A charged particle in a homogeneous magnetic field accelerated by a time periodic Aharonov-Bohm flux

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Abstract

We consider a nonrelativistic quantum charged particle moving on a plane under the influence of a uniform magnetic field and driven by a periodically time-dependent Aharonov-Bohm flux. We observe an acceleration effect in the case when the Aharonov-Bohm flux depends on time as a sinusoidal function whose frequency is in resonance with the cyclotron frequency. In particular, the energy of the particle increases linearly for large times. An explicit formula for the acceleration rate is derived with the aid of the quantum averaging method, and then it is checked against a numerical solution with a very good agreement.

Keywords: electron-cyclotron resonance, Aharonov-Bohm flux, quantum averaging method, acceleration rate

1 Introduction

The problem of acceleration in physical systems driven by time-periodic external forces, both in classical and quantum mechanics, has a rather long history though the results in the latter case are much less complete. One of the most prominent examples which initiated a lot of efforts in this field is the so called Fermi accelerator. On the basis of a theory due to Fermi to explain the acceleration of cosmic rays [8] Ulam formulated a mathematical model describing a massive particle bouncing between two infinitely heavy walls while one of the walls is oscillating [26]. A thorough analysis finally did

not fully confirm the expectations, however [28, 19, 20]. The model has also been reformulated in the framework of quantum mechanics [16].

More models of this sort have been studied in detail so far but we just mention one of them, the so called electron cyclotron resonance. One readily finds that electrons in a uniform magnetic field can gain energy from a microwave electric field whose frequency is equal to the electron cyclotron frequency. Because of an unlimited energy increase the relativistic effects cannot be neglected in a complete analysis. But even the relativistic model admits a quite explicit characterization of the resonant solution for a transverse circularly polarized electromagnetic wave propagating along the uniform magnetic field [21]. In experimental arrangements the heated electrons are confined in a magnetic mirror field. Consequently, as they move along a flux tube of the mirror field they are exposed to the resonance heating only in a restricted region [24, 10, 12]. This acceleration mechanism is widely used in plasma physics.

Here we wish to discuss, on the quantum level, a model sharing some features with the preceding one. We again consider a charged particle placed in a uniform magnetic field. In our model the situation is simplified, however, in the sense that the particle is confined to a plane perpendicular to the magnetic field. Instead of a transverse electromagnetic wave propagating along the uniform field we apply, as an external force, an oscillating Aharonov-Bohm flux. The frequency of oscillations Ω again coincides with the cyclotron frequency ω_c or, more generally, it may be an integer multiple of ω_c.

The Aharonov-Bohm effect itself received a tremendous attention as a genuinely quantum phenomenon [2], and almost all its possible aspects have been studied in the time-independent case. For example, a careful analysis can be found in [22]. On the other hand, the time-dependent case represents an essentially more difficult mathematical problem and it has been treated so far only marginally in a few papers [18, 1, 3, 5].

The model we propose has already been studied in the framework of classical mechanics [4]. It turns out that a resonance acceleration again exists but it has some remarkable new features if compared to the standard electron cyclotron resonance. If Ω is an integer multiple of ω_c, then the classical trajectory eventually reaches an asymptotic domain where it resembles a spiral whose circles pass very closely to the singular flux line and, at the same time, their radii expand with the rate $t^{1/2}$ as t approaches infinity. The particle moves along the circles approximately with frequency ω_c while its energy increases linearly with time. Denoting by $\mathcal{E}(t)$ the energy depending on time, an important characteristic of the dynamics is the acceleration rate which is computed in [4] and is given by the formula

$$\gamma_{\mathrm{acc}} := \lim_{t\to\infty} \frac{\mathcal{E}(t)}{t} = \frac{e\omega_c}{4\pi}\,|\Phi'(\tau)|. \tag{1}$$

Here τ is a real number which is expressible in terms of some asymptotic

parameters of the trajectory.

The purpose of the current paper is to demonstrate that one can derive a formula analogous to (1) also in the framework of quantum mechanics. To this end and because of complexity of the problem, we restrict ourselves to the case when the AB flux depends on time as a sinusoidal function. To this system we apply the quantum averaging method getting this way an approximate time evolution for which we observe a resonance effect whose principal characterization is again a linear increase of energy.

Let us now be more specific. We consider a quantum point particle of mass M and charge e moving on the plane in the presence of a homogeneous magnetic field of magnitude B. For definiteness, all constants M, e, B are supposed to be positive. Assume further that the particle is driven by an Aharonov-Bohm magnetic flux concentrated along a line intersecting the plane in the origin and whose strength $\Phi(t)$ is oscillating with frequency Ω.

In the time-independent case, the Hamiltonian corresponding to a homogeneous magnetic field and a constant Aharonov-Bohm flux of magnitude Φ_0 reads

$$\frac{\hbar^2}{2M}\left(-\frac{1}{r}\,\partial_r r\partial_r+\frac{1}{r^2}\left(-i\partial_\theta-\frac{e\Phi_0}{2\pi\hbar}+\frac{eBr^2}{2\hbar}\right)^2\right)$$

where (r,θ) are polar coordinates on the plane, and the Hilbert space in question is $L^2(\mathbb{R}_+\times S^1, r\mathrm{d}r\mathrm{d}\theta)$. Making use of the rotational symmetry of the model we restrict ourselves to a fixed eigenspace of the angular momentum $J_3=-i\hbar\partial_\theta$ with an eigenvalue $j_3\hbar$, $j_3\in\mathbb{Z}$. Put

$$p:=j_3-e\Phi_0/(2\pi\hbar).$$

Then this restriction leads to the radial Hamiltonian

$$H(p)=\frac{\hbar^2}{2M}\left(-\frac{1}{r}\,\partial_r r\partial_r+\frac{1}{r^2}\left(p+\frac{eBr^2}{2\hbar}\right)^2\right) \tag{2}$$

in $\mathscr{H}=L^2(\mathbb{R}_+, r\mathrm{d}r)$. Without loss of generality, we can assume that $p>0$ (note that $H(-p)-H(p)$ is a constant). The boundary conditions at the origin are chosen to be the regular ones (then $H(p)$ is the so called Friedrichs self-adjoint extension of the symmetric operator defined on compactly supported smooth functions). Let us note that if $0<p<1$, then more general boundary conditions are admissible [7] but here we confine ourselves to the above standard choice.

Let

$$\omega_c=eB/M$$

be the cyclotron frequency. The operator $H(p)$ has a simple discrete spectrum, the eigenvalues are

$$E_n(p)=\hbar\omega_c(n+p+1/2),\quad n=0,1,2,\ldots, \tag{3}$$

with the corresponding normalized eigenfunctions

$$\varphi_n(p;r) = c_n(p)\, r^p\, L_n^{(p)}\left(\frac{eBr^2}{2\hbar}\right) \exp\left(-\frac{eBr^2}{4\hbar}\right) \tag{4}$$

where

$$c_n(p) = \left(\frac{eB}{2\hbar}\right)^{(p+1)/2} \left(\frac{2\,n!}{\Gamma(n+p+1)}\right)^{1/2}$$

are the normalization constants and $L_n^{(p)}$ are the generalized Laguerre polynomials.

