RESONANCE THEORY FOR PERTURBED HILL OPERATOR

EVGENY KOROTYAEV

Abstract. We consider the Schrödinger operator $Hy = -y'' + (p+q)y$ with a periodic potential p plus a compactly supported potential q on the real line. The spectrum of H consists of an absolutely continuous part plus a finite number of simple eigenvalues below the spectrum and in each spectral gap $\gamma_n \neq \emptyset, n \geqslant 1$. We prove the following results: 1) the distribution of resonances in the disk with large radius is determined, 2) the asymptotics of eigenvalues and antibound states are determined at high energy gaps, 3) if H has infinitely many open gaps in the continuous spectrum, then for any sequence $(\varkappa)_1^\infty, \varkappa_n \in \{0,2\}$, there exists a compactly supported potential q with $\int_{\mathbb{R}} q dx = 0$ such that H has $\varkappa_n$ eigenvalues and $2 - \varkappa_n$ antibound states (resonances) in each gap γ_n for n large enough.

1. Introduction

Consider the Schrödinger operator H acting in $L^2(\mathbb{R})$ and given by

$$H = H_0 + q, \qquad \text{where} \qquad H_0 = -\frac{d^2}{dx^2} + p.$$

We assume that $p \in L^2(0,1)$ is a real 1-periodic potential, and q is a real compactly supported potential and belongs to the class $\mathcal{Q}_t^r$ given by

$$\mathcal{Q}_t^r = \left\{ q \in L^r(\mathbb{R}) : \ \operatorname{supp} q \subset [0,t] \right\}, \quad t > 0, \quad r \geqslant 1.$$

The spectrum of H_0 is absolutely continuous and consists of spectral bands $\mathfrak{S}_n$ separated by gaps γ_n, which are given by (see Fig. 1)

$$\sigma(H_0) = \sigma_{ac}(H_0) = \cup_{n \geqslant 1} \mathfrak{S}_n,$$

$$\mathfrak{S}_n = [E_{n-1}^+, E_n^-], \quad \gamma_n = (E_n^-, E_n^+), \quad n \geqslant 1, \quad \text{and} \quad E_0^+ < .. \leqslant E_{n-1}^+ < E_n^- \leqslant E_n^+ < ... \ .$$

We assume that $E_0^+ = 0$. The bands $\mathfrak{S}_n, \mathfrak{S}_{n+1}$ are separated by a gap $\gamma_n = (E_n^-, E_n^+)$. If a gap degenerates, that is $\gamma_n = \emptyset$, then the corresponding bands $\mathfrak{S}_n$ and $\mathfrak{S}_{n+1}$ merge. Here $E_n^\pm$ is the eigenvalue of the boundary value problem

$$-y'' + p(x)y = \lambda y, \quad \lambda \in \mathbb{C}, \qquad y(x+2) = y(x), x \in \mathbb{R}. \tag{1.1}$$

If $E_n^- = E_n^+$ for some n, then this number $E_n^\pm$ is the double eigenvalue of the problem (1.1). The lowest eigenvalue $E_0^+ = 0$ is always simple and the corresponding eigenfunction is 1-periodic. The eigenfunctions, corresponding to the eigenvalue $E_{2n}^\pm$, are 1-periodic, and for the case $E_{2n+1}^\pm$ they are anti-periodic, i.e., $y(x+1) = -y(x), \quad x \in \mathbb{R}$.

It is well known, that the spectrum of H consists of an absolutely continuous part $\sigma_{ac}(H) = \sigma(H_0)$ plus a finite number of simple eigenvalues, both in each gap $\gamma_n \neq \emptyset, n \geqslant 1$ and in the

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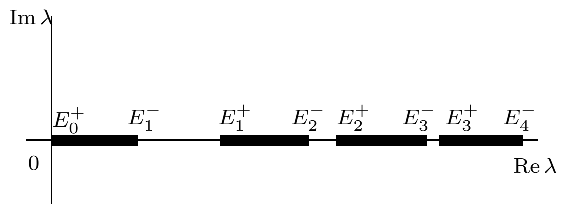

FIGURE 1. The cut domain $\mathbb{C}\setminus\cup\mathfrak{S}_n$ and the cuts (bands) $\mathfrak{S}_n=[E_{n-1}^+,E_n^-], n\geqslant 1$

