COORDINATE-INVARIANT INCREMENTAL LYAPUNOV FUNCTIONS

MAJID ZAMANI[1] AND RUPAK MAJUMDAR[2]

Abstract. The notion of incremental stability was proposed by several researchers as a strong property of dynamical and control systems. In this type of stability, the focus is on the convergence of trajectories with respect to themselves, rather than with respect to an equilibrium point or a particular trajectory. Similarly to stability, Lyapunov functions play an important role in the study of incremental stability. In this paper, we propose coordinate-invariant notions of incremental Lyapunov function and provide the description of incremental stability in terms of existence of the proposed Lyapunov functions. Moreover, we develop a backstepping design approach providing a recursive way of constructing controllers as well as incremental Lyapunov functions. The effectiveness of our method is illustrated by synthesizing a controller rendering a single-machine infinite-bus electrical power system incrementally stable.

1. Introduction

Incremental stability requires that all trajectories converge to each other, rather than to an equilibrium point or a particular trajectory. While it is well-known that for linear systems incremental stability is equivalent to stability, it can be a much stronger property than stability for nonlinear systems. The study of incremental stability goes back to the work of Zames in the 60's [Zam63]; See [ZT11], for a historical discussion and a broad list of applications of incremental stability.

Similarly to stability, Lyapunov functions play an important role in the study of incremental stability. In [Ang02], Angeli proposed the notions of incremental Lyapunov function and incremental input-to-state Lyapunov function, and used these notions to prove characterizations of incremental global asymptotic stability (δ-GAS) and incremental input-to-state stability (δ-ISS). Notions of δ-GAS, δ-ISS and incremental Lyapunov functions, proposed in [Ang02], are not coordinate invariant, in general. Since most of the controller design approaches benefit from changes of coordinates, in [ZT11], authors proposed different notions of δ-GAS and δ-ISS which are coordinate invariant. In this paper, we propose notions of incremental Lyapunov function and incremental input-to-state Lyapunov function that are coordinate invariant. Moreover, we use these new notions of Lyapunov functions to describe the notions of incremental stability, proposed in [ZT11].

The number of applications of incremental Lyapunov functions has increased in the past years. Examples include building explicit bounds on the region of attraction in phase-locking in the Kuramoto system [FCPL10], construction of symbolic models for nonlinear control systems [GPT09, Gir10, CGG11], robust test of hybrid systems [JFA$^+$07], approximation of stochastic hybrid systems [JP09], and source-code model checking for nonlinear dynamical systems [KDL$^+$08]. Note that incremental Lyapunov functions can be used as bisimulation functions, recognized as a key tool for the provided analysis in [JFA$^+$07, JP09, KDL$^+$08]. Our motivation comes from symbolic control, proposed in [GPT09, Gir10, CGG11], where incremental Lyapunov functions were identified as a key property for the construction of finite abstractions of nonlinear control systems. Hence, there is a growing need for design methods providing incremental Lyapunov functions since most of the existing design methods provide Lyapunov functions rather than incremental Lyapunov functions.

We use our proposed notions of Lyapunov functions to develope synthesis tools for incremental stability. As an example, we develop a backstepping design method for incremental stability for strict-feedback[1] form systems. The proposed approach was inspired by the incremental backstepping approach provided in [ZT11, ZT10].

[1]See equation (3.10) or [KKK95] for a definition.

Like the original backstepping method, the proposed approach in this paper provides a recursive way of constructing controllers as well as incremental Lyapunov functions. Our design approach is illustrated by designing a controller rendering a single-machine infinite-bus electrical power system incrementally stable.

2. Control Systems and Stability Notions

2.1. **Notation.** The symbols $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ denote the set of real, positive, and nonnegative real numbers, respectively. The symbol I_m denotes the identity matrix in $\mathbb{R}^{m\times m}$. Given a vector $x \in \mathbb{R}^n$, we denote by x_i the i–th element of x, and by $\|x\|$ the Euclidean norm of x; we recall that $\|x\| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$. Given a measurable function $f : \mathbb{R}_0^+ \to \mathbb{R}^n$, the (essential) supremum of f is denoted by $\|f\|_\infty$; we recall that $\|f\|_\infty := \text{(ess)sup}\{\|f(t)\|, t \geq 0\}$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called radially unbounded if $f(x) \to \infty$ as $\|x\| \to \infty$. A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the map $\beta(r, s)$ belongs to class $\mathcal{K}_\infty$ with respect to r and, for each fixed nonzero r, the map $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$. If $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function with a smooth inverse, and if $X : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map, we denote by $\phi_* X$ the map defined by $(\phi_* X)(y) = \frac{\partial \phi}{\partial x}\big|_{x=\phi^{-1}(y)} X \circ \phi^{-1}(y)$. A function $\mathbf{d} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ is a metric on $\mathbb{R}^n$ if for any $x, y, z \in \mathbb{R}^n$, the following three conditions are satisfied: i) $\mathbf{d}(x, y) = 0$ if and only if $x = y$; ii) $\mathbf{d}(x, y) = \mathbf{d}(y, x)$; and iii) $\mathbf{d}(x, z) \leq \mathbf{d}(x, y) + \mathbf{d}(y, z)$.

2.2. **Control Systems.** The class of control systems that we consider in this paper is formalized in the following definition.

Definition 2.1. *A control system is a quadruple:*

$$\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f),$$

where:

- $\mathbb{R}^n$ *is the state space;*
- $\mathsf{U} \subseteq \mathbb{R}^m$ *is the input set;*
- $\mathcal{U}$ *is the set of all measurable functions of time from intervals of the form* $]a, b[\subseteq \mathbb{R}$ *to* U *with* $a < 0$ *and* $b > 0$*;*
- $f : \mathbb{R}^n \times \mathsf{U} \to \mathbb{R}^n$ *is a continuous map satisfying the following Lipschitz assumption: for every compact set* $Q \subset \mathbb{R}^n$*, there exists a constant* $Z \in \mathbb{R}^+$ *such that* $\|f(x, u) - f(y, u)\| \leq Z\|x - y\|$ *for all* $x, y \in Q$ *and all* $u \in \mathsf{U}$*.*

A curve $\xi :]a, b[\to \mathbb{R}^n$ is said to be a *trajectory* of Σ if there exists $\upsilon \in \mathcal{U}$ satisfying:

$$\dot{\xi}(t) = f\left(\xi(t), \upsilon(t)\right), \tag{2.1}$$

for almost all $t \in \,]a, b[$. We also write $\xi_{x\upsilon}(t)$ to denote the point reached at time t under the input υ from initial condition $x = \xi_{x\upsilon}(0)$; this point is uniquely determined, since the assumptions on f ensure existence and uniqueness of trajectories [Son98]. A control system Σ is said to be forward complete if every trajectory is defined on an interval of the form $]a, \infty[$. Sufficient and necessary conditions for a system to be forward complete can be found in [AS99]. A control system Σ is said to be smooth if f is an infinitely differentiable function of its arguments.