Thus our main goal is to study the time evolution governed by the periodically time-dependent Hamiltonian $H(a(t))$ where

$$a(t) = p + \epsilon f(\Omega t)$$

and $f(t)$ is a 2π-periodic continuously differentiable function, $\Omega > 0$ is a frequency and ϵ is a small parameter. This means that the Aharonov-Bohm flux is supposed to depend on time as

$$\Phi(t) = \Phi_0 - (2\pi\hbar\epsilon/e)\, f(\Omega t). \tag{5}$$

Without loss of generality one can assume that

$$\int_0^{2\pi} f(t)\mathrm{d}t = 0. \tag{6}$$

As discussed in [3], for the values $0 < p < 1$ the domain of $H(a(t))$ in fact depends on t, and this feature makes the discussion from the mathematical point of view a bit more complicated. Nevertheless, the time evolution is still guaranteed to exist.

2 The Floquet operator and the quasienergy

Let $U(t, t_0)$ be the propagator (evolution operator) associated with $H(a(t))$; it is known to exist [3]. An important characteristic of the dynamical properties of the system is the time evolution over a period which is described by the Floquet (monodromy) operator $U(T, 0)$, with $T = 2\pi/\Omega$. We are primarily interested in the asymptotic behavior of the mean value of energy

$$\langle U(T,0)^N\psi, H(p)U(T,0)^N\psi\rangle$$

for an initial condition ψ as N tends to infinity while focusing on the resonant case when

$$\Omega = \mu\omega_c \text{ for some } \mu \in \mathbb{N}. \tag{7}$$

A basic tool in the study of time-dependent quantum systems is the quasienergy operator

$$K = -i\hbar\partial_t + H(a(t))$$

acting in the so called extended Hilbert space which is, in our case,

$$\mathscr{K} = L^2((0,T) \times \mathbb{R}_+, r\mathrm{d}t\mathrm{d}r).$$

The time derivative is taken with the periodic boundary conditions. This approach, very similar to that usually applied in classical mechanics, makes it possible to pass from a time-dependent system to an autonomous one. The price to be paid for it is that one has to work with more complex operators on the extended Hilbert space.

An important property of the quasienergy consists in its close relationship to the Floquet operator [11, 27]. In more detail, if $\psi(t,r) \in \mathscr{K}$ is an eigenfunction or a generalized eigenfunction of K, $K\psi = \eta\psi$, which also implies that $\psi(t+T,r) = \psi(t,r)$, then the wavefunction $e^{-i\eta t/\hbar}\psi(t,r)$ solves the Schrödinger equation with the initial condition $\psi_0(r) = \psi(0,r)$. It follows that $U(T,0)\psi_0 = e^{-i\eta T/\hbar}\psi_0$. Thus from the spectral decomposition of the quasienergy one can deduce the spectral decomposition of the Floquet operator.

Let

$$K_0 = -i\hbar\partial_t + H(p)$$

be the unperturbed quasienergy operator. Its complete set of normalized eigenfunctions is

$$\{T^{-1/2}e^{im\Omega t}\varphi_n(p;r);\ m \in \mathbb{Z}, n \in \mathbb{Z}_+\}$$

(here $\mathbb{Z}_+ = \{0,1,2,\ldots\}$ stands for nonnegative integers, the wave functions $\varphi_n(p;r)$ are defined in (4)), with the corresponding eigenvalues $m\hbar\Omega + E_n(p)$. Thus K_0 has a pure point spectrum which is in the resonant case (7) infinitely degenerated.

To take into account these degeneracies we perform the following transformation of indices. Denote by $[x]$ and $\{x\}$ the integer and the fractional part of a real number x, respectively, i.e. $x = [x] + \{x\}$, $[x] \in \mathbb{Z}$ and $0 \le \{x\} < 1$. Furthermore, let

$$\rho(\mu,k) = \mu\,\{k/\mu\}$$

be the remainder in division of an integer k by μ. The transformation of indices is a one-to-one map of $\mathbb{Z} \times \mathbb{Z}_+$ onto itself sending (m,n) to (k,ℓ), with

$$k = k(m,n) := \mu m + n,\ \ \ell = \ell(m,n) := [n/\mu], \tag{8}$$

and, conversely,

$$m = m(k,\ell) := [k/\mu] - \ell,\ \ n = n(k,\ell) := \mu\ell + \rho(\mu,k). \tag{9}$$

Using the new indices (k, ℓ) we put

$$\Psi_{k,\ell}(p; t, r) = T^{-1/2}\, e^{im(k,\ell)\Omega t}\, \varphi_{n(k,\ell)}(p; r). \tag{10}$$

Then the vectors $\Psi_{k,\ell}$, $(k, \ell) \in \mathbb{Z} \times \mathbb{Z}_+$, form an orthonormal basis in the extended Hilbert space $\mathscr{K}$. For a fixed integer $k \in \mathbb{Z}$ let P_k be the orthogonal projection onto the subspace in $\mathscr{K}$ spanned by the vectors $\Psi_{k,\ell}$, $\ell \in \mathbb{Z}_+$. Then

$$K_0 = \sum_{k\in\mathbb{Z}} \lambda_k P_k \text{ where } \lambda_k = \hbar\omega_c(k + p + 1/2). \tag{11}$$

Furthermore, using the basis $\{\Psi_{k,\ell}\}$ one can identify $\mathscr{K}$ with the Hilbert space $\ell^2(\mathbb{Z} \times \mathbb{Z}_+)$. In particular, partial differential operators in the variables t and r like the quasienergy are identified in this way with matrix operators. In the sequel we denote matrix operators by bold uppercase letters.

3 The quantum averaging method

The full quasienergy operator $K = K(\epsilon)$ depends on the small parameter ϵ. Let us write $K(\epsilon)$ as a formal power series, $K(\epsilon) = K_0 + \epsilon K_1 + \epsilon^2 K_2 + \ldots$. In our case,

$$K_1 = f(\Omega t)\hbar\omega_c\left(\frac{\hbar p}{M\omega_c r^2} + \frac{1}{2}\right),\ K_2 = \frac{f(\Omega t)^2\hbar^2}{2Mr^2}\,, \tag{12}$$

and $K_3 = K_4 = \ldots = 0$. The ultimate goal of the quantum averaging method in the case of resonances is a unitary transformation resulting in a partial (block-wise) diagonalization of $K(\epsilon)$. Thus one seeks a skew-Hermitian operator $W(\epsilon)$ so that $e^{W(\epsilon)}K(\epsilon)e^{-W(\epsilon)}$ commutes with K_0 which is the same as saying that it commutes with all projections P_k. This goal is achievable in principle through an infinite recurrence which in practice should be interrupted at some step. Here we shall be content with the first order approximation.