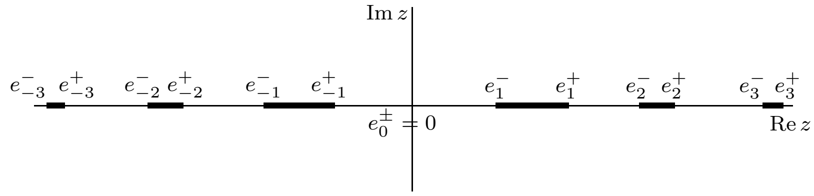

FIGURE 2. The cut domain $\mathcal{Z}=\mathbb{C}\setminus\cup\overline{g}_n$ and the cuts $g_n=(e_n^-,e_n^+)$ in the z-plane.

half-line $(-\infty,E_0^+)$, see [Rb], [F1]. Moreover, in a remote open gap γ_n the operator H has at most two eigenvalues [Rb] and precisely one eigenvalue in the case $\int_{\mathbb{R}}q(x)dx\neq 0$ [Zh1], [F2]. Note that the potential q in [Rb],[Zh1] belongs to the more general class, see also [F3]-[F4], [GS], [So], [Zh2], [Zh3].

The resonance theory for the multidimensional Schrödinger operator with a periodic potential plus a real compactly supported potential has been much less studied, see [D], [G] and references therein. Some results for the case of a slowly varying perturbations of a 1D periodic Schrödinger operator have been announced in [KM].

Introduce the two-sheeted Riemann surface Λ obtained by joining the upper and lower rims of two copies of the cut plane $\mathbb{C}\setminus\sigma_{ac}(H_0)$ in the usual (crosswise) way. The n-th gap on the first physical sheet Λ_1 we will denote by $\gamma_n^{(1)}$ and whereas the same gap on the second nonphysical sheet Λ_2 we will denote by $\gamma_n^{(2)}$. Let γ_n^c be the union of $\overline{\gamma}_n^{(1)}$ and $\overline{\gamma}_n^{(2)}$, i.e.,

$$\gamma_n^c=\overline{\gamma}_n^{(1)}\cup\overline{\gamma}_n^{(2)}.$$

In what follows we will use the momentum variable $z=\sqrt{\lambda},\lambda\in\Lambda$ and the corresponding Riemann surface $\mathcal{M}$, which is more convenient for us, than the Riemann surface Λ. The mapping $\lambda\mapsto z=\sqrt{\lambda}$ is a bijection between the cut Riemann surface $\Lambda\setminus\cup\gamma_n^c$ and the cut momentum domain $\mathcal{Z}$ (see Fig.2.) given by

$$\mathcal{Z}=\mathbb{C}\setminus\bigcup_{n\neq 0}\overline{g}_n,\quad \text{where}\ \ g_n=(e_n^-,e_n^+),\quad e_n^\pm=-e_{-n}^\mp=\sqrt{E_n^\pm}>0,\quad n\geqslant 1. \tag{1.2}$$

Here $\mathbb{R}\setminus\bigcup_{n\neq 0}\overline{g}_n$ is the momentum spectrum and $g_n\neq\emptyset$ is the momentum gap. Slitting the n-th momentum gap g_n (suppose it is nontrivial) we obtain a cut g_n^c with an upper rim g_n^+ and lower rim g_n^-. Below we will identify this cut g_n^c and the union of of the upper rim (gap)

$\overline{g}_n^+$ and the lower rim (gap) $\overline{g}_n^-$, i.e.,

$$g_n^c = \overline{g}_n^+ \cup \overline{g}_n^-. \tag{1.3}$$

In order to construct the Riemann surface $\mathcal{M}$ we take the cut domain $\mathcal{Z}$ and identify (i.e. we glue) the upper rim g_n^+ of the cut g_n^c with the upper rim g_{-n}^+ of the cut g_{-n}^c and correspondingly the lower rim g_n^- of the cut g_n^c with the lower rim g_{-n}^- of the cut g_{-n}^c for all nontrivial gaps. The mapping $\lambda \mapsto z = \sqrt{\lambda}$ from Λ onto $\mathcal{M}$ is one-to-one and onto and we have the following.

1) The physical gap $\gamma_n^{(1)} \subset \Lambda_1$ is mapped onto the physical "gap" (the upper rim) $g_n^+ \subset \mathcal{M}_1$ and the half-line $(-\infty, 0) \subset \Lambda_1$ is mapped onto $i\mathbb{R}_+$.