2.3. **Stability notions.** Here, we recall the notions of incremental global asymptotic stability ($\delta_\exists$-GAS) and incremental input-to-state stability ($\delta_\exists$-ISS), presented in [ZT11].

Definition 2.2 ([ZT11]). *A control system Σ is incrementally globally asymptotically stable ($\delta_\exists$-GAS) if it is forward complete and there exist a metric $\mathbf{d}$ and a $\mathcal{KL}$ function β such that for any $t \in \mathbb{R}_0^+$, any $x, x' \in \mathbb{R}^n$ and any $\upsilon \in \mathcal{U}$ the following condition is satisfied:*

$$\mathbf{d}\left(\xi_{x\upsilon}(t), \xi_{x'\upsilon}(t)\right) \leq \beta\left(\mathbf{d}\left(x, x'\right), t\right). \tag{2.2}$$

As defined in [Ang02], δ-GAS requires the metric $\mathbf{d}$ to be the Euclidean metric. However, Definition 2.2 only requires the existence of a metric. We note that while δ-GAS is not generally invariant under changes of coordinates, $\delta_\exists$-GAS is. When the origin is an equilibrium point for Σ and the map $\psi : \mathbb{R}^n \to \mathbb{R}_0^+$, defined by $\psi(x) = \mathbf{d}(x, 0)$, is continuous and radially unbounded[2], both $\delta_\exists$-GAS and δ-GAS imply global asymptotic stability.

Definition 2.3 ([ZT11]). *A control system Σ is incrementally input-to-state stable ($\delta_\exists$-ISS) if it is forward complete and there exist a metric $\mathbf{d}$, a $\mathcal{KL}$ function β, and a $\mathcal{K}_\infty$ function γ such that for any $t \in \mathbb{R}_0^+$, any $x, x' \in \mathbb{R}^n$, and any υ, $\upsilon' \in \mathcal{U}$ the following condition is satisfied:*

$$\mathbf{d}\left(\xi_{x\upsilon}(t), \xi_{x'\upsilon'}(t)\right) \leq \beta\left(\mathbf{d}\left(x, x'\right), t\right) + \gamma\left(\|\upsilon - \upsilon'\|_\infty\right). \tag{2.3}$$

By observing (2.2) and (2.3), it is readily seen that $\delta_\exists$-ISS implies $\delta_\exists$-GAS while the converse is not true in general. Moreover, whenever the metric $\mathbf{d}$ is the Euclidean metric, $\delta_\exists$-ISS becomes δ-ISS as defined in [Ang02]. We note that while δ-ISS is not generally invariant under changes of coordinates, $\delta_\exists$-ISS is. When the origin is an equilibrium point for Σ and the map $\psi : \mathbb{R}^n \to \mathbb{R}_0^+$, defined by $\psi(x) = \mathbf{d}(x, 0)$, is continuous and radially unbounded, both $\delta_\exists$-ISS and δ-ISS imply input-to-state stability.

We discuss in the next section descriptions of $\delta_\exists$-GAS and $\delta_\exists$-ISS in terms of existence of incremental Lyapunov functions.

2.4. **Descriptions of incremental stability.** This section contains descriptions of $\delta_\exists$-GAS and $\delta_\exists$-ISS in terms of existence of incremental Lyapunov functions. We start by introducing the following definition which was inspired by the notions of incremental global asymptotic stability (δ-GAS) Lyapunov function and incremental input-to-state stability (δ-ISS) Lyapunov function presented in [Ang02].

Definition 2.4. *Consider a control system Σ and a smooth function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$. Function V is called a $\delta_\exists$-GAS Lyapunov function for Σ, if there exist a metric $\mathbf{d}$, $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, and $\kappa \in \mathbb{R}^+$ such that:*

(i) *for any $x, x' \in \mathbb{R}^n$*
$\underline{\alpha}(\mathbf{d}(x, x')) \leq V(x, x') \leq \overline{\alpha}(\mathbf{d}(x, x'))$;
(ii) *for any $x, x' \in \mathbb{R}^n$ and any $u \in \mathsf{U}$*
$\frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial x'} f(x', u) \leq -\kappa V(x, x')$.

Function V is called a $\delta_\exists$-ISS Lyapunov function for Σ, if there exist a metric $\mathbf{d}$, $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, σ, and $\kappa \in \mathbb{R}^+$ satisfying conditions (i) and:

(iii) *for any $x, x' \in \mathbb{R}^n$ and for any $u, u' \in \mathsf{U}$*
$\frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial x'} f(x', u') \leq -\kappa V(x, x') + \sigma(\|u - u'\|)$.

While δ-GAS and δ-ISS Lyapunov functions, as defined in [Ang02], require the metric $\mathbf{d}$ in condition (i) in Definition 2.4 to be the Euclidean metric, Definition 2.4 only requires the existence of a metric. We note that while δ-GAS and δ-ISS Lyapunov functions are not invariant under changes of coordinates in general, $\delta_\exists$-GAS and $\delta_\exists$-ISS Lyapunov functions are.

Remark 2.5. *Condition (iii) of Definition 2.4 can be replaced by:*

$$\frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial x'} f(x', u') \leq -\rho(\mathbf{d}(x, x')) + \sigma(\|u - u'\|),$$

[2]Under the stated assumptions it can be shown that $\underline{\alpha}(\|x\|) \leq \psi(x) \leq \overline{\alpha}(\|x\|)$ for $\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\overline{\alpha}$.

where ρ is a $\mathcal{K}_\infty$ function. It is known that there is no loss of generality in considering $\rho(\mathbf{d}(x,x'))=\kappa V(x,y)$, by appropriately modifying the $\delta_\exists$-ISS Lyapunov function V (see Lemma 11 in [PW96]*).*

In the next lemma, we show that $\delta_\exists$-GAS and $\delta_\exists$-ISS Lyapunov functions, defined in Definition 2.4, are invariant under changes of coordinates.