Let us introduce the block-wise diagonal part of an operator A in $\mathscr{K}$ as

$$\operatorname{diag} A := \sum_{k\in\mathbb{Z}} P_k A P_k.$$

Thus $\operatorname{diag} A$ surely commutes with K_0. The off-diagonal part is then defined as $\operatorname{offdiag} A := A - \operatorname{diag} A$. Developing formally in ϵ one has $W(\epsilon) = \epsilon W_1 + O(\epsilon^2)$ and

$$e^{W(\epsilon)}K(\epsilon)e^{-W(\epsilon)} = K_0 + \epsilon K_1 + \epsilon\,[W_1, K_0] + O(\epsilon^2).$$

Choosing W_1 as

$$W_1 = \sum_{k_1,k_2,k_1\neq k_2} (\lambda_{k_1} - \lambda_{k_2})^{-1} P_{k_1} K_1 P_{k_2}$$

one has

$$[W_1, K_0] = -\operatorname{offdiag} K_1 \tag{13}$$

and

$$e^{W(\epsilon)} K(\epsilon) e^{-W(\epsilon)} = K_0 + \epsilon \operatorname{diag} K_1 + O(\epsilon^2).$$

Let us note that the solution is also expressible in terms of averaging integrals, and this explains the name of the method [23, 15]. In more detail, one has

$$\operatorname{diag} A = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau e^{-iuK_0/\hbar} A\, e^{iuK_0/\hbar}\, \mathrm{d}u \tag{14}$$

and

$$W_1 = \lim_{\tau\to\infty} \frac{i}{\hbar\tau} \int_0^\tau (\tau - u) e^{-iuK_0/\hbar} \operatorname{offdiag}(K_1) e^{iuK_0/\hbar}\, \mathrm{d}u. \tag{15}$$

After switching on the perturbation, any unperturbed eigenvalue λ_k gives rise to a perturbed spectrum which, in the first order approximation, equals the spectrum of the operator $\lambda_k P_k + \epsilon P_k K_1 P_k$ restricted to the subspace $\operatorname{Ran} P_k \subset \mathscr{K}$. If the degeneracy of λ_k is infinite then the character of the perturbed spectrum may be arbitrary, depending on the properties of $P_k K_1 P_k$. The corresponding perturbed (generalized) eigenvectors span a subspace which is the range of the orthogonal projection

$$\begin{aligned} P_k(\epsilon) &:= e^{-W(\epsilon)} P_k e^{W(\epsilon)} = P_k - \epsilon\, [W_1, P_k] + O(\epsilon^2) \\ &= P_k - \epsilon\, (\hat{S}_k K_1 P_k + P_k K_1 \hat{S}_k) + O(\epsilon^2) \end{aligned}$$

where

$$\hat{S}_k = \sum_{\ell, \ell\neq k} (\lambda_\ell - \lambda_k)^{-1} P_\ell$$

is the reduced resolvent of K_0 taken at the isolated eigenvalue λ_k. Thus the first order averaging method is in fact nothing but the standard quantum perturbation method in the first order but accomplished on the extended Hilbert space simultaneously for all eigenvalues of K_0 (compare to [17, Chp. II§2]).

Our strategy in the remainder of the paper is based on replacing the true quasienergy $K(\epsilon)$ by its first order approximation

$$K_{(1)} := K_0 + \epsilon \operatorname{diag} K_1 \tag{16}$$

and, consequently, $U(T, 0)$ is replaced by an approximate Floquet operator $U_{(1)}$ associated with $K_{(1)}$. To determine the approximate Floquet operator $U_{(1)}$ one has to solve the spectral problem for $K_{(1)}$. To this end, as already pointed out above, one can employ the orthonormal basis $\{\Psi_{k\ell}\}$ in order to identify operators in $\mathscr{K}$ with infinite matrices indexed by $\mathbb{Z} \times \mathbb{Z}_+$.

Let $\{e^1_k; k \in \mathbb{Z}\}$ denote the standard basis in $\ell^2(\mathbb{Z})$, and $\{e^2_\ell; \ell \in \mathbb{Z}_+\}$ denote the standard basis in $\ell^2(\mathbb{Z}_+)$. It is convenient to write $\ell^2(\mathbb{Z} \times \mathbb{Z}_+)$ as the tensor product of Hilbert spaces $\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}_+)$ which also means

identification of the standard basis in $\ell^2(\mathbb{Z}\times\mathbb{Z}_+)$ with the set of vectors $\{e^1_k\otimes e^2_\ell;\, k\in\mathbb{Z}, \ell\in\mathbb{Z}_+\}$.

Let $\boldsymbol{P}_k$ be the orthogonal projection onto the one-dimensional subspace $\mathbb{C}e^1_k\subset\ell^2(\mathbb{Z})$. Recalling (16), (11) and (12), the matrix $\boldsymbol{K}_{(1)}$ of the operator $K_{(1)}$ expressed in the basis (10) takes the form

$$\boldsymbol{K}_{(1)}=\sum_{k\in\mathbb{Z}}\boldsymbol{P}_k\otimes(\lambda_k+\epsilon\boldsymbol{A}_k) \tag{17}$$

where $\boldsymbol{A}_k$ is the matrix operator in $\ell^2(\mathbb{Z}_+)$ with the entries

$$(\boldsymbol{A}_k)_{\ell_1,\ell_2}=\langle\,\Psi_{k,\ell_1},K_1\Psi_{k,\ell_2}\,\rangle_{\mathscr{K}}\,. \tag{18}$$

To compute the matrix entries of $\boldsymbol{A}_k$ one observes that formally (see (2))

$$K_1=f(\Omega t)\,\partial H(p)/\partial p \tag{19}$$

and so

$$\langle\,\Psi_{k,\ell_1},K_1\Psi_{k,\ell_2}\,\rangle_{\mathscr{K}}=\mathscr{F}[f](\ell_2-\ell_1)\,\left\langle\varphi_{n(k,\ell_1)}(p),(\partial H(p)/\partial p)\,\varphi_{n(k,\ell_2)}(p)\right\rangle$$

where

$$\mathscr{F}[f](j)=(2\pi)^{-1}\int_0^{2\pi}e^{-ij\cdot s}f(s)\,\mathrm{d}s$$

stands for the jth Fourier coefficient of f. Recall that, by the assumption (6), $\mathscr{F}[f](0)=0$. Moreover, for $\ell_1\neq\ell_2$ one has $n(k,\ell_1)\neq n(k,\ell_2)$, hence

$$\left\langle\varphi_{n(k,\ell_1)}(p),H(p)\varphi_{n(k,\ell_2)}(p)\right\rangle=0. \tag{20}$$

In [3] it is derived that, for $n_1\neq n_2$,

$$\left\langle\varphi_{n_1}(p),\frac{\partial\varphi_{n_2}(p)}{\partial p}\right\rangle=\frac{1}{2(n_2-n_1)}\,\min\left\{\frac{\gamma(p;n_2)}{\gamma(p;n_1)},\frac{\gamma(p;n_1)}{\gamma(p;n_2)}\right\} \tag{21}$$

where

$$\gamma(p;n)=\big(\Gamma(n+p+1)/n!\big)^{1/2}.$$

Differentiating (20) with respect to p and using (21) one finally obtains the relation

$$(\boldsymbol{A}_k)_{\ell_1,\ell_2}=\frac{\hbar\omega_c}{2}\,\mathscr{F}[f](\ell_2-\ell_1)\,\min\left\{\frac{\gamma(p;n(k,\ell_2))}{\gamma(p;n(k,\ell_1))},\frac{\gamma(p;n(k,\ell_1))}{\gamma(p;n(k,\ell_2))}\right\}. \tag{22}$$

Note that $n(k,\ell)$, as defined in (9), is μ–periodic in the integer variable k, and so is the matrix $\boldsymbol{A}_k$, i.e. $\boldsymbol{A}_{k+\mu}=\boldsymbol{A}_k$. Moreover, since $\mu\omega_c=2\pi/T$ one also has $e^{-i\lambda_{k+\mu}T/\hbar}=e^{-i\lambda_kT/\hbar}$ (see (11)). For an integer s, $0\le s<\mu$, let $\mathscr{H}_s$ be the closed subspace in the original Hilbert space $\mathscr{H}=L^2(\mathbb{R}_+,r\mathrm{d}r)$

spanned by the vectors $\varphi_{s+j\mu}(r)$, $j = 0, 1, 2, \ldots$. Then $\mathscr{H}$ decomposes into the orthogonal sum

$$\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \ldots \oplus \mathscr{H}_{\mu-1},$$

and from the relationship between $K_{(1)}$ and $U_{(1)}$, as recalled in Section 2, it follows that every subspace $\mathscr{H}_s$ is invariant with respect to $U_{(1)}$.

