Some integrals and series involving the Gegenbauer polynomials and the Legendre functions on the cut $(-1,1)$

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Abstract

We use the recent findings of Cohl [arXiv:1105.2735] and evaluate the principal and the residual values of the integral $\int_{-1}^{1}\mathrm{d}t\,(1-t^2)^{\lambda-1/2}(x-t)^{-\kappa-1/2}C_n^{\lambda}(t)$, with $\operatorname{Re}\lambda>-\frac{1}{2}$, $\kappa\in\mathbb{C}$, $-1<x<1$, where $C_n^{\lambda}(t)$ is the Gegenbauer polynomial. For $\operatorname{Re}\kappa<\frac{1}{2}$, we also evaluate the integrals $\int_{-1}^{x}\mathrm{d}t\,(1-t^2)^{\lambda-1/2}(x-t)^{-\kappa-1/2}C_n^{\lambda}(t)$ and $\int_{x}^{1}\mathrm{d}t\,(1-t^2)^{\lambda-1/2}(x-t)^{-\kappa-1/2}C_n^{\lambda}(t)$, both with $\operatorname{Re}\lambda>-\frac{1}{2}$, $-1<x<1$. The results are expressed in terms of the on-the-cut associated Legendre functions $P_{n+\lambda-1/2}^{\kappa-\lambda}(\pm x)$ and $Q_{n+\lambda-1/2}^{\kappa-\lambda}(x)$. In addition, we find closed-form representations of the series $\sum_{n=0}^{\infty}(\pm)^n[(n+\lambda)/\lambda]P_{n+\lambda-1/2}^{\kappa-\lambda}(\pm x)C_n^{\lambda}(t)$ and $\sum_{n=0}^{\infty}(\pm)^n[(n+\lambda)/\lambda]Q_{n+\lambda-1/2}^{\kappa-\lambda}(\pm x)C_n^{\lambda}(t)$, both with $\operatorname{Re}\lambda>-\frac{1}{2}$, $\kappa\in\mathbb{C}$, $-1\leqslant t\leqslant 1$, $-1<x<1$.

Key words: Special functions; Legendre functions; Gegenbauer polynomials; Fourier expansions

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1 Introduction

Recently, Cohl [1] has derived the integral formula

$$\frac{n+\lambda}{\lambda}\int_{-1}^{1}\mathrm{d}t\,\frac{(1-t^2)^{\lambda-1/2}C_n^{\lambda}(t)}{(z-t)^{\kappa+1/2}}=\mathrm{e}^{\mathrm{i}\pi(\lambda-\kappa)}\frac{\sqrt{\pi}\,(n+\lambda)\Gamma(n+2\lambda)}{2^{\lambda-3/2}n!\Gamma(\lambda+1)\Gamma(\kappa+\frac{1}{2})}(z^2-1)^{(\lambda-\kappa)/2}\mathfrak{Q}_{n+\lambda-1/2}^{\kappa-\lambda}(z)$$
$$(\operatorname{Re}\lambda>-\tfrac{1}{2},\ \kappa\in\mathbb{C},\ z\in\mathbb{C}\setminus(-\infty,1]),\qquad(1.1)$$

where $C_n^{\lambda}(t)$ is the Gegenbauer polynomial, while $\mathfrak{Q}_{\nu}^{\mu}(z)$ is the associated Legendre function of the second kind. The integral (1.1) generalizes the Gormley's result [2]

$$\frac{n+\lambda}{\lambda}\int_{-1}^{1}\mathrm{d}t\,\frac{(1-t^2)^{\lambda-1/2}C_n^{\lambda}(t)}{z-t}=\mathrm{e}^{\mathrm{i}\pi(\lambda-1/2)}\frac{\sqrt{\pi}\,(n+\lambda)\Gamma(n+2\lambda)}{2^{\lambda-3/2}n!\Gamma(\lambda+1)}(z^2-1)^{(\lambda-1/2)/2}\mathfrak{Q}_{n+\lambda-1/2}^{1/2-\lambda}(z)$$
$$(\operatorname{Re}\lambda>-\tfrac{1}{2},\ z\in\mathbb{C}\setminus(-\infty,1]),\qquad(1.2)$$

which, in turn, is an extension of the celebrated Neumann's integral formula [3]

$$\int_{-1}^{1}\mathrm{d}t\,\frac{P_n(t)}{z-t}=2\mathfrak{Q}_n(z)\qquad(z\in\mathbb{C}\setminus[-1,1]),\qquad(1.3)$$

where $P_n(t)$ is the Legendre polynomial. The factor $(n+\lambda)/\lambda$ appearing in front of the integrals in Eqs. (1.1) and (1.2) (and also at some other places in the text) is only seemingly awkward and has been introduced to avoid the difficulty one otherwise encounters when $\lambda \to 0$.