2) The nonphysical gap $\gamma_n^{(2)} \subset \Lambda_2$ is mapped onto the nonphysical "gap" (the lower rim) $g_n^- \subset \mathcal{M}_2$ and the half-line $(-\infty, 0) \subset \Lambda_2$ is mapped onto $i\mathbb{R}_-$.

3) $\mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}$ plus all physical gaps g_n^+ is a so-called physical "sheet" $\mathcal{M}_1$.

4) $\mathbb{C}_- = \{z : \operatorname{Im} z < 0\}$ plus all nonphysical gaps g_n^- is a so-called nonphysical "sheet" $\mathcal{M}_2$.

5) The momentum spectrum $\sigma_M = \mathbb{R} \setminus \cup [e_n^-, e_n^+]$ joints the first $\mathcal{M}_1$ and second sheets $\mathcal{M}_2$.

Note that if $p = 0$, then Λ is a Riemann surface of the function $\sqrt{\lambda}$, $\mathcal{M} = \mathbb{C}$ is the momentum plane, $\mathcal{M}_1 = \mathbb{C}_+$ is the physical "sheet" and $\mathcal{M}_2 = \mathbb{C}_-$ is the nonphysical "sheet".

We introduce the determinant

$$D(z) = \det(I + q(H_0 - z^2)^{-1}), \qquad z \in \mathbb{C}_+,$$

which is analytic in $\mathbb{C}_+$ and continuous up to $\mathbb{R} \setminus \{z : z = e_n^\pm, n \in \mathbb{Z}\}$, where $e_0^\pm = 0$, see [F4], [F1]. It is well known that if $D(z) = 0$ for some zero $z \in \mathcal{M}_1$, then z^2 is an eigenvalue of H and $z \in \cup_{n \neq 0} g_n^+$ or $z \in i\mathbb{R}_+$. We introduce our basic function ξ by

$$\xi(z) = 2i \sin k(z) D(z), \qquad z \in \mathbb{C}_+. \tag{1.4}$$

Here $k(z)$ is the quasimomentum for the operator H_0 introduced by Firsova [F3] and Marchenko-Ostrovski [MO], see Section 2 for a precise definition of k. In Section 2 we describe the properties of the function k, which is analytic in $z \in \mathcal{Z}$. Moreover, we show that $\sin k(z), z \in \mathcal{Z}$ is analytic in $\mathcal{M}$ and $\mathcal{M}$ is the Riemann surface of $\sin k(z)$. All zeros of $\sin k(z), z \in \mathcal{M}$ have the form $e_n^\pm, n \in \mathbb{Z}$, where $e_0^\pm = 0$. In Theorem 1.1 we will show that ξ has an analytic extension from $\mathbb{C}_+$ into the Riemann surface $\mathcal{M}$.

Definition S. Let $\zeta \in \mathcal{M}$ be a zero of $\xi(z), z \in \mathcal{M}$ and assume that $\zeta \neq e_n^+$ for any $e_n^+ = e_n^-, n \neq 0$.

1) If $\zeta \in i\mathbb{R}_+$ or $\zeta \in \bigcup\limits_{n \neq 0} g_n^+$, we call ζ a *bound state.*

2) If $\zeta \in \mathcal{M}_2$ and $\zeta \neq e_n^\pm, n \in \mathbb{Z}$, we call ζ a *resonance.*

3) Let $e_0^\pm = 0$. If $\zeta = 0$ or $\zeta = e_n^\pm$ for the open gap $|g_n| > 0, n \neq 0$, we call ζ a *virtual state.*

4) A point $\zeta \in \mathcal{M}$ is called a *state* if it is either a bound state or a resonance or a virtual state. We denote by $\mathfrak{S}_{st}(H)$ the set of all states. If $\zeta \in g_n^-, n \neq 0$ or $\zeta \in i\mathbb{R}_-$, then we call ζ an *antibound state.*

5) The *multiplicity* of a bound state, a resonance or the point 0 is the multiplicity of the corresponding zero. The *multiplicity* of the virtual state $\zeta \neq 0$ is the multiplicity of the zero $z = 0$ of the function $\xi(\zeta + z^2)$. A state with multiplicity one is called simple.

Of course, z^2 is really the energy, but since the momentum z is the natural parameter, we will abuse the terminology.