In the example which we study in more detail in the following section (for a sinusoidal function $f(t)$), the matrix operators $\boldsymbol{A}_s$ have purely absolutely continuous spectra. For the sake of simplicity of the notation let us confine ourselves to this case. For a fixed index s, $0 \le s < \mu$, suppose that all generalized eigenvectors and eigenvalues of $\boldsymbol{A}_s$ are parametrized by a parameter $\theta \in (a_s, b_s)$. Let us call them $\boldsymbol{x}_s(\theta)$ and $\eta_s(\theta)$, respectively, i.e.

$$\boldsymbol{A}_s \boldsymbol{x}_s(\theta) = \eta_s(\theta) \boldsymbol{x}_s(\theta),$$

and write

$$\boldsymbol{x}_s(\theta) = (\xi_{s;0}(\theta), \xi_{s;1}(\theta), \xi_{s;2}(\theta), \ldots).$$

The generalized eigenvectors $\boldsymbol{x}_s(\theta)$ are supposed to be normalized to the δ function, i.e.

$$\langle \boldsymbol{x}_s(\theta_1), \boldsymbol{x}_s(\theta_2) \rangle = \delta(\theta_1 - \theta_2),$$

which in fact means that $\xi_{s;\ell}(\theta)$ as a function in the variables $\ell \in \mathbb{Z}_+$ and $\theta \in (a_s, b_s)$ is a kernel of a unitary mapping between the Hilbert spaces $\ell^2(\mathbb{Z}_+)$ and $L^2((a_s, b_s), \mathrm{d}\theta)$. Thus the spectral decomposition of $\boldsymbol{A}_s$ reads:

$$\forall \boldsymbol{v} \in \ell^2(\mathbb{Z}_+), \ \boldsymbol{A}_s \boldsymbol{v} = \int_{a_s}^{b_s} \eta_s(\theta) \langle \boldsymbol{x}_s(\theta), \boldsymbol{v} \rangle \, \boldsymbol{x}_s(\theta) \, \mathrm{d}\theta.$$

Put

$$\Xi_s(\theta, r) = \sum_{j=0}^{\infty} \xi_{s;j}(\theta) \, \varphi_{s+j\mu}(p; r). \tag{23}$$

Then again,

$$\int_0^\infty \overline{\Xi_s(\theta_1, r)} \, \Xi_s(\theta_2, r) \, r \mathrm{d}r = \delta(\theta_1 - \theta_2)$$

and, for all $\psi(r) \in \mathscr{H}_s$,

$$U_{(1)} \psi(r) = e^{-2\pi i (s+p+1/2)/\mu} \int_{a_s}^{b_s} e^{-i\epsilon \, \eta_s(\theta) T/\hbar} \, \langle \, \Xi_s(\theta), \psi \rangle \, \Xi_s(\theta, r) \, \mathrm{d}\theta. \tag{24}$$

To get a correct approximation in the first order of the propagator one further has to take into account the transformation which is inverse to that generated by $W(\epsilon) \approx \epsilon \, W_1$. First observe that W_1 is a multiplication operator on the Hilbert space $\mathscr{K}$ in the following sense. Let S be the unitary operator on $\mathscr{K}$ acting as

$$S\psi(t, r) = e^{i\Omega t} \psi(t, r), \ \forall \psi \in \mathscr{K}.$$

An operator L on $\mathscr{K}$ commutes with S if and only if there exists a one-parameter T-periodic family of operators $\mathcal{L}(t)$ on $L^2(\mathbb{R}_+, r\mathrm{d}r)$ such that $L\psi(t,r) = \mathcal{L}(t)\psi(t,r)$. Notice that

$$S^{-1}K_0S = K_0 + \hbar\Omega.$$

With this equality, it is obvious from (14) that if A commutes with S then the same is true for diag A. Furthermore, as one can see from (12), K_1 commutes with S, and from (15) one infers that W_1 commutes with S as well. Hence there exists a one-parameter T-periodic family of skew-Hermitian operators $\mathcal{W}_1(t)$ on $L^2(\mathbb{R}_+, r\mathrm{d}r)$ such that

$$W_1\psi(t,r) = \mathcal{W}_1(t)\psi(t,r),\ \forall\psi \in \mathscr{K}.$$

Next notice that a transformation of the quasienergy operator of the form $\tilde{K} = e^{\mathcal{W}(t)}Ke^{-\mathcal{W}(t)}$, where again $\mathcal{W}(t)$ is a T-periodic family of skew-Hermitian operators on $L^2(\mathbb{R}_+, r\mathrm{d}r)$, implies a transformation of the associated propagators according to the rule

$$\tilde{U}(t_1,t_2) = e^{\mathcal{W}(t_1)}U(t_1,t_2)e^{-\mathcal{W}(t_2)}.$$

Hence the correct approximation of the Floquet operator reads

$$U(T,0) \approx U_{\text{approx}} = e^{-\epsilon\mathcal{W}_1(0)}U_{(1)}e^{\epsilon\mathcal{W}_1(0)}. \tag{25}$$

Let us note, however, that one has, for $N \in \mathbb{N}$ and $\psi \in \mathscr{K}$,

$$\langle U_{\text{approx}}^N\psi, H(p)U_{\text{approx}}^N\psi\rangle = \langle U_{(1)}^N\psi_1, (H(p) + \epsilon\,[\mathcal{W}_1(0), H(p)])\,U_{(1)}^N\psi_1\rangle + O(\epsilon^2) \tag{26}$$

where $\psi_1 = e^{\epsilon\mathcal{W}_1(0)}\psi$. If the commutator $[\mathcal{W}_1(0), H(p)]$ happens to be bounded then it does not contribute to the acceleration rate.

Finally let us indicate how to compute the operator-valued function $\mathcal{W}_1(t)$. One has (here $\varphi_n = \varphi_n(p;r)$)

$$\mathcal{W}_1(t) = \sum_{j=-\infty}^{\infty}\sum_{n_1,n_2=0}^{\infty} e^{i\Omega jt}\,w(j,n_1,n_2)\,\langle\varphi_{n_2},\cdot\,\rangle\,\varphi_{n_1} \tag{27}$$

where

$$w(j,n_1,n_2) = \frac{1}{T}\int_0^T e^{-i\Omega jt}\,\langle\varphi_{n_1}, \mathcal{W}_1(t)\varphi_{n_2}\rangle\,\mathrm{d}t.$$

The commutator equation (13) is equivalent to the differential equation

$$-\,i\hbar\,\mathcal{W}_1{}'(t) + [H(p), \mathcal{W}_1(t)] = \text{offdiag}\,K_1. \tag{28}$$

Substituting (27) into (28) and using (19) jointly with (14) one finds that

$$w(j,n_1,n_2) = \frac{\mathscr{F}[f](j)}{\hbar\omega_c(\mu j + n_1 - n_2)}\left\langle\varphi_{n_1}, \frac{\partial H(p)}{\partial p}\,\varphi_{n_2}\right\rangle \text{ if } \mu j + n_1 - n_2 \neq 0 \tag{29}$$

and $w(j,n_1,n_2) = 0$ otherwise.