Lemma 2.6. *Let $\Sigma=(\mathbb{R}^n,\mathsf{U},\mathcal{U},f)$ be a control system and let $\phi:\mathbb{R}^n\to\mathbb{R}^n$ be a smooth map with a smooth inverse. If the function V is a $\delta_\exists$-GAS (resp. $\delta_\exists$-ISS) Lyapunov function for Σ, then the function $V\circ\phi^{-1}=V\left(\phi^{-1}(\cdot),\phi^{-1}(\cdot)\right)$ is a $\delta_\exists$-GAS (resp. $\delta_\exists$-ISS) Lyapunov function for $\Sigma'=(\mathbb{R}^n,\mathsf{U},\mathcal{U},\phi_*f)$.*

Proof. Inequalities (i) in Definition 2.4, transforms under ϕ to:

$$\underline{\alpha}\left(\mathbf{d}\left(\phi^{-1}(y),\phi^{-1}(y')\right)\right)\leq V\left(\phi^{-1}(y),\phi^{-1}(y')\right)\leq\overline{\alpha}\left(\mathbf{d}\left(\phi^{-1}(y),\phi^{-1}(y')\right)\right), \tag{2.4}$$

where $y=\phi(x)$ and $y'=\phi(x')$. Therefore, the function $V\circ\phi^{-1}$ satisfies the inequalities (i) in Definition 2.4 by $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, and the metric $\mathbf{d}'(y,y')=\mathbf{d}\left(\phi^{-1}(y),\phi^{-1}(y')\right)$. Let us now show that condition (ii) in Definition 2.4 holds for Σ' using $V\circ\phi^{-1}$. Since ϕ has a smooth inverse, it can be readily checked that $\frac{\partial\phi^{-1}}{\partial y}\frac{\partial\phi}{\partial x}\left(\phi^{-1}(y)\right)=I_n$ by just taking derivative of $\phi^{-1}(\phi(x))=x$ with respect to $x=\phi^{-1}(y)$. Hence, we obtain:

$$\begin{aligned}
&\frac{\partial\left(V\circ\phi^{-1}\right)}{\partial y}(\phi_*f)(y,u)+\frac{\partial\left(V\circ\phi^{-1}\right)}{\partial y'}(\phi_*f)(y',u)=\\
&\frac{\partial V}{\partial x}\Big|_{x=\phi^{-1}(y)}\frac{\partial\phi^{-1}}{\partial y}\left(\phi_*f\right)(y,u)+\frac{\partial V}{\partial x'}\Big|_{x'=\phi^{-1}(y')}\frac{\partial\phi^{-1}}{\partial y}\left(\phi_*f\right)(y',u)=\\
&\frac{\partial V}{\partial x}\Big|_{x=\phi^{-1}(y)}\frac{\partial\phi^{-1}}{\partial y}\frac{\partial\phi}{\partial x}\Big|_{x=\phi^{-1}(y)}f\left(\phi^{-1}(y),u\right)+\frac{\partial V}{\partial x'}\Big|_{x'=\phi^{-1}(y')}\frac{\partial\phi^{-1}}{\partial y}\frac{\partial\phi}{\partial x}\Big|_{x'=\phi^{-1}(y')}f\left(\phi^{-1}(y'),u\right)=\\
&\frac{\partial V}{\partial x}\Big|_{x=\phi^{-1}(y)}f\left(\phi^{-1}(y),u\right)+\frac{\partial V}{\partial x'}\Big|_{x'=\phi^{-1}(y')}f\left(\phi^{-1}(y'),u\right)\leq-\kappa V\left(\phi^{-1}(y),\phi^{-1}(y')\right),
\end{aligned}\tag{2.5}$$

which completes the proof. Similarly, it can be shown that $V\circ\phi^{-1}$ satisfies condition (iii) in Definition 2.4 for Σ' if V satisfies it for Σ. □

The following theorem describes $\delta_\exists$-ISS (resp. $\delta_\exists$-GAS) in terms of existence of a $\delta_\exists$-ISS (resp. $\delta_\exists$-GAS) Lyapunov function.

Theorem 2.7. *A control system Σ is $\delta_\exists$-ISS (resp. $\delta_\exists$-GAS) if it admits a $\delta_\exists$-ISS (resp. $\delta_\exists$-GAS) Lyapunov function.*

Proof. The proof is inspired by the proof of Theorem 5.2 in [ZPJT10]. By using property (i) in Definition 2.4, we obtain:

$$\mathbf{d}\left(\xi_{x\upsilon}(t),\xi_{x'\upsilon'}(t)\right)\leq\underline{\alpha}^{-1}\left(V\left(\xi_{x\upsilon}(t),\xi_{x'\upsilon'}(t)\right)\right). \tag{2.6}$$

By using property (iii) and the comparison lemma [Kha96], one gets:

$$V(\xi_{x\upsilon}(t),\xi_{x'\upsilon'}(t))\leq e^{-\kappa t}V(\xi_{x\upsilon}(0),\xi_{x'\upsilon'}(0))+e^{-\kappa t}*\sigma(\|\upsilon(t)-\upsilon'(t)\|), \tag{2.7}$$

where $*$ denotes the convolution integral[3]. By combining inequalities (2.6) and (2.7), one gets:

$$\begin{aligned}
\mathbf{d}\left(\xi_{x\upsilon}(t),\xi_{x'\upsilon'}(t)\right)&\leq\underline{\alpha}^{-1}\left(e^{-\kappa t}V(x,x')+e^{-\kappa t}*\sigma(\|\upsilon(t)-\upsilon'(t)\|)\right)\\
&\leq\underline{\alpha}^{-1}\left(e^{-\kappa t}V(x,x')+\frac{1-e^{-\kappa t}}{\kappa}\sigma\left(\|\upsilon-\upsilon'\|_\infty\right)\right)\\
&\leq\underline{\alpha}^{-1}\left(e^{-\kappa t}V(x,x')+\frac{1}{\kappa}\sigma\left(\|\upsilon-\upsilon'\|_\infty\right)\right)=\gamma(\rho,\phi)
\end{aligned}$$

[3] $e^{-\kappa t}*\sigma(\|\upsilon(t)-\upsilon'(t)\|)=\int_0^t e^{-\kappa(t-\tau)}\sigma(\|\upsilon(\tau)-\upsilon'(\tau)\|)d\tau$.

