \tag{13}$$

The derivative is

$$\frac{d}{dk}J[I_k] = A(\bar{F}(k) - \bar{F}(k+q))\left[\int\limits_0^k (k-x)\bar{F}(x)dx + \int\limits_k^\infty (k-x)\bar{F}(x+q)dx\right],$$

where $A = 2\Bigg/\left(\alpha\left[\int\limits_0^k \bar{F}(x)dx + \int\limits_k^\infty \bar{F}(x+q)dx\right]^2\right) > 0$. If we denote the expression in square brackets by $\psi(k)$, we arrive at the representation

$$\frac{d}{dk}J[I_k] = \phi(k)\psi(k), \tag{14}$$

where $\phi(k) \geq 0$ and, as differentiation readily yields, $\psi(k)$ is an increasing function changing its sign on $[0,\infty)$. Then the root k^* of the equation $\psi(k) = 0$, i.e., Eq. (10), is the minimum point $J[I_k]$.

The function $I^*(x)$ obtained in Theorem 2 is, in a sense, a generalization of a deductible $I(x) = (x-q)_+$, since the tail of the damage distribution is left to the insurer; small damages are divided according to stop-loss insurance, $I(x) = x \wedge k$. This form of a risk-bearing function has been introduced in [11] under the name SD-insurance in order to solve the insurer utility maximization problem on a certain set of risk-bearing functions. The form of $I^*(x)$ is shown on Fig. 2.

The client only has a "middle" part of his initial risk left, $Y - I^*(Y) = (Y - k^*)_+ \wedge q$. This situation is preferable for a policyholder with possible catastrophic values of damages than stop-loss insurance, when for a large damage the insurer only pays the limit sum k^*.

3. MAXIMIZING THE UTILITY FUNCTIONAL

As the optimality criterion, we consider unit utility, or utility per unit of time

$$J[I] = \varliminf_{t\to\infty} \frac{1}{t}[E\,X_t - \theta D\,X_t].$$

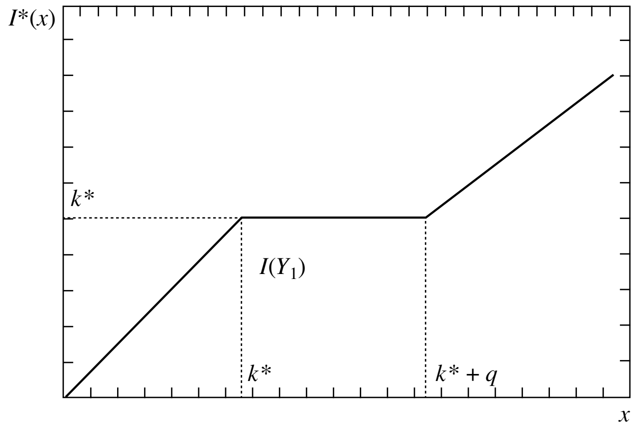

Fig. 2. Optimal risk-bearing function $I^*(x)$ for insurance.

The term "utility" is understood here in the sense of Markowitz theory [14], as a function depending on risk mean and variance, increasing in the first argument and decreasing in the second. In this case, we have selected a linear function $U[X_t] = E\,X_t - \theta D\,X_t$, where $\theta > 0$ is a given weight. Note that the variation coefficient minimization problem can also be thought of as a problem of this type, but with a different utility function $V[X_t] = E\,X_t / D\,X_t$.

As above, the subject of our study is the problem with a restriction on average risk and the problem with a probability one restriction:

$$J[I] \to \max, \quad E\,I_t(Y) \geq M, \tag{15}$$

$$J[I] \to \max, \quad I_t(x) \geq (x-q)_+. \tag{16}$$

Theorem 3. *Maximum values in* (15), (16) *are achieved on stationary strategies independent of the current process state.*

(1) *Optimal risk-bearing function in* (15) *is a stop-loss insurance*

$$I_1^*(x) = x \wedge [k^* \vee \{\alpha/(2\theta)\}],$$

where k^* is as defined in Theorem 1.

(2) *Optimal risk-bearing function in* (16) *is an SD-insurance*

$$I_2^*(x) = [x \wedge \{\alpha/(2\theta)\}] \vee (x-q).$$

Proof. The same arguments as in Theorem 1 prove that stationary strategies independent of the current process state suffice, $I_t(\cdot) = I(\cdot)$. Substituting expressions (6) for the mean and variance of X_t, we get that on the class of these strategies the maximized functional is

$$J[I] = \alpha E\,I(Y) - \theta E\,I^2(Y). \tag{17}$$

(1) Repeating the arguments from the proof of Theorem 1, we arrive at the conclusion that for problem (15), a stop-loss risk-bearing function $I_k(x) = x \wedge k$ is optimal, and it remains to find $\max\limits_{k \geq k_M} J[I_k] = \max\limits_{k \geq k_M} \alpha \int\limits_0^k \bar{F}(x)dx + 2\theta \int\limits_0^k x\bar{F}(x)dx$, where k_M is defined as the root of equation $E\,Y \wedge k = M$. The derivative is

$$\frac{d}{dk} J[I_k] = \bar{F}(k)[\alpha - 2\theta k],$$

so the maximum point will be a zero of the derivative $k^0 = \alpha/(2\theta)$ if $k^0 \geq k_M$, and the left boundary point k_M $(= k^*)$ otherwise.