We recall the results about the resonances from [F1]:

1) Let $q = 0$. Thus we have $D = 1$ and $\xi(z) = 2i \sin k(z)$, $z \in \mathcal{Z}$. Then the operator H_0 has only virtual states $e_n^\pm$ for all $e_n^- \neq e_n^+, n \neq 0$ and $e_0^+ = 0$. There are no other states.

2) Let a gap $g_n = \emptyset$ for some $n \neq 0$. Then for any $h \in C_0^\infty(\mathbb{R}), h \neq 0$ the function $((H - z^2)^{-1}h, h)$ is analytic at the point $e_n^+ = e_n^- \in \mathcal{Z}$. Roughly speaking there is no difference between these points and other points inside the spectrum $\sigma_{ac}(H)$. The point $e_n^+ = e_n^-$ is not the state.

3) If $\int_\mathbb{R} q(x)dx \neq 0$, then H has precisely one bound state on each open physical gap and an odd number $\geqslant 1$ of antibound states on each open non-physical gap for n large enough.

Define the coefficients for all $n \geqslant 1$ by

$$\widehat{q}_0 = \int_\mathbb{R} q(x)dx, \quad \widehat{q}_n = \widehat{q}_{cn} + i\widehat{q}_{sn}, \quad \widehat{q}_{cn} = \int_\mathbb{R} q(x)\cos 2\pi nx dx, \quad \widehat{q}_{sn} = \int_\mathbb{R} q(x)\sin 2\pi nx dx. \quad (1.5)$$

Let $\mu_n^2, n \geqslant 1$ be eigenvalues and y_n be the corresponding eigenfunctions of the Sturm-Liouville problem

$$-y_n'' + py_n = \mu_n^2 y_n, \qquad y_n(0) = y_n(1) = 0, \quad (1.6)$$

on the interval $[0, 1]$. It is well known that each $\mu_n^2 \in [E_n^-, E_n^+]$ for all $n \geqslant 1$.

In order to formulate Theorem 1.1 we define c_n, s_n the angles $\phi_n \in [0, 2\pi)$

$$c_n = \cos\phi_n, \quad s_n = \sin\phi_n \in [-1, 1], \qquad \phi_n \in [0, 2\pi) \quad (1.7)$$

by the identities

$$\frac{E_n^- + E_n^+}{2} - \mu_n^2 = \frac{|\gamma_n|}{2}c_n, \quad |1 - c_n^2|^{\frac{1}{2}} \operatorname{sign}(|y_n'(1)| - 1) = s_n, \quad \text{if} \quad |\gamma_n| > 0, \quad (1.8)$$

where all eigenfunctions satisfy $y_n'(0) = 1$. We describe our first main results about states.

Theorem 1.1. *Let potentials $(p, q) \in L^2(0, 1) \times \mathcal{Q}_t^1, t > 0$. Then we have*

i) ξ has an analytic extension from $\mathbb{C}_+$ into the Riemann surface $\mathcal{M}$ and the function $J(z) = \operatorname{Re}\xi(z), z \in \sigma_M = \mathbb{R} \setminus \cup[e_n^-, e_n^+]$ has an analytic extension into the whole plane $\mathbb{C}$.

ii) There exist an even number $\geqslant 0$ of states (counted with multiplicity) on each set $g_n^c \neq \emptyset, n \neq 0$, where g_n^c is a union of the physical gap $\overline{g}_n^{\,+} \subset \mathcal{M}_1$ and the non-physical gap $\overline{g}_n^{\,-} \subset \mathcal{M}_2$.

iii) There are no states in the "forbidden" domain $\mathcal{D} \subset \mathbb{C}_-$ given by

$$\mathcal{D} = \Big\{ z \in \mathbb{C}_- : |z| > \max\{180 e^{2\|p\|_1}, \ C_0 e^{2t|\operatorname{Im} z|}\} \Big\}, \quad C_0 = 12\|q\|_t e^{\|p\|_1 + \|q\|_t + 2\|p\|_t}, \quad (1.9)$$

where $\|q\|_t = \int_0^t |q(x)|dx$.