where $\gamma(\rho,\phi)=\underline{\alpha}^{-1}(\rho+\phi)$, $\rho=e^{-\kappa t}V(x,x')$, and $\phi=\frac{1}{\kappa}\sigma\left(\|\upsilon-\upsilon'\|_\infty\right)$. Since γ is nondecreasing in each variable, we have:

$$\mathbf{d}\left(\xi_{x\upsilon}(t),\xi_{x'\upsilon'}(t)\right)\leq h\left(e^{-\kappa t}V(x,x')\right)+h\left(\frac{1}{\kappa}\sigma\left(\|\upsilon-\upsilon'\|_\infty\right)\right)$$

where $h(r)=\gamma(r,r)=\underline{\alpha}^{-1}(2r)$ and $h:\mathbb{R}_0^+\to\mathbb{R}_0^+$ is a $\mathcal{K}_\infty$ function. Moreover, using $V(x,x')\leq\overline{\alpha}(\mathbf{d}\left(x,x'\right))$, one obtains:

$$\mathbf{d}(\xi_{x\upsilon}(t),\xi_{x'\upsilon'}(t))\leq\underline{\alpha}^{-1}\left(2e^{-\kappa t}\overline{\alpha}(\mathbf{d}(x,x'))\right)+\underline{\alpha}^{-1}\left(\frac{2}{\kappa}\sigma(\|\upsilon-\upsilon'\|_\infty)\right).$$

Therefore, by defining functions β and γ as

$$\begin{aligned}\beta(\mathbf{d}(x,x'),t)&=\underline{\alpha}^{-1}\left(2e^{-\kappa t}\overline{\alpha}\left(\mathbf{d}(x,x')\right)\right)\\ \gamma(\|\upsilon-\upsilon'\|_\infty)&=\underline{\alpha}^{-1}\left(\frac{2}{\kappa}\sigma(\|\upsilon-\upsilon'\|_\infty)\right),\end{aligned}$$

condition (2.3) is satisfied. Hence, the system Σ is $\delta_\exists$-ISS. The proof also works for $\delta_\exists$-GAS case by simply forcing $\upsilon=\upsilon'$. □

Remark 2.8. *It can be checked that existence of a contraction metric, with respect to states, or, with respect to states and inputs, defined in* [ZT11]*, implies existence of a $\delta_\exists$-GAS or $\delta_\exists$-ISS Lyapunov function, respectively. Furthermore, the Riemannian distance function provided by the contraction metric is the $\delta_\exists$-GAS or $\delta_\exists$-ISS Lyapunov function. Detailed information can be found in the proof of Theorem 5.5 in* [ZPJT10]*.*

In the next section, we propose a backstepping design procedure, providing a recursive way of constructing controllers as well as incremental Lyapunov functions, to render control systems incrementally stable .

3. Backstepping Design Procedure

The method described here was inspired by the incremental backstepping approach provided in [ZT11]. While the approach proposed in [ZT11] provides a recursive way of constructing contraction metrics, the proposed approach in this paper provides a recursive way of constructing incremental Lyapunov functions.

Consider the class of control systems $\Sigma=(\mathbb{R}^n,\mathsf{U},\mathcal{U},f)$ with f of the parametric-strict-feedback form [KKK95]:

$$\begin{aligned}f_1(x,u)&=h_1(x_1)+b_1x_2,\\ f_2(x,u)&=h_2(x_1,x_2)+b_2x_3,\\ &\vdots\\ f_{n-1}(x,u)&=h_{n-1}(x_1,\cdots,x_{n-1})+b_{n-1}x_n,\\ f_n(x,u)&=h_n(x)+g(x)u,\end{aligned}\tag{3.1}$$

where $x\in\mathbb{R}^n$ is the state and $u\in\mathsf{U}\subseteq\mathbb{R}$ is the control input. The functions $h_i:\mathbb{R}^i\to\mathbb{R}$, for $i=1,\ldots,n$, and $g:\mathbb{R}^n\to\mathbb{R}$ are smooth, $g(x)\neq0$ over the domain of interest, and $b_i\in\mathbb{R}$, for $i=1,\ldots,n$, are nonzero constants.

We can now state one of the results, describing a backstepping controller for the control system (3.1).

Theorem 3.1. *For any control system $\Sigma=(\mathbb{R}^n,\mathsf{U},\mathcal{U},f)$ with f of the form (3.1) and for any $\lambda\in\mathbb{R}^+$, the state feedback control law:*

$$k(x,\widehat{u})=\frac{1}{g(x)}\Big[k_n(x)-h_n(x)\Big]+\frac{1}{g(x)}\widehat{u},\tag{3.2}$$

where

$$
\begin{aligned}
k_l(x) &= -b_{l-1}\left(x_{l-1} - \phi_{l-2}(x)\right) - \lambda\left(x_l - \phi_{l-1}(x)\right) \\
&\quad + \frac{\partial \phi_{l-1}}{\partial x} f(x, k(x)),\ \textit{for } l = 1, \cdots, n, \qquad (3.3) \\
\phi_l(x) &= \frac{1}{b_l}\left[k_l(x) - h_l(x)\right],\ \textit{for } l = 1, \cdots, n-1, \\
\phi_{-1}(x) &= \phi_0(x) = 0\ \forall x \in \mathbb{R}^n,\ b_0 = 0,\ \textit{and } x_0 = 0,
\end{aligned}
$$

renders the control system Σ $\delta_\exists$*-ISS with respect to the input* $\widehat{\upsilon}$ *and the function*

$$
\widehat{V}(x, x') = \sqrt{\sum_{l=0}^{n-1}\left[\left(x_{l+1} - \phi_l(x)\right) - \left(x'_{l+1} - \phi_l(x')\right)\right]^2},
$$

is a $\delta_\exists$*-ISS Lyapunov function for* Σ.

Proof. The proposed control law (3.2) transforms a control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ with f of the form (3.1) into:

$$
\begin{array}{rcl}
f_1(x, k(x, \widehat{u})) & = & h_1(x_1) + b_1 x_2, \\
f_2(x, k(x, \widehat{u})) & = & h_2(x_1, x_2) + b_2 x_3, \\
& \vdots & \\
f_{n-1}(x, k(x, \widehat{u})) & = & h_{n-1}(x_1, \cdots, x_{n-1}) + b_{n-1} x_n, \\
f_n(x, k(x, \widehat{u})) & = & k_n(x) + \widehat{u}.
\end{array} \qquad (3.4)
$$