4 A sinusoidally time-dependent AB flux

In the remainder of the paper we discuss the example when $f(t) = \sin(t)$. The goal of the current section is to provide more details on the spectral decomposition of the averaged quasienergy $K_{(1)}$ derived in (16). Naturally, rather than directly with the quasienergy we shall deal with its matrix, as given in (17) and (18).

We still assume that $s \in \{0, 1, \ldots, \mu - 1\}$ is fixed. For this choice of $f(t)$, an immediate evaluation of formula (22) gives

$$(\boldsymbol{A}_s)_{j_1,j_2} = \frac{\hbar\omega_c}{4i}\,\delta_{|j_2-j_1|,1}\,\mathrm{sign}(j_2 - j_1)\left(\prod_{\nu=1}^{\mu}\frac{\mu j_< + s + \nu}{\mu j_< + s + p + \nu}\right)^{1/2}.$$

where $j_< = \min\{j_1, j_2\}$. Thus one has

$$\boldsymbol{A}_s = (\hbar\omega_c/4)\,\boldsymbol{D}\boldsymbol{J}\boldsymbol{D}^{-1}$$

where $\boldsymbol{J}$ is the Jacobi (tridiagonal) matrix with zero diagonal,

$$\boldsymbol{J} = \begin{pmatrix} 0 & \alpha_0 & 0 & 0 & \ldots \\ \alpha_0 & 0 & \alpha_1 & 0 & \ldots \\ 0 & \alpha_1 & 0 & \alpha_2 & \ldots \\ 0 & 0 & \alpha_2 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{30}$$

and with the positive entries

$$\alpha_j = \left(\prod_{\nu=1}^{\mu}\frac{\mu j + s + \nu}{\mu j + s + p + \nu}\right)^{1/2},$$

and $\boldsymbol{D}$ is the unitary diagonal matrix with the diagonal $(1, i, i^2, i^3, \ldots)$.

This is an elementary fact that the spectrum of $\boldsymbol{J}$ is simple since any eigenvector or generalized eigenvector is unambiguously determined by its first entry. Moreover, one readily observes that the matrices $\boldsymbol{J}$ and $-\boldsymbol{J}$ are unitarily equivalent, and so the spectrum of $\boldsymbol{J}$ is symmetric with respect to the origin.

In our case,

$$\alpha_j = 1 - p/(2j) + O(j^{-2}) \quad \text{as } j \to \infty.$$

Hence $\boldsymbol{J}$ is rather close to the "free" Jacobi matrix $\boldsymbol{J}_0$ for which $\alpha_{0,j} = 1$ for all j. The spectral problem for $\boldsymbol{J}_0$ is readily solvable explicitly (see below). It turns out that the spectral properties of $\boldsymbol{J}$ are close to those of $\boldsymbol{J}_0$ as well [13], see also [25]. In particular, it is known that the singular continuous spectrum of $\boldsymbol{J}$ is empty, the essential spectrum coincides with the absolutely

continuous spectrum and equals the interval $[-2, 2]$. Furthermore, there are no embedded eigenvalues, i.e. if η is an eigenvalue of $\boldsymbol{J}$ then $|\eta| \geq 2$.

Splitting $\boldsymbol{J}$ into the sum of the upper triangular and the lower triangular part, one notes that $\|\boldsymbol{J}\| \leq 2 \sup\{\alpha_0, \alpha_1, \alpha_2, \ldots\}$. In our example, $\alpha_j \leq 1$ for all j and so $\|\boldsymbol{J}\| \leq 2$ and, consequently, the spectrum of $\boldsymbol{J}$ is contained in the interval $[-2, 2]$. This means that the only possible eigenvalues of $\boldsymbol{J}$ are ± 2. But one can exclude even this possibility. In fact, suppose that $\boldsymbol{J}\boldsymbol{u} = 2\boldsymbol{u}$, with $\boldsymbol{u} = (u_0, u_1, u_2, \ldots)$ and $u_0 = 1$. Then

$$\alpha_{j-1} u_{j-1} + \alpha_j u_{j+1} = 2u_j \text{ for } j = 0, 1, 2, \ldots$$

(while putting $u_{-1} = 0$). Summing this equality for $j = 0, 1, \ldots, n$, and using that $\alpha_j \leq 1$, one finds that $u_{n+1} \geq u_n + 1$ for $n = 0, 1, 2, \ldots$. Hence $u_j \geq j+1$ for all j, and so $\boldsymbol{u}$ is not square summable. Thus one can summarize that the spectrum of $\boldsymbol{J}$ is simple, purely absolutely continuous and equals $[-2, 2]$.

Let us parametrize the spectrum of $\boldsymbol{A}_s = \boldsymbol{A}_s(p)$ by a continuous parameter θ, $0 < \theta < \pi$, so that

$$\eta(\theta) := (\hbar\omega_c/2)\cos(\theta)$$

is a point from the spectrum and $\boldsymbol{x}(p;\theta)$ is the corresponding normalized generalized eigenvector with components $\xi_j(p;\theta)$, $j = 0, 1, 2, \ldots$ (here we drop the index s at $\boldsymbol{x}$ and ξ in order to simplify the notation). The asymptotic behavior of the components ξ_j is known [14, 6]; one has

$$\xi_j(p;\theta) \sim A(p;\theta)\, i^j \cos\bigl(j\theta - (p/2)\cot(\theta)\log(j+1) + \phi(p;\theta)\bigr) \tag{31}$$

for $j \gg 0$. Here $A(p;\theta)$ is a normalization constant and $\phi(p;\theta)$ is a phase which depends on the initial conditions imposed on the sequence $\{\xi_j\}$ (the initial condition is simply $\xi_{-1} = 0$) but the asymptotic methods employed in the cited articles do not provide an explicit value for it. In the limit case $p = 0$ the generalized eigenvectors are known explicitly, namely

$$\xi_j(0;\theta) = \sqrt{2/\pi}\, i^j \sin((j+1)\theta)$$

for all j. Hence $\phi(0;\theta) = \theta - \pi/2$.

The generalized eigenvectors are supposed to be normalized so that

$$\langle \boldsymbol{x}(p;\theta_1), \boldsymbol{x}(p;\theta_2)\rangle = \delta(\theta_1 - \theta_2).$$

For $p = 0$, one can use the equality

$$\sum_{n=1}^{\infty} e^{inx} = \pi\delta(x) - \mathcal{P}\frac{1}{1 - e^{-ix}}$$

which is valid for $x = \theta_1 - \theta_2 \in (-\pi, \pi)$ and where the symbol $\mathcal{P}$ indicates the regularization of a nonintegrable singularity in the sense of the principal value. The normalization is an immediate consequence of this identity.