iv) In each $g_n^c \neq \emptyset, n \geqslant 1 + \frac{e^{t\pi/2}}{\pi}C_0$ there exist exactly two simple real states $z_n^\pm \in g_n^c$ such that $e_n^- \leqslant z_n^- < e_n < z_n^+ \leqslant e_n^+$ (for the definition of e_n see (2.10)) and satisfy

$$z_n^\pm = e_n^\pm \mp \frac{2|\gamma_n|}{(4\pi n)^3}(\mp\widehat{q}_0 - c_n\widehat{q}_{cn} + s_n\widehat{q}_{sn} + O(1/n))^2,$$

$$(-1)^n J(z_n^\pm) = \frac{|\gamma_n|}{(2\pi n)^2}(\mp\widehat{q}_0 - c_n\widehat{q}_{cn} + s_n\widehat{q}_{sn} + O(1/n)) \qquad \text{as} \qquad n \to \infty. \quad (1.10)$$

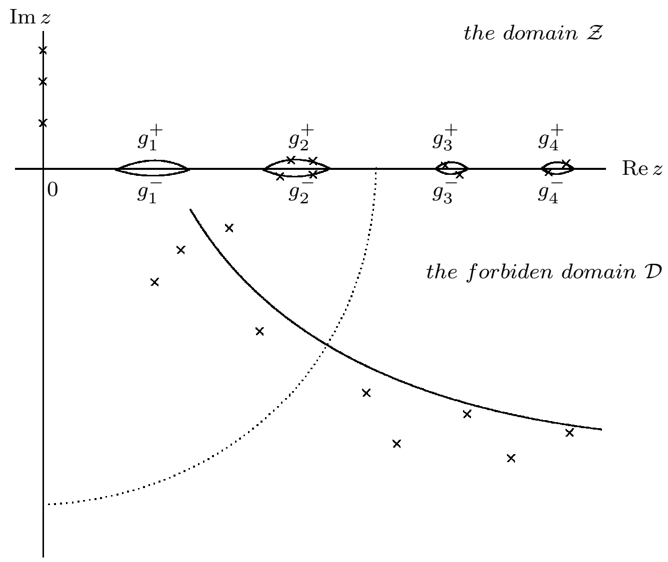

FIGURE 3. The bound states and resonances on the $\mathcal{Z}$ domain with the physical rims g_n^+ and the nonphysical rims g_n^-

Moreover, if a state $\zeta \in \{z_n^-, z_n^+\}$ satisfies $(-1)^n J(\zeta) > 0$, $(-1)^n J(\zeta) < 0$, $J(\zeta) = 0$), then ζ is a bound state, an antibound state or a virtual state correspondingly and, in particular,

$$\begin{aligned} &if \quad \widehat{q}_0 > 0 \ \Rightarrow \quad z_n^- \in \mathcal{M}_1 \ is\ bound\ state, \qquad z_n^+ \in \mathcal{M}_2 \ is\ antibound\ state, \\ &\qquad if \quad \widehat{q}_0 < 0 \ \Rightarrow \quad z_n^- \in \mathcal{M}_2 \ is\ antibound\ state, \qquad z_n^+ \in \mathcal{M}_1 \ is\ bound\ state. \end{aligned} \tag{1.11}$$

Remark. 1) The forbidden domain $\mathcal{D}$ is similar to the case $p = 0$, see [K2] and Fig. 3.

2) In the proof of Theorem 1.3 the estimates $z_n^- < e_n < z_n^+$ and asymptotics (1.10) are important.

Recall that p is even, i.e., $p \in L^2_{even}(0,1) = \{p \in L^2(0,1), p(x) = p(1-x), x \in (0,1)\}$ iff $\mu_n^2 \in \{E_n^-, E_n^+\}$ for all $n \geqslant 1$, see [GT], [KK1]. Note that $\mu_n^2 \in \{E_n^-, E_n^+\} \Leftrightarrow s_n = 0$.

Corollary 1.2. *Let $\widehat{q}_0 = 0$ and $\widehat{q}_n = |\widehat{q}_n| e^{i\tau_n}, n \geqslant 1$ for some $\tau_n \in [0, 2\pi)$. Assume that $|\cos(\phi_n + \tau_n)| > \varepsilon > 0$ and $|\widehat{q}_n| > n^{-\alpha}$ for n large enough and for some $\varepsilon, \alpha \in (0,1)$, where ϕ_n is defined by (1.8). Then we have*

i) The operator H has $\varkappa_n = 1 - \operatorname{sign}\cos(\phi_n + \tau_n)$ bound states in the physical gap $g_n^+ \neq \emptyset$ and $2 - \varkappa_n$ antibound states inside the nonphysical gap $g_n^- \neq \emptyset$ for n large enough.