The coordinate transformation $z = \psi(x)$, where $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is the smooth map (with the smooth inverse) defined by:

$$
z = \psi(x) = \begin{bmatrix} x_1 \\ x_2 - \phi_1(x) \\ x_3 - \phi_2(x) \\ \vdots \\ x_n - \phi_{n-1}(x) \end{bmatrix}, \qquad (3.5)
$$

transforms the control system $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f)$ with f of the form (3.4) to the control system $\Sigma' = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, f')$, where $f' = \psi_* f$ has the following form:

$$
f'(z, \widehat{u}) = Az + B\widehat{u}, \qquad (3.6)
$$

where

$$
A = \begin{bmatrix}
-\lambda & b_1 & 0 & 0 & \cdots & 0 \\
-b_1 & -\lambda & b_2 & 0 & \cdots & 0 \\
0 & -b_2 & -\lambda & b_3 & \cdots & 0 \\
\vdots & & & & & \vdots \\
0 & & & & \ddots & b_{n-1} \\
0 & & \cdots & 0 & -b_{n-1} & -\lambda
\end{bmatrix},\ B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \qquad (3.7)
$$

It can be easily checked that the function

$$
V(z, z') = \sqrt{(z - z')^T (z - z')}, \qquad (3.8)
$$

satisfies

$$
\frac{\partial V}{\partial z}(Az + B\widehat{u}) + \frac{\partial V}{\partial z'}(Az' + B\widehat{u}') \le -\lambda V(z, z') + |\widehat{u} - \widehat{u}'|. \qquad (3.9)
$$

Hence the function (3.8) satisfies conditions (i) and (iii) in Definition 2.4, implying that it is a $\delta_\exists$-ISS Lyapunov function for Σ'. Using Theorem 2.7, we obtain that Σ' is $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$. Using Lemma 2.6, we conclude that the following function:

$$\widehat{V}(x,x') = V\left(\psi(x),\psi(x')\right) = \sqrt{\left(\psi(x)-\psi(x')\right)^T\left(\psi(x)-\psi(x')\right)} = \sqrt{\sum_{l=0}^{n-1}\left[\left(x_{l+1}-\phi_l(x)\right)-\left(x'_{l+1}-\phi_l(x')\right)\right]^2},$$

is a $\delta_\exists$-ISS Lyapunov function for Σ. Therefore, using Theorem 2.7, we obtain that Σ is $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$. The $\delta_\exists$-ISS condition (2.3), as shown in Theorem 2.7, is given by:

$$\mathbf{d}\left(\xi_{x\widehat{\upsilon}}(t),\xi_{x'\widehat{\upsilon}'}(t)\right) \leq 2e^{-\lambda t}\mathbf{d}(x,x') + \frac{2}{\lambda}|\widehat{\upsilon}-\widehat{\upsilon}'|_\infty,$$

where $\mathbf{d}(x,x') = \|\psi(x)-\psi(x')\|$, for any $x,x'\in\mathbb{R}^n$. $\square$

Remark 3.2. *It can be readily seen that the state feedback control law (3.2) renders the control system Σ $\delta_\exists$-GAS and the function*

$$\widehat{V}(x,x') = \sqrt{\sum_{l=0}^{n-1}\left[\left(x_{l+1}-\phi_l(x)\right)-\left(x'_{l+1}-\phi_l(x')\right)\right]^2},$$

is a $\delta_\exists$-GAS Lyapunov function for Σ.

Remark 3.3. *It can be checked that:*

$$G_n(x) = \frac{1}{2}\frac{\partial^2\widehat{V}(x,0)}{\partial x^2} =$$

$$\begin{bmatrix} \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} [1]+\left(\frac{\partial\phi_1}{\partial y_2}\right)^T\frac{\partial\phi_1}{\partial y_2} & -\left(\frac{\partial\phi_1}{\partial y_2}\right)^T \\ -\frac{\partial\phi_1}{\partial y_2} & 1 \end{bmatrix} + \left(\frac{\partial\phi_2}{\partial y_3}\right)^T\frac{\partial\phi_2}{\partial y_3} & -\left(\frac{\partial\phi_2}{\partial y_3}\right)^T \\ -\frac{\partial\phi_2}{\partial y_3} & 1 \end{bmatrix} + \cdots \\ \vdots \end{bmatrix} + \left(\frac{\partial\phi_{n-1}}{\partial y_n}\right)^T\frac{\partial\phi_{n-1}}{\partial y_n} & -\left(\frac{\partial\phi_{n-1}}{\partial y_n}\right)^T \\ -\frac{\partial\phi_{n-1}}{\partial y_n} & 1 \end{bmatrix},$$

where $y_l = [x_1,\cdots,x_{l-1}]^T$, for $l=2,\cdots,n$. As showed in [ZT11]*, G_n is a contraction metric, for the control system (3.1), equipped with the state feedback control law (3.2).*

Now, we extend the result in Theorem 3.1 to the class of control systems $\Sigma = (\mathbb{R}^n,\mathsf{U},\mathcal{U},f)$ with f of the strict-feedback form [KKK95]:

$$\begin{array}{rcl} f_1(x,u) & = & h_1(x_1)+g_1(x_1)x_2, \\ f_2(x,u) & = & h_2(x_1,x_2)+g_2(x_1,x_2)x_3, \\ & \vdots & \\ f_{n-1}(x,u) & = & h_{n-1}(x_1,\cdots,x_{n-1})+g_{n-1}(x_1,\cdots,x_{n-1})x_n, \\ f_n(x,u) & = & h_n(x)+g_n(x)u, \end{array} \tag{3.10}$$

where $x\in\mathbb{R}^n$ is the state and $u\in\mathsf{U}\subseteq\mathbb{R}$ is the control input. The functions $h_i:\mathbb{R}^i\to\mathbb{R}$, and $g_i:\mathbb{R}^i\to\mathbb{R}$, for $i=1,\ldots,n$, are smooth, and $g_i(x_1,\cdots,x_i)\neq 0$ over the domain of interest.

Theorem 3.4. *Let $\Sigma = (\mathbb{R}^n,\mathsf{U},\mathcal{U},f)$ be a control system where f is of the form (3.10). The state feedback control law $u = k(\varphi(x),\widehat{u})$, where k was defined in (3.2) and $\varphi:\mathbb{R}^n\to\mathbb{R}^n$ is a smooth map (with the smooth inverse) defined by:*

$$\varphi(x) = \begin{bmatrix} x_1 \\ g_1(x_1)x_2 \\ g_1(x_1)g_2(x_1,x_2)x_3 \\ \vdots \\ \prod_{i=1}^{n-1} g_i(x_1,\cdots,x_i)x_n \end{bmatrix}, \tag{3.11}$$

renders control system Σ $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$ and the function

$$\widetilde{V}(x,x') = \sqrt{\left(\psi\circ\varphi(x)-\psi\circ\varphi(x')\right)^T\left(\psi\circ\varphi(x)-\psi\circ\varphi(x')\right)},$$

where ψ was defined in (3.5), is a $\delta_\exists$-ISS Lyapunov function for Σ.