For general p, the contribution to the δ function should come from the most singular and, at the same time, the leading term in the asymptotic expansion of $\xi_j(p;\theta)$, as given in (31). This time, when investigating the singularity near the diagonal $\theta_1 = \theta_2$ in the scalar product of two generalized eigenvectors, one is lead to considering the sum

$$\sum_{n=1}^{\infty} n^{iax} e^{inx}$$

where $a = p/(2\sin^2\theta_1)$ is a real constant. Using the Lerch function $\Phi(z,s,v)$ one has for $|z| < 1$ (see [9, § 9.55]),

$$\sum_{n=1}^{\infty} n^s z^n = z\,\Phi(z,s,1) = \Gamma(1-s) \sum_{n=-\infty}^{\infty} (-\log(z) + 2\pi n i)^{-1+s}.$$

From here one deduces that, for any real a,

$$\sum_{n=1}^{\infty} n^{iax} e^{inx} = \pi\delta(x) + i\mathcal{P}\frac{1}{x} + g_a(x) \tag{32}$$

where $g_a(x)$ is a regular distribution, i.e. a locally integrable function. Hence in the general case, too, the normalization constant is given by

$$A(p;\theta) = \sqrt{2/\pi}\,.$$

As already mentioned, the phase $\phi(p;\theta)$ in the asymptotic solution (31) remains undetermined. But we remark that a bit more can be said about the behavior of the phase near the spectral point 0 (the center of the spectrum) which corresponds to the value of the parameter $\theta = \pi/2$. More precisely, one can compute the derivative $\partial\phi(p;\pi/2)/\partial\theta$. Though this result is not directly used in the sequel it represents an additional information about generalized eigenfunctions of $\boldsymbol{J}$. We briefly indicate basic steps of the computation in Appendix.

5 The acceleration rate

In the case when $f(t) = \sin(t)$ the commutator $[\mathcal{W}_1(0), H(p)]$ occurring in (26) can be shown to be bounded. This implies that instead of the approximate Floquet operator U_{approx}, as given in (25), one can work directly with $U_{(1)}$ defined in (24) when deriving a formula for the acceleration rate. On the other hand, one should not forget about the transformation of the initial state, i.e. ψ_0 has to be replaced by $e^{\epsilon\mathcal{W}_1(0)}\psi_0$, see (26).

First let us shortly discuss the boundedness of the commutator. From (27) and (29) while using also (21) one derives that for $n_1, n_2 \in \mathbb{Z}_+$, $n_1 - n_2 \neq \pm\mu$,

$$\begin{aligned}&\langle \varphi_{n_1}(p), [H(p), \mathcal{W}_1(0)]\varphi_{n_2}(p)\rangle \\ &\quad = \frac{\hbar\omega_c}{4i}\,\frac{n_1 - n_2}{(n_1-n_2)^2 - \mu^2}\,\min\left\{\frac{\gamma(p;n_2)}{\gamma(p;n_1)}, \frac{\gamma(p;n_1)}{\gamma(p;n_2)}\right\}.\end{aligned} \tag{33}$$

Of course, the parallels to the diagonal determined by $n_1 - n_2 = \pm\mu$ can be explicitly evaluated as well but for our purposes it is sufficient to know that they are bounded. In [3, Lemma 6] it is shown that the matrix operator $\boldsymbol{Q}$ in $\ell^2(\mathbb{Z}_+)$ with the entries

$$\boldsymbol{Q}_{n_1,n_2} = \frac{\hbar\omega_c}{4i\,(n_1-n_2)}\,\min\left\{\frac{\gamma(p;n_2)}{\gamma(p;n_1)}\,,\frac{\gamma(p;n_1)}{\gamma(p;n_2)}\right\} \tag{34}$$

for $n_1 \neq n_2$ and 0 otherwise is bounded. Thus to verify the boundedness of the commutator it suffices to show that the difference of matrices (33) and (34) has a finite operator norm. This can be readily done, for example, with the aid of the following estimate for the norm of a Hermitian matrix operator $\boldsymbol{B}$ [17, § I.4.3],

$$\|\boldsymbol{B}\| \leq \sup_{n_1\in\mathbb{Z}_+}\sum_{n_2=0}^{\infty}|\boldsymbol{B}_{n_1,n_2}|.$$

To proceed further, we again fix an integer s, $0 \leq s < \mu$. Suppose one is given a function $\varrho(\theta) \in C_0^\infty((0,\pi))$. Recalling (23) we put

$$\psi(r) = \int_0^\pi \Xi_s(\theta,r)\varrho(\theta)\,\mathrm{d}\theta. \tag{35}$$

In what follows, we drop the index s and, whenever convenient, write simply H instead of $H(p)$. Using (24), one has, for $N \in \mathbb{N}$,

$$\begin{aligned}\langle U_{(1)}^N\psi, HU_{(1)}^N\psi\rangle &= \int_0^\pi\int_0^\pi e^{i\epsilon(\cos\theta_1-\cos\theta_2)\omega_c TN/2}\\ &\qquad\times\langle\,\Xi(\theta_1), H\Xi(\theta_2)\rangle\,\overline{\varrho(\theta_1)}\varrho(\theta_2)\,\mathrm{d}\theta_1\mathrm{d}\theta_2\\ &= \sum_{j=0}^\infty E_{s+j\mu}(p)\left|\int_0^\pi e^{-i\epsilon\cos(\theta)\omega_c TN/2}\xi_j(p;\theta)\varrho(\theta)\,\mathrm{d}\theta\right|^2.\end{aligned}$$

Note that $\{\xi_j(p;\theta)\}_{j=0}^\infty$ is an orthonormal basis in $L^2((0,\pi),\mathrm{d}\theta)$ and so

$$\sum_{j=0}^\infty\left|\int_0^\pi e^{-i\epsilon\cos(\theta)\omega_c TN/2}\xi_j(p;\theta)\varrho(\theta)\,\mathrm{d}\theta\right|^2 = \int_0^\pi|\varrho(\theta)|^2\,\mathrm{d}\theta.$$

Hence, in view of (3), the leading contribution to the acceleration rate comes from the expression

$$\mu\hbar\omega_c\sum_{j=0}^\infty(j+1)\left|\int_0^\pi e^{-i\epsilon\cos(\theta)\omega_c TN/2}\xi_j(p;\theta)\varrho(\theta)\,\mathrm{d}\theta\right|^2.$$

Furthermore, restricting this sum to an arbitrarily large but finite number of summands results in an expression which is uniformly bounded in N. This justifies replacement of $\xi_j(p;\theta)$ by the leading asymptotic term, as given in

(31) (with $A(p;\theta) = \sqrt{2/\pi}$). Hence the leading contribution to the acceleration rate is expressible as

$$\frac{2\hbar\Omega}{\pi} \int_0^\pi \int_0^\pi h(\theta_1,\theta_2) e^{i\epsilon(\cos\theta_1 - \cos\theta_2)\omega_c TN/2}\, \overline{\varrho(\theta_1)}\varrho(\theta_2)\, \mathrm{d}\theta_1 \mathrm{d}\theta_2$$

where

$$h(\theta_1,\theta_2) = \sum_{j=0}^{\infty} (j+1) \cos\Big(j\theta_1 - \frac{p}{2}\cot(\theta_1)\log(j+1) + \phi(p;\theta_1)\Big) \\ \times \cos\Big(j\theta_2 - \frac{p}{2}\cot(\theta_2)\log(j+1) + \phi(p;\theta_2)\Big). \qquad (36)$$

The singular part of the distribution $h(\theta_1,\theta_2)$ is supported on the diagonal $\theta_1 = \theta_2$. The sum in (36) can be evaluated analogously as that in (32) with the result

$$h(\theta_1,\theta_2) = -\frac{1}{2}\frac{\partial}{\partial\theta_2}\mathcal{P}\frac{1}{\theta_1-\theta_2} - \frac{\pi}{2}\Big(\frac{\partial\phi(p;\theta_1)}{\partial\theta} - 1\Big)\delta(\theta_1-\theta_2) \\ + \text{ a regular distribution.}$$