(2) Solving problem (16), we, similar to the proof of Theorem 2, get that the optimal risk-bearing function is an SD-insurance $I_k(x) = x \wedge k \vee (x - q)$, where $k \geq 0$. Substituting into (17) expressions for $E\, I_k(Y)$ and $E\, I_k^2(Y)$, we obtain a problem of maximizing by k the function

$$\max_{k \geq 0} \alpha \left(\int\limits_0^k \bar{F}(x)dx + \int\limits_k^\infty \bar{F}(x+q)dx \right) - 2\theta \left(\int\limits_0^k x\bar{F}(x)dx + \int\limits_k^\infty x\bar{F}(x+q)dx \right).$$

The derivative

$$\frac{d}{dk} J[I_k] = \left(\bar{F}(k) - \bar{F}(k+q)\right)[\alpha - 2\theta k]$$

equals the product $\phi(k)\psi(k)$ (see (14)), where $\phi(k) \geq 0$ and $\psi(k)$ decreases, changing sign at point $k^0 = \alpha/(2\theta)$, which is the maximal point of $J[I_k]$.

The found optimal risk-bearing functions have the same form as solutions of coefficient of variation minimization problems (4) and (9). But while in (4) optimal risk-bearing employs the minimal possible average insurer risk ($E\, I^*(Y) = M$), now in (10), for a large enough load coefficient α, the value $k^0 = \alpha/(2\theta) > k^*$ and the average risk $E\, I_1^*(Y)$ turn out to be shifted from the left boundary. In this case, the first level $\alpha/(2\theta)$ in the SD-insurance I_2^* coincides with the hold level in the stop-loss insurance I_1^*.

4. MINIMIZING THE PROBABILITY OF DEVIATING FROM AN "AVERAGE" TRAJECTORY

In this section, we study the situation when an insurer would like to minimize the "spread" of values of X_t around the desired average value $m(t) = E\, X_t$ at a remote moment of time. As a measure of this spread we take the probability of getting out of the ε-neighborhood of the time normalized process. The minimized functional looks like

$$J[I] = \overline{\lim_{t \to \infty}}\, P\left\{ \left| \frac{X_t - E\, X_t}{\sqrt{t}} \right| > \varepsilon \right\}.$$

Consider both types of constraints, on average risk and with probability one:

$$J[I] \to \min, \quad E\, I_t(Y) \geq M, \tag{18}$$
$$J[I] \to \min, \quad I_t(x) \geq (x-q)_+. \tag{19}$$

Theorem 4. *Minimum values in problems* (18), (19) *are achieved on stationary strategies independent of the current process state.*

(1) *The optimal risk-bearing function in* (18) *is a stop-loss insurance*

$$I_1^*(x) = x \wedge k^*,$$

where k^* is as defined in Theorem 1.

(2) *The optimal risk-bearing function in* (19) *is a deductible*

$$I_2^*(x) = (x-q)_+.$$

Proof. As above, after the arguments completely similar to the proof of the first part of Theorem 1 we get hat minimal values in (18) and (19) are achieved on constant strategies, i.e., on stationary strategies independent of the current process state, $I_t(\cdot) = I(\cdot)$. For this control, the random process X_t for each fixed t is a complex Poisson random value (r.v.), and the normal convergence theorem (see, e.g., [15]) implies that the distribution limit as $t \to \infty$ for $(X_t - E\,X_t)/\sqrt{t}$, where $E\,X_t = x_0 + \alpha\lambda t E\,I(Y)$, will be a normal r.v. with parameters $\mu = 0$ and $\sigma^2 = \lambda E\,I^2(Y)$. Therefore, the expression for $J[I]$ can be rewritten as $1 - (\Phi_{0,\sigma}(\varepsilon) - \Phi_{0,\sigma}(-\varepsilon))$ or, passing to the standard normal distribution function, $J[I] = 1 - \Phi(\varepsilon/[\lambda E\,I^2(Y)]) + \Phi(-\varepsilon/[\lambda E\,I^2(Y)])$. Since $\Phi(x)$ increases, the minimization in $J[I]$ reduces to minimizing $E\,I^2(Y)$.