Proof. As showed in [ZT11], the coordinate transformation $\eta=\varphi(\xi)$ transforms the control system $\Sigma = (\mathbb{R}^n,\mathsf{U},\mathcal{U},f)$ with f of the form (3.10) to the control system $\Sigma'=(\mathbb{R}^n,\mathsf{U},\mathcal{U},f')$, where $f'=\varphi_*f$ has the following form:

$$\begin{array}{rcl} f_1'(y,u) &=& h_1'(y_1)+y_2,\\ f_2'(y,u) &=& h_2'(y_1,y_2)+y_3,\\ &\vdots& \\ f_{n-1}'(y,u) &=& h_{n-1}'(y_1,\cdots,y_{n-1})+y_n,\\ f_n'(y,u) &=& h_n'(y)+g'(y)u, \end{array} \tag{3.12}$$

where $h_i':\mathbb{R}^i\rightarrow\mathbb{R}$, for $i=1,\cdots,n$, are smooth functions, $g'=\prod_{i=1}^{i=n}g_i$, and $y\in\mathbb{R}^n$ is the state of Σ'. As proved in Theorem 3.1, the state feedback control law k, defined in (3.2), makes the function

$$\widehat{V}(y,y') = \sqrt{\left(\psi(y)-\psi(y')\right)^T\left(\psi(y)-\psi(y')\right)}, \tag{3.13}$$

a $\delta_\exists$-ISS Lyapunov function, for the control system Σ'. As proved in Lemma 2.6, the function

$$\widetilde{V}(x,x') = \widehat{V}(\varphi(x),\varphi(x')) = \sqrt{\left(\psi\circ\varphi(x)-\psi\circ\varphi(x')\right)^T\left(\psi\circ\varphi(x)-\psi\circ\varphi(x')\right)},$$

is a $\delta_\exists$-ISS Lyapunov function , for the control system Σ, equipped with the state feedback control law $k(\varphi(x),\widehat{u})$. Therefore, the state feedback control law $k(\varphi(x),\widehat{u})$ makes the control system Σ $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$. The $\delta_\exists$-ISS condition (2.3), as shown in Theorem 2.7, is given by:

$$\mathbf{d}\left(\xi_{x\widehat{\upsilon}}(t),\xi_{x'\widehat{\upsilon}'}(t)\right)\leq 2e^{-\lambda t}\mathbf{d}(x,x')+\frac{2}{\lambda}|\widehat{\upsilon}-\widehat{\upsilon}'|_\infty,$$

where $\mathbf{d}(x,x')=\|\psi\circ\varphi(x)-\psi\circ\varphi(x')\|$, for any $x,x'\in\mathbb{R}^n$.

□

Remark 3.5. *It can be readily seen that the state feedback control law $k(\varphi(x),\widehat{u})$, where k was defined in (3.2), renders the control system Σ $\delta_\exists$-GAS and the function*

$$\widetilde{V}(x,x') = \sqrt{\left(\psi\circ\varphi(x)-\psi\circ\varphi(x')\right)^T\left(\psi\circ\varphi(x)-\psi\circ\varphi(x')\right)},$$

is a $\delta_\exists$-GAS Lyapunov function for Σ.

4. Backstepping Controller Design for an Infinite-Bus Power System

We illustrate the results in this paper on a single-machine infinite-bus power system with static VAR compensator [SL10]. The control system $\Sigma=\left(\mathbb{R}^3,\mathsf{U},\mathcal{U},f\right)$ with f of the form:

$$\begin{array}{rcl} f_1(x,u) &=& x_2,\\ f_2(x,u) &=& -\dfrac{\omega_0}{H}E_q'V_sy_{svc0}\sin(x_1+\delta_0)-\dfrac{D}{H}x_2+\dfrac{\omega_0}{H}P_m-\dfrac{\omega_0}{H}E_q'V_s\sin(x_1+\delta_0)x_3,\\ f_3(x,u) &=& -\dfrac{1}{T_{svc}}x_3+\dfrac{1}{T_{svc}}u, \end{array} \tag{4.1}$$

models a single-machine infinite-bus (SIMB) electrical power system with static VAR compensator (SVC). In the mentioned model, x_1 is the deviation of the generator rotor angle, x_2 is the relative speed of the generator rotor, x_3 is the deviation of the susceptance of the overall system, δ_0 is the operating point of the generator rotor angle, ω_0 is the operating point of the speed of the generator rotor, H is the inertia constant, P_m is the mechanical power on the generator shaft, D is the damping coefficient, E'_q is the inner generator voltage, V_s is the infinite bus voltage, y_{svc0} is the operating point of the susceptance of the overall system, T_{svc} is the time constant of SVC regulator, and u is the input of SVC regulator. We assume that $\sin(x_1+\delta_0)$ is nonzero over the domain of the interest.

A typical specification for an infinite-bus generator is to recover frequency after a fault as well as maintaining the voltage at the terminals within an acceptable range. Even though recovering frequency can be seen as an output regulation problem, the terminal voltage is a nonlinear function of the state variables thus making this control problem quite challenging. However, these kind of control problems can easily be solved by using symbolic control methods [GPT09, Gir10, CGG11], which, however, apply only to incrementally stable systems, and benefit from using incremental Lyapunov functions. Hence, in this example we use the results presented in this paper to render the infinite-bus generator incrementally stable as well as constructing an incremental Lyapunov function.

The control system (4.1) is of the form (3.10). The coordinate transformation (3.11), given by:

$$\begin{bmatrix}\eta_1\\ \eta_2\\ \eta_3\end{bmatrix} = \varphi(\xi) = \begin{bmatrix}\xi_1\\ \xi_2\\ -\frac{\omega_0}{H}E'_qV_s\sin(\delta_0+\xi_1)\xi_3\end{bmatrix}, \tag{4.2}$$

transforms the control system $\Sigma = \left(\mathbb{R}^3, \mathsf{U}, \mathcal{U}, f\right)$ to the control system $\Sigma' = \left(\mathbb{R}^3, \mathsf{U}, \mathcal{U}, f'\right)$ with $f' = \varphi_* f$ of the form:

$$\begin{aligned} f'_1(y,u) &= h'_1(y_1) + y_2 = y_2,\\ f'_2(y,u) &= h'_2(y_1,y_2) + y_3 = -\frac{D}{H}y_2 + \frac{\omega_0}{H}P_m - \frac{\omega_0}{H}E'_qV_sy_{svc0}\sin(y_1+\delta_0) + y_3,\\ f'_3(y,u) &= h'_3(y) + g'(y)u = y_2\cot(y_1+\delta_0)y_3 - \frac{1}{T_{svc}}y_3 - \frac{\omega_0}{HT_{svc}}E'_qV_s\sin(y_1+\delta_0)u. \end{aligned} \tag{4.3}$$