Estimating the acceleration rate we can restrict ourselves to a sufficiently small but fixed neighborhood of the diagonal with a radius $d > 0$. Thus we arrive at the expression

$$\frac{\hbar\Omega}{\pi}\mathcal{P}\int_{\substack{0\\|\theta_1-\theta_2|<d}}^{\pi}\int_0^\pi \frac{1}{\theta_1-\theta_2}\frac{\partial}{\partial\theta_2}\Big(e^{-i\epsilon\sin(\theta_1)(\theta_1-\theta_2)\omega_c TN/2}\,\overline{\varrho(\theta_1)}\varrho(\theta_2)\Big)\mathrm{d}\theta_1\mathrm{d}\theta_2.$$

Further we carry out the differentiation, as indicated in the integrand, and get rid of the terms which are not proportional to N or which are non-singular. Moreover, we use the substitution $\theta_2 = \theta_1 + u$. Thus we obtain the expression

$$-\frac{i\epsilon\hbar\Omega\omega_c TN}{2\pi}\int_0^\pi \sin(\theta_1)|\varrho(\theta_1)|^2\Big(\mathcal{P}\int_{-d}^{d}\frac{1}{u}e^{i\epsilon\sin(\theta_1)\omega_c TNu/2}\,\mathrm{d}u\Big)\mathrm{d}\theta_1 \\ = \frac{\epsilon\hbar\Omega\omega_c TN}{\pi}\int_0^\pi \sin(\theta)|\varrho(\theta)|^2\Big(\int_0^d \frac{1}{u}\sin\Big(\frac{\epsilon}{2}\sin(\theta)\omega_c TNu\Big)\mathrm{d}u\Big)\mathrm{d}\theta.$$

Finally note that, for any a real,

$$\lim_{N\to\infty}\int_0^d \frac{1}{u}\sin(aNu)\,\mathrm{d}u = \frac{\pi}{2}\,\mathrm{sign}\,a.$$

Suppose that the initial state is chosen as $e^{-\epsilon\mathcal{W}_1(0)}\psi$. Then we conclude that the formula for the acceleration rate in the first-order approximation reads

$$\gamma_{\mathrm{acc}} := \lim_{N\to\infty}\big\langle U_{(1)}^N\psi, H(p)U_{(1)}^N\psi\big\rangle \Big/ (NT\|\psi\|^2) \\ = \frac{|\epsilon|\hbar\omega_c\Omega}{2}\int_0^\pi \sin(\theta)|\varrho(\theta)|^2\,\mathrm{d}\theta \Big/ \int_0^\pi |\varrho(\theta)|^2\,\mathrm{d}\theta. \qquad (37)$$

Here we have used that

$$\|\psi\|^2 = \int_0^\pi |\varrho(\theta)|^2 \, \mathrm{d}\theta.$$

Formula (37) can be compared to formula (1), as derived for a classical particle, in the case when $\Phi(t)$ is given by (5) and $f(t) = \sin(t)$. Then (1) gives the acceleration rate

$$\gamma_{\mathrm{acc}} = |\epsilon| \hbar \omega_c \Omega \sin(\xi)/2$$

where $\xi \in (0, \pi)$ depends on some data which can be learned from the asymptotic behavior of the classical trajectory. Let us finally note that, according to the analysis and discussion of the classical case presented in [4], the first-order averaging approximation may in fact yield the correct acceleration rate (valid for the original system), and this is so even if the parameter ϵ is not necessarily assumed to be very small.

6 A numerical test

We conclude our discussion by a presentation of a numerical result that agree quite nicely with the predicted acceleration rate (37). For the sake of simplicity we put $\Omega = \omega_c = 1$, and so $\mu = 1$ and $s = 0$. We still assume that $f(t) = \sin(t)$. Concerning the physical constants, we set $\hbar = 1$, $e = 1$ and $M = 1$. Furthermore, we choose $p = 2.5$, $\epsilon = 0.4$, and for the density $\varrho(\theta)$ determining an initial state according to (35) we take the Gaussian function

$$\varrho(\theta) = \left(\frac{20}{\pi}\right)^{1/4} \exp\big(- 10(2-\theta)^2 + 8i\theta \big)$$

restricted to the interval $\theta \in (0, \pi)$. Its values near the limit points of the interval are in fact numerically indistinguishable from 0. Particularly, $\varrho(\theta)$ is normalized to unity with a negligible error, i.e.

$$\int_0^\pi |\varrho(\theta)|^2 \, \mathrm{d}\theta = 1.$$

The numerical method we use is based on expanding a solution of the time-dependent Schrödinger equation with respect to the time-dependent basis $\{\varphi_n(a(t)); \, n \in \mathbb{Z}_+\}$, with $\varphi_n(p)$ being defined in (4). Below we call the solution of the Schrödinger equation $\psi(t)$. Recalling (23) we put

$$\psi_0(r) = \int_0^\pi \Xi_0(\theta, r) \varrho(\theta) \, \mathrm{d}\theta,$$

and we have $\|\psi_0\| = 1$. The task is to solve the Cauchy problem for the time-dependent Schrödinger equation

$$i\partial_t \psi(t) = H(a(t))\psi(t), \;\; \psi(0) = \tilde{\psi}_0 := e^{-\epsilon \mathcal{W}_1(0)} \psi_0.$$

Let us note that in the case of $f(t) = \sin(t)$ the matrix entries of $\mathcal{W}_1(0)$ are expressed as the finite sum

$$\langle \varphi_{n_1}(p), \mathcal{W}_1(0)\varphi_{n_2}(p)\rangle = w(1, n_1, n_2) + w(-1, n_1, n_2),$$

with $w(j, n_1, n_2)$ being given in (29) (with $\mu = 1$). To carry out the computations we truncate the Fourier expansion of $\psi(t)$,

$$\psi(t) = \sum_{n=0}^{\infty} x_n(t)\varphi_n(a(t)),\ x_n(t) = \langle \varphi_n(a(t)), \psi(t)\rangle,\ n = 0, 1, \ldots,$$

at some fixed order $n_{\max}$. In this way we obtain a system of ordinary differential equations for the Fourier coefficients

$$\begin{aligned} ix_n'(t) &= E_n(a(t))x_n(t) - ia'(t)\sum_{j=0}^{n_{\max}} \langle \varphi_n(a(t)), \varphi_j'(a(t))\rangle\, x_j(t), \\ x_n(0) &= \langle \varphi_n(a(0)), \tilde{\psi}_0\rangle,\ n = 0, 1, \ldots, n_{\max}. \end{aligned}$$

Explicit formulas for the scalar products are known from [3] (see (21)). In order to approximately solve this system we employ the explicit Runge-Kutta method of order 4 (RK4) with an adaptive step-size control, and we choose $n_{\max} = 120$.