From Eq. (1.1) and from the closure relation for the Gegenbauer polynomials, which is

$$\frac{2^{2\lambda-1}\Gamma^2(\lambda)}{\pi}\sum_{n=0}^{\infty}\frac{n!(n+\lambda)}{\Gamma(n+2\lambda)}C_n^{\lambda}(t)C_n^{\lambda}(t') = \frac{\delta(t-t')}{(1-t^2)^{(\lambda-1/2)/2}(1-t'^2)^{(\lambda-1/2)/2}}$$
$$(\operatorname{Re}\lambda > -\tfrac{1}{2},\ -1<t,t'<1) \qquad (1.4)$$

(here $\delta(t-t')$ is the Dirac delta function), one may deduce the summation formula [1]

$$\sum_{n=0}^{\infty}\frac{n+\lambda}{\lambda}\mathfrak{Q}_{n+\lambda-1/2}^{\kappa-\lambda}(z)C_n^{\lambda}(t) = \mathrm{e}^{\mathrm{i}\pi(\kappa-\lambda)}\frac{\sqrt{\pi}\,\Gamma(\kappa+\frac{1}{2})}{2^{\lambda+1/2}\Gamma(\lambda+1)}\frac{(z^2-1)^{(\kappa-\lambda)/2}}{(z-t)^{\kappa+1/2}}$$
$$(\operatorname{Re}\lambda > -\tfrac{1}{2},\ \kappa\in\mathbb{C},\ -1\leqslant t,t'\leqslant 1,\ x\in\mathbb{C}\setminus(-\infty,1]). \qquad (1.5)$$

The particular case of this relation with $\kappa=\lambda$ has been known before [4, p. 183]. For $\kappa=\lambda=1/2$, Eq. (1.5) reduces to the Heine's identity

$$\sum_{n=0}^{\infty}(2n+1)\mathfrak{Q}_n(z)P_n(t) = \frac{1}{z-t}. \qquad (1.6)$$

In Sec. 2 of this communication, we show that one may use the relation in Eq. (1.1) to evaluate some further definite integrals involving the Gegenbauer polynomials. Furthermore, in Sec. 3, we exploit the identity (1.5) to determine closed-form representations of some series involving the Gegenbauer polynomials and the associated Legendre functions on the cut. While particular cases of the relations we arrive at in the present work, corresponding to specific choices of the parameters λ and κ, may be found in the literature on special functions, we are not aware of any appearance of these relations in their most general forms derived below.

Throughout the paper, it is understood that

$$(z^2-1)^{\alpha} \equiv (z-1)^{\alpha}(z+1)^{\alpha} \qquad (|\arg(z\pm 1)|<\pi). \qquad (1.7)$$

2 Evaluation of some definite integrals involving the Gegenbauer polynomials

At first, let us investigate the limit of the formula in Eq. (1.1) as $z\to x\pm\mathrm{i}0$, with $-1<x<1$. Exploiting the identities

$$x\pm\mathrm{i}0+1 = 1+x, \qquad x\pm\mathrm{i}0-1 = \mathrm{e}^{\pm\mathrm{i}\pi}(1-x) \qquad (-1<x<1) \qquad (2.1)$$

and

$$\mathrm{e}^{-\mathrm{i}\pi\mu}\mathfrak{Q}_{\nu}^{\mu}(x\pm\mathrm{i}0) = \mathrm{e}^{\pm\mathrm{i}\pi\mu/2}\left[Q_{\nu}^{\mu}(x)\mp\frac{\mathrm{i}\pi}{2}P_{\nu}^{\mu}(x)\right] \qquad (-1<x<1), \qquad (2.2)$$