By using the results in Theorem 3.1 for a control system of the form (4.3) and for $\lambda = 2$, we have:

$$\begin{aligned} \phi_1(y_1) &= -2y_1,\\ \phi_2(y_1,y_2) &= -5y_1 + \frac{\omega_0}{H}E'_qV_sy_{svc0}\sin(y_1+\delta_0) - \frac{\omega_0}{H}P_m + \left(\frac{D}{H}-4\right)y_2,\\ k_3(y) &= -12y_1 + \left(\frac{D}{H}-6\right)y_3 + \left(\frac{D}{H}-6\right)\frac{\omega_0}{H}P_m + \left(6-\frac{D}{H}\right)\frac{\omega_0}{H}E'_qV_sy_{svc0}\sin(y_1+\delta_0)\\ &\quad + \frac{\omega_0}{H}E'_qV_sy_{svc0}\cos(y_1+\delta_0)y_2 + \left(6\frac{D}{H}-\frac{D^2}{H^2}-14\right)y_2. \end{aligned}$$

Therefore, the state feedback control law:

$$\begin{aligned} k(y,\widehat{u}) &= \frac{1}{g'(y)}\left[k_3(y) - h'_3(y)\right] + \frac{1}{g'(y)}\widehat{u}\\ &= -\frac{HT_{svc}}{\omega_0E'_qV_s\sin(\delta_0+y_1)}\Bigg[-12y_1 + \left(\frac{D}{H}-6+\frac{1}{T_{svc}}\right)y_3\\ &\quad + \left(6-\frac{D}{H}\right)\frac{\omega_0}{H}E'_qV_sy_{svc0}\sin(y_1+\delta_0) + \frac{\omega_0}{H}E'_qV_sy_{svc0}\cos(y_1+\delta_0)y_2\\ &\quad + \left(6\frac{D}{H}-\frac{D^2}{H^2}-14\right)y_2 + \left(\frac{D}{H}-6\right)\frac{\omega_0}{H}P_m - y_2\cot(y_1+\delta_0)y_3\Bigg] - \frac{HT_{svc}\widehat{u}}{\omega_0E'_qV_s\sin(\delta_0+y_1)}, \end{aligned} \tag{4.4}$$

makes the control system Σ' $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$. The corresponding $\delta_\exists$-ISS Lyapunov function for the control system (4.3) is given by:

$$\begin{aligned}\widehat{V}(y,y') &= \Bigg[(y_1-y_1')^2+\left(2(y_1-y_1')+(y_2-y_2')\right)^2+\Bigg[(y_3-y_3')+\left(4-\frac{D}{H}\right)(y_2-y_2')\\&\quad+5(y_1-y_1')-\frac{\omega_0}{H}E_q'V_s y_{svc0}(\sin(y_1+\delta_0)-\sin(y_1'+\delta_0))\Bigg]^2\Bigg]^{\frac{1}{2}}.\end{aligned}$$

By using Theorem 3.4, the state feedback control law (4.4), and the coordinate transformation (4.2), we obtain the state feedback control law $k(\varphi(x),\widehat{u})$ making Σ $\delta_\exists$-ISS with respect to the input $\widehat{\upsilon}$. The corresponding $\delta_\exists$-ISS Lyapunov function for the control system Σ is given by:

$$\begin{aligned}\widetilde{V}(x,x') &= \Bigg[(x_1-x_1')^2+\left(2(x_1-x_1')+(x_2-x_2')\right)^2+\Bigg[-\frac{\omega_0}{H}E_q'V_s\Big(\sin(\delta_0+x_1)x_3\\&\quad-\sin(\delta_0+x_1')x_3'\Big)+\left(4-\frac{D}{H}\right)(x_2-x_2')+5(x_1-x_1')-\frac{\omega_0}{H}E_q'V_s y_{svc0}(\sin(x_1+\delta_0)\\&\quad-\sin(x_1'+\delta_0))\Bigg]^2\Bigg]^{\frac{1}{2}}.\end{aligned}$$

We simulate the closed-loop system with $\widehat{u}=0$, and the following parameters: $\omega_0=314.159$, $H=5.9$, $E_q'=1$, $V_s=1$, $y_{svc0}=0.814$, $\delta_0=57.3°$, $D=5$, $P_m=1$, and $T_{svc}=0.02$. In Figure 1, we show the closed-loop trajectories stemming from the initial conditions $(0.2217,\ -4.159,\ 0.086)$, $(0.0471,\ -4.159,\ -0.214)$, and $(-0.3019,\ 2.841,\ 0.186)$, respectively.

5. Discussion

In this paper we introduced coordinate-invariant notions of incremental Lyapunov function and provided the description of incremental stability, defined in [ZT11], in terms of existence of proposed incremental Lyapunov functions. Furthermore, we developed a backstepping procedure to design controllers, enforcing incremental stability, for strict-feedback form systems. The proposed backstepping procedure provides a recursive way of constructing controllers as well as incremental Lyapunov functions. An example was provided to illustrate the proposed technique. The authors are currently trying to extend the synthesis results in this paper to more general class of control systems and develop new synthesis tools such as control-Lyapunov functions.

References

[Ang02] D. Angeli. A Lyapunov approach to incremental stability properties. *IEEE Transactions on Automatic Control*, 47(3):410–21, 2002.

[AS99] D. Angeli and E. D. Sontag. Forward completeness, unboundedness observability, and their Lyapunov characterizations. *Systems and Control Letters*, 38:209–217, 1999.

[CGG11] J. Camara, A. Girard, and G. Gossler. Synthesis of switching controllers using approximately bisimilar multiscale abstractions. *in Proc. of 14th Int. Conf. Hybrid Systems: Computation and Control (HSCC)*, April 2011.

[FCPL10] A. Franci, A. Chaillet, and W. Pasillas-Lepine. Phase-locking between kuramoto oscillators: robustness to time-varying natural frequencies. *in Proceedings of the 49th IEEE Conference on Decision and Control*, pages 1587–1592, December 2010.

[Gir10] A. Girard. Controller synthesis for safety and reachability via approximate bisimulation. *Submitted for publication*, arXiv: 1010.4672., October 2010.