From the computational point of view it is convenient to introduce the mean value of energy at time t as

$$\mathcal{E}(t) := \langle \psi(t), H(a(t))\psi(t)\rangle.$$

$\mathcal{E}(t)$ is then approximated by the sum

$$\mathcal{E}(t) \approx \sum_{n=0}^{n_{\max}} E_n(a(t))|x_n(t)|^2.$$

The acceleration rate is computed according to formula (37) in which one has to substitute ψ_0 for ψ. Let us point out that this formula depends only on the time evolution over the intervals which are integer multiples of the period T, and clearly, $H(a(NT)) = H(p)$ for $N = 0, 1, 2, \ldots$. The predicted acceleration rate for the above particular values of parameters is $\gamma_{\text{acc}} = 0.1796$. The numerically computed function $\mathcal{E}(t)/t$ is compared to this value in Fig. 1.

Appendix. The phase $\phi(p;\theta)$ near the spectral point 0

Here we compute the derivative $\partial\phi(p;\pi/2)/\partial\theta$ of the phase $\phi(p;\theta)$ introduced in (31). We know that 0 always belongs to the spectrum of the Jacobi matrix

$\boldsymbol{J}$ introduced in (30). Putting $\boldsymbol{u} = (u_0, u_1, u_2, \ldots)$, with $u_{2j+1} = 0$ and

$$u_{2j} = (-1)^j \prod_{k=0}^{j-1} \frac{\alpha_{2k}}{\alpha_{2k+1}} \tag{A.1}$$

for $j = 0, 1, 2, \ldots$, one has $\boldsymbol{J}\boldsymbol{u} = \boldsymbol{0}$ and $u_0 = 1$. Recalling that, in our example, $\alpha_j = 1 - p/(2j) + O(j^{-2})$ one derives that

$$u_{2j} = (-1)^j u_\infty \big(1 + p/(8j) + O(j^{-2})\big) \quad \text{as } j \to \infty,$$

where

$$u_\infty = \lim_{j\to\infty} (-1)^j u_{2j}$$

is a finite constant (depending on p, however). Comparing to (31), with $A(p;\theta) = \sqrt{2/\pi}$ and $\theta = \pi/2$, one finds that

$$\boldsymbol{x}(p;\pi/2) = (\sqrt{2/\pi}/u_\infty)\, \boldsymbol{u}.$$

Moreover, $\phi(p;\pi/2) = 0$.

Differentiating the equality

$$\boldsymbol{J}\boldsymbol{x}(p;\theta) = 2\cos(\theta)\boldsymbol{x}(p;\theta)$$

with respect to θ at the point $\pi/2$ and using the substitution

$$\partial\boldsymbol{x}(p;\pi/2)/\partial\theta = -\Big(2\sqrt{2/\pi}/u_\infty\Big)\boldsymbol{v},$$

with $\boldsymbol{v} = (v_0, v_1, v_2, \ldots)$, one arrives at the equation $\boldsymbol{J}\boldsymbol{v} = \boldsymbol{u}$. From (31) one deduces that

$$v_j \sim \frac{1}{2}\, u_\infty \sin\Big(j\,\frac{\pi}{2}\Big)\bigg(j + \frac{p}{2}\,\log(j+1) + \frac{\partial\phi(p;\pi/2)}{\partial\theta}\bigg) \tag{A.2}$$

for $j \gg 0$. This suggests that one can seek a solution $\boldsymbol{v}$ such that $v_{2j} = 0$ for all j. This assumption on $\boldsymbol{v}$ is in fact necessary and makes the solution unambiguous since otherwise one could add to $\boldsymbol{v}$ any nonzero multiple of $\boldsymbol{u}$ thus violating the asymptotic behavior (A.2). Given that all odd elements of the vector $\boldsymbol{u}$ and all even elements of $\boldsymbol{v}$ vanish the equation $\boldsymbol{J}\boldsymbol{v} = \boldsymbol{u}$ effectively reduces to a linear system with a lower triangular matrix which is explicitly solvable. Using (A.1) one can express the solution as

$$v_{2j+1} = \frac{1}{\alpha_{2j}u_{2j}} \sum_{k=0}^{j} (u_{2k})^2, \quad j = 0, 1, 2, \ldots. \tag{A.3}$$

Noting that

$$\sum_{k=0}^{j} \bigg(1 + \frac{p}{4k+2}\bigg) = j + 1 + \frac{p}{4}\big(\log(4j+4) + \gamma_E\big) + O(j^{-1}),$$

where γ_E is the Euler constant, and that

$$(u_{2k})^2 = u_\infty^2(1 + p/(4k) + O(k^{-2}))$$

one derives

$$\sum_{k=0}^{j} \left(\frac{u_{2k}}{u_\infty}\right)^2 = j + 1 + \frac{p}{4}\big(\log(4j+4) + \gamma_E\big) + \sum_{k=0}^{\infty}\left(\left(\frac{u_{2k}}{u_\infty}\right)^2 - 1 - \frac{p}{4k+2}\right) + O(j^{-1}).$$

Using (A.3) and comparing to (A.2) one finally arrives at the relation

$$\frac{\partial\phi(p;\pi/2)}{\partial\theta} = 1 + \frac{p}{2}\big(\log(2) + \gamma_E\big) + 2\sum_{j=0}^{\infty}\left(\left(\frac{u_{2j}}{u_\infty}\right)^2 - 1 - \frac{p}{4j+2}\right).$$

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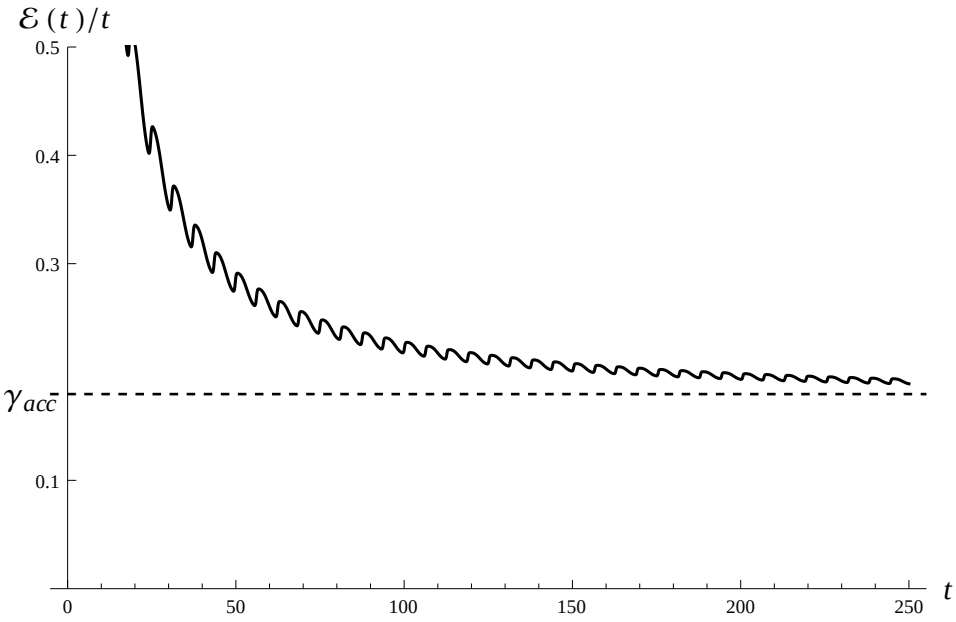

Figure 1. The function $\mathcal{E}(t)/t$, with $\mathcal{E}(t) = \langle \psi(t), H(a(t))\psi(t) \rangle$ and $\psi(t)$ being a normalized solution of the time-dependent Schrödinger equation, compared to the value of the acceleration rate $\gamma_{\rm acc}$ derived in eq. (37).