where $P_{\nu}^{\mu}(x)$ and $Q_{\nu}^{\mu}(x)$ are the associated Legendre functions *on the cut*, we obtain

$$\frac{n+\lambda}{\lambda}\int_{-1}^{1}\mathrm{d}t\,\frac{(1-t^2)^{\lambda-1/2}C_n^{\lambda}(t)}{(x\pm\mathrm{i}0-t)^{\kappa+1/2}} = \frac{\sqrt{\pi}\,(n+\lambda)\Gamma(n+2\lambda)}{2^{\lambda-3/2}n!\Gamma(\lambda+1)\Gamma(\kappa+\frac{1}{2})}$$
$$\times(1-x^2)^{(\lambda-\kappa)/2}\left[Q_{n+\lambda-1/2}^{\kappa-\lambda}(x)\mp\frac{\mathrm{i}\pi}{2}P_{n+\lambda-1/2}^{\kappa-\lambda}(x)\right]$$
$$(\operatorname{Re}\lambda > -\tfrac{1}{2},\ -1<x<1). \qquad (2.3)$$

Hence, for the principal (PV) and the residual (RV) values of the integral

$$\frac{n+\lambda}{\lambda}\int_{-1}^{1}\mathrm{d}t\,\frac{(1-t^2)^{\lambda-1/2}C_n^{\lambda}(t)}{(x-t)^{\kappa+1/2}} \qquad (\operatorname{Re}\lambda > -\tfrac{1}{2},\ -1<x<1), \qquad (2.4)$$

defined respectively as[1]

$$\mathrm{PV}\left\{\frac{n+\lambda}{\lambda}\int_{-1}^{1}\mathrm{d}t\,\frac{(1-t^2)^{\lambda-1/2}C_n^{\lambda}(t)}{(x-t)^{\kappa+1/2}}\right\}=\frac{n+\lambda}{2\lambda}\int_{-1}^{1}\mathrm{d}t\,\frac{(1-t^2)^{\lambda-1/2}C_n^{\lambda}(t)}{(x+\mathrm{i}0-t)^{\kappa+1/2}}+\frac{n+\lambda}{2\lambda}\int_{-1}^{1}\mathrm{d}t\,\frac{(1-t^2)^{\lambda-1/2}C_n^{\lambda}(t)}{(x-\mathrm{i}0-t)^{\kappa+1/2}}\qquad(\mathrm{Re}\,\lambda>-\tfrac{1}{2},\ -1<x<1)\tag{2.5}$$

and

$$\mathrm{RV}\left\{\frac{n+\lambda}{\lambda}\int_{-1}^{1}\mathrm{d}t\,\frac{(1-t^2)^{\lambda-1/2}C_n^{\lambda}(t)}{(x-t)^{\kappa+1/2}}\right\}=\frac{n+\lambda}{2\lambda}\int_{-1}^{1}\mathrm{d}t\,\frac{(1-t^2)^{\lambda-1/2}C_n^{\lambda}(t)}{(x+\mathrm{i}0-t)^{\kappa+1/2}}-\frac{n+\lambda}{2\lambda}\int_{-1}^{1}\mathrm{d}t\,\frac{(1-t^2)^{\lambda-1/2}C_n^{\lambda}(t)}{(x-\mathrm{i}0-t)^{\kappa+1/2}}\qquad(\mathrm{Re}\,\lambda>-\tfrac{1}{2},\ -1<x<1),\tag{2.6}$$

from Eq. (2.3) we have

$$\mathrm{PV}\left\{\frac{n+\lambda}{\lambda}\int_{-1}^{1}\mathrm{d}t\,\frac{(1-t^2)^{\lambda-1/2}C_n^{\lambda}(t)}{(x-t)^{\kappa+1/2}}\right\}=\frac{\sqrt{\pi}\,(n+\lambda)\Gamma(n+2\lambda)}{2^{\lambda-3/2}n!\Gamma(\lambda+1)\Gamma(\kappa+\frac{1}{2})}(1-x^2)^{(\lambda-\kappa)/2}Q_{n+\lambda-1/2}^{\kappa-\lambda}(x)\qquad(\mathrm{Re}\,\lambda>-\tfrac{1}{2},\ -1<x<1)\tag{2.7}$$