[GPT09] A. Girard, G. Pola, and P. Tabuada. Approximately bisimilar symbolic models for incrementally stable switched systems. *IEEE Transactions on Automatic Control*, 55(1):116–126, January 2009.

[JFA+07] A. A. Julius, G. E. Fainekos, M. Anand, I. Lee, and G. J. Pappas. Robust test generation and coverage for hybrid systems. *in Proc. of 10th Int. Conf. Hybrid Systems: Computation and Control (HSCC)*, April 2007.

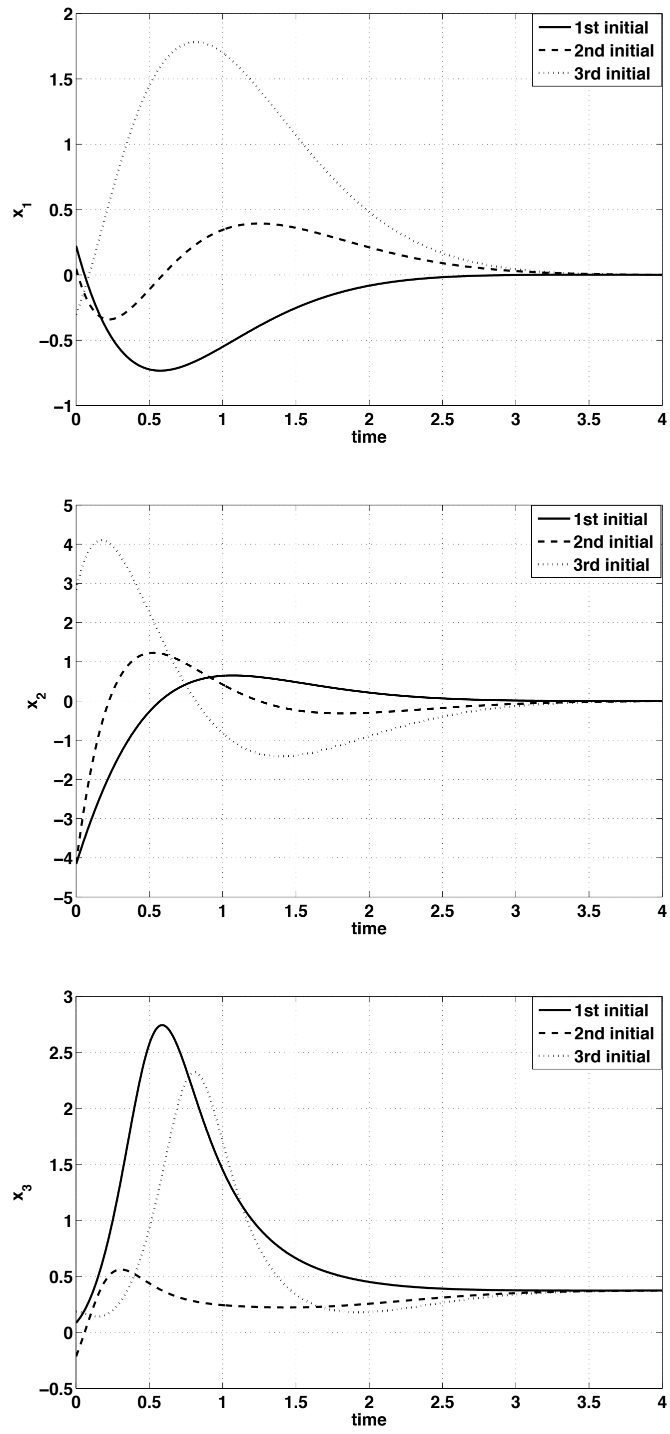

FIGURE 1. Evolution of x_1, x_2, and x_3 with initial conditions $(0.2217,\ -4.159,\ 0.086)$, $(0.0471,\ -4.159,\ -0.214)$, and $(-0.3019,\ 2.841,\ 0.186)$, respectively.

[JP09] A. A. Julius and G. J. Pappas. Approximations of stochastic hybrid systems. *IEEE Transaction on Automatic Control*, 54(6):1193–1203, 2009.

[KDL+08] J. Kapinski, A. Donze, F. Lerda, H. Maka, S. Wagner, and B. H. Krogh. Control software model checking using bisimulation functions for nonlinear systems. *in Proceedings of the 47th IEEE Conference on Decision and Control*, pages 4024–4029, 2008.

[Kha96] H. K. Khalil. *Nonlinear systems*. Prentice-Hall, Inc., New Jersey, 2nd edition, 1996.

[KKK95] M. Krstic, I. Kanellakopoulos, and P. P. Kokotovic. *Nonlinear and adaptive control design*. John Wiley and Sons, 1995.

[PW96] L. Praly and Y. Wang. Stabilization in spite of matched unmodeled dynamics and an equivalent definition of input-to-state stability. *Math. Control Signal Systems*, 9:1–33, 1996.

[SL10] L. Y. Sun and Y. Liu. Nonlinear adaptive backstepping controller design for static var compensator. *in Proceedings of Chinese Control and Decision Conference*, pages 3013–3018, 2010.

[Son98] E. D. Sontag. *Mathematical control theory*, volume 6. Springer-Verlag, New York, 2nd edition, 1998.

[Zam63] G. Zames. Functional analysis applied to nonlinear feedback systems. *IEEE Transaction on Circuit Theory*, 10(3):392–404, September 1963.

[ZPJT10] M. Zamani, G. Pola, M. Mazo Jr., and P. Tabuada. Symbolic models for nonlinear control systems without stability assumptions. *Provisionally accepted for publication in IEEE Transaction on Automatic Control*, arXiv:1002.0822, February 2010.

[ZT10] M. Zamani and P. Tabuada. Towards backstepping design for incremental stability. *in Proceedings of the 49th IEEE Conference on Decision and Control*, pages 2426–2431, 2010.

[ZT11] M. Zamani and P. Tabuada. Backstepping design for incremental stability. *IEEE Transaction on Automatic Control*, 56(9), 2011.

[1]Department of Electrical Engineering, University of California at Los Angeles, Los Angeles, CA 90095

E-mail address: zamani@ee.ucla.edu

URL: http://www.ee.ucla.edu/~zamani

[2]Max Planck Institute for Software Systems, Kaiserslautern, Germany and Department of Computer Science, University of California at Los Angeles, Los Angeles, CA 90095

E-mail address: rupak@mpi-sws.org

URL: http://www.cs.ucla.edu/~rupak