(1) For the solution of (18), passing to the problem $\min E\,I^2(Y)$, $E\,I(Y) = m$, we find the optimal risk-bearing function $I_k(x) = x \wedge k$, where k is defined by the equation $E\,I_k(Y) = m$. Taking into account that

$$\frac{d}{dk} E\,I_k^2(Y) = \frac{d}{dk} 2\int_0^k x\bar{F}(x)dx = 2k\bar{F}(k)$$

and, therefore, $E\,I_k^2(Y)$ increases in k, we get the solution of the problem (18): $I_1^*(x) = x \wedge k^*$. As in Theorem 1, k^* is defined by the equation $\int_0^k \bar{F}(x)dx = M$.

(2) The problem $\min E\,I^2(Y)$, $I(x) \geq (x - q)_+$ is analyzed similar to Theorem 2. In the end, the optimal insurance is an SD-insurance $I_k(x) = [x \wedge k] \vee (x - q)$. Since $E\,I_k^2(Y) = 2\left[\int_0^k x\bar{F}(x)dx + \int_k^\infty x\bar{F}(x+q)dx\right]$, the derivative

$$\frac{d}{dk} E\,I_k^2(Y) = (\bar{F}(k) - \bar{F}(k+q))2k$$

has the form similar to (14). Consequently, the minimum point of $J[I_k]$ over $k \in [0, \infty)$ will be $k^* = 0$. As a result, the optimal risk-bearing function is $I_2^*(x) = (x - q)_+$.

Note that the optimal stop-loss insurance in (18) coincides with the risk-bearing function found in Theorem 1, which minimizes the coefficient of variation. However, in problem (19) with a "hard" constraint, unlike Theorem 2, the optimal solution is a deductible, a degenerate case of SD-insurance where on the first level $k^* = 0$. The policyholder selects a lower bound on the set of admissible risk-bearing functions, and in this regard the chosen strategy is maximally cautious.

Remark 1. If in target functionals of problems (18), (19) we replace the inequality in deviation probability $P\left\{\left|(X_t - E\,X_t)/\sqrt{t}\right| > \varepsilon\right\}$ by a single-side inequality $(X_t - E\,X_t)/\sqrt{t} < -\varepsilon$, the answer will not change since minimizing $J[I] = \Phi(-\varepsilon/[\lambda E\,I^2(Y)])$ reduces to minimizing $E\,I^2(Y)$. If, on the other hand, we use the inequality $(X_t - E\,X_t)/\sqrt{t} < \varepsilon$, which means that the insurer would like to maximize the probability of X_t exceeding the level $E\,X_t + \varepsilon\sqrt{t}$, then the optimal risk-bearing function changes. In this case, minimizing $J[I] = 1 - \Phi(\varepsilon/[\lambda E\,I^2(Y)])$ is equivalent to minimizing $E\,I^2(Y)$. It is easy to see that the latter problem has the same solution $I^*(x) = x$ for restrictions on average risk and with probability one. For this strategy, the insurance company takes up all risk of the clients hoping that the premium accumulation speed will be high enough so that his capital exceeds (probabilistically) the value $E\,X_t + \varepsilon\sqrt{t}$. Compared to optimal risk-bearing functions found in Theorem 4, this kind of insurance is the least cautious, brought about by the very problem setting in which the optimal risk-bearing function is the one that leads to maximal deviation of X_t around its mean value.

5. EXAMPLE

As an example illustrating our results, we consider the problem of Section 2 about minimizing the stationary coefficient of variation $J[I] = \overline{\lim}_{t\to\infty} D\,X_t/E\,X_t$ under both kinds of constraints for the case when payment distribution is exponential, $F(x) \overset{def}{=} P\{Y \leq x\} = 1 - e^{-x/\mu}$.

(1) Suppose that a restriction has been imposed on the insurer's average risk (see Subsection 2.1), $E\,Z \overset{def}{=} E\,\{Y - I(Y)\} \leq C$, where $0 \leq C < \mu$ is a constant given by the policyholder. This restriction can be rewritten as $E\,I(Y) \geq M$, where $M = E\,Y - C$. By Theorem 1, the optimal risk-bearing function for the problem $\min_I J[I]$ is a stop-loss insurance $I^*(x) = x \wedge k^*$, and the parameter k^* (the holding level) is defined as the root of the equation $\int_0^k \bar{F}(x)dx = M$. In case of exponential distribution $F(x)$, we get an equation $\mu(1 - e^{-k/\mu}) = M$, which implies

$$k^* = \mu \ln\left(\frac{\mu}{\mu - M}\right). \tag{20}$$

Now (6) yields an expression for the optimal stationary coefficient of variation $J[I^*] = 2\int_0^{k*} x\bar{F}(x)dx/\left(\alpha\int_0^{k^*} \bar{F}(x)dx\right)$ and, taking (20) into account,

$$J[I^*] = \frac{2\mu}{\alpha}\left(\frac{M-\mu}{M}\ln\left(\frac{\mu}{\mu - M}\right) + 1\right).$$

We set the average payment value $\mu = 10$ and the load coefficient $\alpha = 0.5$. Results of computing the optimal holding level k^* and $J^* = J[I^*]$ for different values of the upper bound C on the insurer average risk are shown in Table 1.