and

$$\mathrm{RV}\left\{\frac{n+\lambda}{\lambda}\int_{-1}^{1}\mathrm{d}t\,\frac{(1-t^2)^{\lambda-1/2}C_n^{\lambda}(t)}{(x-t)^{\kappa+1/2}}\right\}=-\frac{\mathrm{i}\pi^{3/2}(n+\lambda)\Gamma(n+2\lambda)}{2^{\lambda-1/2}n!\Gamma(\lambda+1)\Gamma(\kappa+\frac{1}{2})}(1-x^2)^{(\lambda-\kappa)/2}P_{n+\lambda-1/2}^{\kappa-\lambda}(x)\qquad(\mathrm{Re}\,\lambda>-\tfrac{1}{2},\ -1<x<1).\tag{2.8}$$

It should be stressed that since we admit both λ and κ to be complex, in general neither the principal value displayed in Eq. (2.7) is real nor the residual value given in Eq. (2.8) is purely imaginary.

For $\mathrm{Re}\,\kappa<\frac{1}{2}$, using

$$x\pm\mathrm{i}0-t=\begin{cases}x-t & (-1\leqslant t<x)\\ \mathrm{e}^{\pm\mathrm{i}\pi}(t-x) & (x<t\leqslant 1),\end{cases}\tag{2.9}$$

we may split the integrals appearing on the left-hand side of Eq. (2.3) as follows:

$$\frac{n+\lambda}{\lambda}\int_{-1}^{1}\mathrm{d}t\,\frac{(1-t^2)^{\lambda-1/2}C_n^{\lambda}(t)}{(x\pm\mathrm{i}0-t)^{\kappa+1/2}}=\frac{n+\lambda}{\lambda}\int_{-1}^{x}\mathrm{d}t\,\frac{(1-t^2)^{\lambda-1/2}C_n^{\lambda}(t)}{(x-t)^{\kappa+1/2}}+\mathrm{e}^{\mp\mathrm{i}\pi(\kappa+1/2)}\frac{n+\lambda}{\lambda}\int_{x}^{1}\mathrm{d}t\,\frac{(1-t^2)^{\lambda-1/2}C_n^{\lambda}(t)}{(t-x)^{\kappa+1/2}}\qquad(\mathrm{Re}\,\lambda>-\tfrac{1}{2},\ \mathrm{Re}\,\kappa<\tfrac{1}{2},\ -1<x<1).\tag{2.10}$$

Combining this with Eq. (2.3), we arrive at an inhomogeneous algebraic system for the two integrals appearing on the right-hand side of Eq. (2.10). Solving this system, and using the well-known identity

$$\Gamma(\zeta)\Gamma(1-\zeta)=\frac{\pi}{\sin(\pi\zeta)},\tag{2.11}$$

we obtain

$$\frac{n+\lambda}{\lambda}\int_{x}^{1}\mathrm{d}t\,\frac{(1-t^2)^{\lambda-1/2}C_n^{\lambda}(t)}{(t-x)^{\kappa+1/2}}=\frac{\sqrt{\pi}\,(n+\lambda)\Gamma(n+2\lambda)\Gamma(\frac{1}{2}-\kappa)}{2^{\lambda-1/2}n!\Gamma(\lambda+1)}(1-x^2)^{(\lambda-\kappa)/2}P_{n+\lambda-1/2}^{\kappa-\lambda}(x)\qquad(\mathrm{Re}\,\lambda>-\tfrac{1}{2},\ \mathrm{Re}\,\kappa<\tfrac{1}{2},\ -1<x<1)\tag{2.12}$$

[1] It should be observed that the definition (2.5) of the principal value of the integral (2.4) is superior to the commonly used Cauchy's definition since, contrary to the latter, it is also applicable when the point $t=x$ is a branch point or an even-order pole of the integrand.

and

$$\frac{n+\lambda}{\lambda}\int_{-1}^{x}\mathrm{d}t\,\frac{(1-t^2)^{\lambda-1/2}C_n^{\lambda}(t)}{(x-t)^{\kappa+1/2}}$$
$$=\frac{\sqrt{\pi}\,(n+\lambda)\Gamma(n+2\lambda)}{2^{\lambda-3/2}n!\Gamma(\lambda+1)\Gamma(\kappa+\frac{1}{2})}(1-x^2)^{(\lambda-\kappa)/2}\left\{Q_{n+\lambda-1/2}^{\kappa-\lambda}(x)-\frac{\pi}{2}P_{n+\lambda-1/2}^{\kappa-\lambda}(x)\cot[\pi(\kappa+\tfrac{1}{2})]\right\}$$
$$(\operatorname{Re}\lambda>-\tfrac{1}{2},\ \operatorname{Re}\kappa<\tfrac{1}{2},\ -1<x<1). \qquad (2.13)$$

The right-hand side of the latter equation may be simplified considerably after one exploits the known relation[2]

$$P_{\nu}^{\mu}(-x)=P_{\nu}^{\mu}(x)\cos[\pi(\nu+\mu)]-\frac{2}{\pi}Q_{\nu}^{\mu}(x)\sin[\pi(\nu+\mu)] \qquad (-1<x<1) \qquad (2.14)$$

and the identity (2.11). This yields

$$\frac{n+\lambda}{\lambda}\int_{-1}^{x}\mathrm{d}t\,\frac{(1-t^2)^{\lambda-1/2}C_n^{\lambda}(t)}{(x-t)^{\kappa+1/2}}$$
$$=(-)^n\frac{\sqrt{\pi}\,(n+\lambda)\Gamma(n+2\lambda)\Gamma(\frac{1}{2}-\kappa)}{2^{\lambda-1/2}n!\Gamma(\lambda+1)}(1-x^2)^{(\lambda-\kappa)/2}P_{n+\lambda-1/2}^{\kappa-\lambda}(-x)$$
$$(\operatorname{Re}\lambda>-\tfrac{1}{2},\ \operatorname{Re}\kappa<\tfrac{1}{2},\ -1<x<1). \qquad (2.15)$$

Equation (2.15) may be also derived from Eq. (2.12) after in the latter one makes the simultaneous replacements $t\to -t$ and $x\to -x$, and subsequently exploits the property $C_n^{\lambda}(-t)=(-)^nC_n^{\lambda}(t)$.

Equations (2.7), (2.8), (2.12) and (2.15) constitute the main result of this section. Particular cases of the two latter, resulting after one makes specific choices of λ and κ, may be found in Refs. [4, p. 261] and [6, p. 187].

3 Evaluation of closed forms of some series involving the Gegenbauer polynomials and the associated Legendre functions on the cut

Next, we shall draw inferences from the expansion (1.5). Approaching to the limit $z\to x\pm\mathrm{i}0$, with $-1<x<1$, after subsequent employment of Eqs. (2.1), (2.2) and (2.9), we arrive at

$$\sum_{n=0}^{\infty}\frac{n+\lambda}{\lambda}\left[Q_{n+\lambda-1/2}^{\kappa-\lambda}(x)\mp\frac{\mathrm{i}\pi}{2}P_{n+\lambda-1/2}^{\kappa-\lambda}(x)\right]C_n^{\lambda}(t)$$
$$=\frac{\sqrt{\pi}\,\Gamma(\kappa+\frac{1}{2})}{2^{\lambda+1/2}\Gamma(\lambda+1)}(1-x^2)^{(\kappa-\lambda)/2}\times\begin{cases}(x-t)^{-\kappa-1/2} & (-1\leqslant t<x)\\ \mathrm{e}^{\mp\mathrm{i}\pi(\kappa+1/2)}(t-x)^{-\kappa-1/2} & (x<t\leqslant 1)\end{cases}$$
$$(\operatorname{Re}\lambda>-\tfrac{1}{2}). \qquad (3.1)$$

Hence, subtracting or adding the two relations embodied in Eq. (3.1), we deduce that

$$\sum_{n=0}^{\infty}\frac{n+\lambda}{\lambda}P_{n+\lambda-1/2}^{\kappa-\lambda}(x)C_n^{\lambda}(t)$$
$$=\frac{\sqrt{\pi}}{2^{\lambda-1/2}\Gamma(\lambda+1)\Gamma(\frac{1}{2}-\kappa)}(1-x^2)^{(\kappa-\lambda)/2}\times\begin{cases}0 & (-1\leqslant t<x)\\ (t-x)^{-\kappa-1/2} & (x<t\leqslant 1)\end{cases}$$
$$(\operatorname{Re}\lambda>-\tfrac{1}{2}) \qquad (3.2)$$