Table 1

C	0	2	4	6	8	9
$M = \mu - C$	10	8	6	4	2	1
k^*	∞	16.094	9.163	5.108	2.231	1.054
J^*	40	23.905	15.565	9.350	4.297	2.071

The case $k^* = \infty$ means that $I^*(Y) = Y \wedge k^* = Y$, i.e., the insurance company has to take up all client's risk under the constraint $E\,I^*(Y) \geq M$ with $M = E\,Y\,(= 10)$.

(2) Suppose now that the policyholder requires that the restriction on his remaining risk $Z \overset{def}{=} Y - I(Y)$ must hold with probability one: $Z \leq q$ or, equivalently, $I(x) \geq (x - q)_+$. By Theorem 2, a risk-bearing function I^* minimizing the stationary coefficient of variation is an SD-insurance $I^*(x) = (x \wedge k^*) \vee (x - q)$ with two holding levels k^* and $k^* + q$. The value of parameter k^* is defined by Eq. (10), which in this case looks like

$$\mu(1 - e^{-q/\mu}) + (k - \mu)e^{k/\mu} = 0.$$

Substituting the exponential distribution in (13), we get an expression for the optimal value of the coefficient of variation

$$J[I^*] = \frac{2\left(\int\limits_0^k x\bar{F}(x)dx + \int\limits_k^\infty x\bar{F}(x+q)dx\right)}{\alpha\left(\int\limits_0^k \bar{F}(x)dx + \int\limits_k^\infty \bar{F}(x+q)dx\right)}$$

$$= \frac{2\mu[1-\delta\exp(-k/\mu)+\delta\exp(-(k+q)/\mu)]}{\alpha[1-\exp(-k/\mu)+\exp(-(k+q)/\mu)]}, \quad \text{where} \quad \delta = 1+k/\mu.$$

For the same values $\mu = 10$ and $\alpha = 0.5$, computation results for the optimal holding level k^* and the value of the functional $J^* = J[I^*]$ are shown in Table 2.

Table 2

q	0	4	8	12	16	20
k^*	∞	8.606	7.363	6.265	5.304	4.470
J^*	40	34.423	29.452	25.061	21.217	17.881

Increasing q means reducing the lower bound $(x-q)_+$ of the set of admissible risk-bearing function values, i.e., extending the minimization region $J[I]$. Therefore, the minimal value $J^* = J[I^*]$ drops as q increases. The case $q = 0$, as in the first part of the example, means $I^*(Y) = Y$, that is, the insurer takes up all client's risk. The value of the level k^*, on the other hand, drops as q decreases, which leads to reducing the share of risk $I^*(Y) = (Y \wedge k^*) \vee (Y - q)$ held by the insurer. In other works, the insurer becomes more cautious is the policyholder's restrictions are relaxed. This effect is similar to the results from the first part of the example (see Table 1), when along with an increase in the upper bound C on an average client's risk the level k^* in the insurer's share of risk $I^*(Y) = Y \wedge k^*$ decreased.

6. CONCLUSION

In this work, we have solved the problems of choosing optimal insurance strategies for the insurer in a risk process of Cramer–Lundberg type on an infinite time interval for three optimality criteria: stationary coefficient of variation, unit utility, and probability of the normalized process deviating from the mean value. In all cases, we have considered two kinds of constraints on the risk remaining with the client after insurance, namely a restriction on the mean value and a restriction with probability one. The novelty in problem settings is, first, in the choice of target functionals, not used earlier for a risk process, and, second, in residual risk constraints natural from the policyholder's point of view. Optimal strategy construction, unlike the traditional approach, is based not on solving of the Hamilton–Jacobi–Bellman equation, but on reducing the problem to the static case and then applying a well-known Neyman–Pearson lemma from the theory of moments. We have shown that for restrictions of the first kind stop-loss insurance turns out to be optimal, while for restrictions with probability one the optimal risk-bearing function is a special form of SD-insurance, a combination of stop-loss insurance and a deductible. We have found optimality equations defining parameters of insurance strategies. Practical applications of our result imply optimization of risk-bearing schemes of an insurance company functioning over a long interval of time. The most important here is the risk-bearing of "large" risks, including the choice of insurance tariffs.

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