[2] It is stated in Refs. [4, p. 170] and [5, p. 144] that the domain of validity of the relation displayed in our Eq. (2.14), and also of the counterpart expression for $Q_{\nu}^{\mu}(-x)$ in terms of $P_{\nu}^{\mu}(x)$ and $Q_{\nu}^{\mu}(x)$, is $0<x<1$. However, it is not difficult to show that if both relations hold on that interval, they must be valid for $-1<x\leqslant 0$ as well.

and

$$\sum_{n=0}^{\infty}\frac{n+\lambda}{\lambda}Q^{\kappa-\lambda}_{n+\lambda-1/2}(x)C^{\lambda}_{n}(t)$$
$$=\frac{\sqrt{\pi}\,\Gamma(\kappa+\frac{1}{2})}{2^{\lambda+1/2}\Gamma(\lambda+1)}(1-x^2)^{(\kappa-\lambda)/2}\times\begin{cases}(x-t)^{-\kappa-1/2} & (-1\leqslant t<x)\\ (t-x)^{-\kappa-1/2}\cos[\pi(\kappa+\frac{1}{2})] & (x<t\leqslant 1)\end{cases}$$
$$(\mathrm{Re}\,\lambda>-\tfrac{1}{2}). \qquad (3.3)$$

If in Eqs. (3.2) and (3.3) we make the simultaneous replacements $t\to -t$ and $x\to -x$, we obtain two further summation formulas:

$$\sum_{n=0}^{\infty}(-)^n\frac{n+\lambda}{\lambda}P^{\kappa-\lambda}_{n+\lambda-1/2}(-x)C^{\lambda}_{n}(t)$$
$$=\frac{\sqrt{\pi}}{2^{\lambda-1/2}\Gamma(\lambda+1)\Gamma(\frac{1}{2}-\kappa)}(1-x^2)^{(\kappa-\lambda)/2}\times\begin{cases}(x-t)^{-\kappa-1/2} & (-1\leqslant t<x)\\ 0 & (x<t\leqslant 1)\end{cases}$$
$$(\mathrm{Re}\,\lambda>-\tfrac{1}{2}), \qquad (3.4)$$

$$\sum_{n=0}^{\infty}(-)^n\frac{n+\lambda}{\lambda}Q^{\kappa-\lambda}_{n+\lambda-1/2}(-x)C^{\lambda}_{n}(t)$$
$$=\frac{\sqrt{\pi}\,\Gamma(\kappa+\frac{1}{2})}{2^{\lambda+1/2}\Gamma(\lambda+1)}(1-x^2)^{(\kappa-\lambda)/2}\times\begin{cases}(x-t)^{-\kappa-1/2}\cos[\pi(\kappa+\frac{1}{2})] & (-1\leqslant t<x)\\ (t-x)^{-\kappa-1/2} & (x<t\leqslant 1)\end{cases}$$
$$(\mathrm{Re}\,\lambda>-\tfrac{1}{2}). \qquad (3.5)$$

Some particular cases of the expansions (3.1)–(3.5), corresponding to specific choices of κ and/or λ, may be found[3] in Refs. [4, pp. 182–183], [5, p. 166] and [7, pp. 341–342].

We leave as an open question for now whether the expansions (1.5) and (3.1)–(3.5) are also valid for $\mathrm{Re}\,\lambda\leqslant -\frac{1}{2}$.

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[3] In the first two series in Sec. 4.5.4 in Ref. [4, p. 182], the constraint $\mathrm{Re}\,\mu<\frac{1}{2}$ is superfluous and may be removed. The same applies to the first two series in Ref. [5, p. 166]. In the second series in Sec. 4.5.4 in Ref. [4, p. 182], $P^{\mu}_{m-1/2}(\cos\vartheta)$ should be replaced by $P^{\mu}_{m-1/2}(-\cos\vartheta)$. In the first formula in Ref. [4, p. 183], the constraint $x<\cos\varphi$ should be replaced by $-1\leqslant\cos\varphi<x<1$. Moreover, it follows from our Eq. (3.4) that the latter formula is valid at least for $\mathrm{Re}\,\nu>-\frac{1}{2}$.

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