ii) Let in addition, a real potential $V \in \mathcal{Q}_t^1$ and let $|\widehat{V}_n| = o(n^{-\alpha})$ as $n \to \infty$. Then the operator $H + V$ has $\varkappa_n$ bound states in the gap $g_n^+ \neq \emptyset$ and $2 - \varkappa_n$ antibound states inside the gap $g_n^- \neq \emptyset$ for n large enough.

iii) Let in addition, $p \in L^2_{even}(0,1)$. Then the following asymptotics hold true:

$$z_n^{\pm} = e_n^{\pm} \mp \frac{2|\gamma_n|}{(4\pi n)^3} \Big(c_n \widehat{q}_{cn} + \frac{O(1)}{n} \Big)^2 \qquad \text{as} \quad n \to \infty. \tag{1.12}$$

Moreover, if $|\widehat{q}_{cn}| > n^{-\alpha}$ for n large enough, then H has exactly $\varkappa_n = 1 - \operatorname{sign} c_n \widehat{q}_{cn}$ bound states in each open gap g_n^+ and $2 - \varkappa_n$ antibound inside gap $g_n^- \neq \emptyset$ for n large enough.

Remark. Let all conditions in Corollary 1.2 iii) hold true and let $\widehat{q}_{cn} > n^{-\alpha}$ for all n large enough. Then
if $\mu_n = e_n^-$, then H has exactly 2 bound states in the open gap g_n^+ for n large enough,
if $\mu_n = e_n^+$, then H has not bound states in the open gap g_n^+ for n large enough.

Let $\#(H, r, X)$ be the total number of state of H in the set $X \subseteq \mathcal{M}$ having modulus $\leqslant r$, each state being counted according to its multiplicity.

Theorem 1.3. *Let the real potential $q \in L^2(\mathbb{R})$ and let $[0, t]$ be the convex hull of the support of q for some $t > 0$. Then the following asymptotics hold true:*

$$\#(H, r, \mathbb{C}_-) = r\frac{2t + o(1)}{\pi} \quad as \quad r \to \infty. \tag{1.13}$$

Remark. 1) The first term in (1.13) does not depend on the periodic potential p.

2) The distribution of the resonances for the case $p \neq \text{const}$ in the domain $\mathbb{C}_-$ is similar to the case $p = 0$, obtained by Zworski [Z1].

3) In the proof of (1.13) we use the Paley-Wiener type Theorem from [Fr], the Levinson Theorem (see Sect. 5) and a priori estimates from [KK], [MO].

Consider some inverse problems for the operator H.

Theorem 1.4. *i) Let the spectrum of the operator H_0 have infinitely many gaps $\gamma_n \neq \emptyset$ for some $p \in L^2(0, 1)$. Then for any sequence $(\varkappa_n)_1^\infty$, where $\varkappa_n \in \{0, 2\}$, there exists some potential $q \in \mathcal{Q}_1^1$ (defined by (5.9)) such that H has exactly $\varkappa_n$ bound states in each gap $g_n^+ \neq \emptyset$ and $2 - \varkappa_n$ antibound states inside each gap $g_n^- \neq \emptyset$ for n large enough.*
ii) Let $q \in \mathcal{Q}_t^1$ satisfy $\widehat{q}_0 = 0$ and let $|\widehat{q}_n| > n^{-\alpha}$ for all n large enough and some $\alpha \in (0, 1)$. Then for any sequences $(\varkappa_n)_1^\infty$, where $\varkappa_n \in \{0, 2\}$ and $(\delta_n)_1^\infty \in \ell^2$, where all $\delta_n \geqslant 0$ and infinitely many $\delta_n > 0$, there exists a potential $p \in L^2(0, 1)$ such that each gap length $|\gamma_n| = \delta_n, n \geqslant 1$. Moreover, H has exactly $\varkappa_n$ bound states in each physical gap $g_n^+ \neq \emptyset$ and $2 - \varkappa_n$ antibound states inside each non-physical gap $g_n^- \neq \emptyset$ for n large enough.

Remark . The proof of ii) is more difficult and here we use results from the inverse spectral theory from [K5].

A lot of papers are devoted to the resonances for the Schrödinger operator with $p = 0$, see [Fr], [H], [K1], [K2], [S], [Z1], [Z3] and references therein. Although resonances have been studied in many settings, but there are relatively few cases where the asymptotics of the resonance counting function are known, mainly the one dimensional case [Fr], [K1], [K2], [S], and [Z1]. We recall that Zworski [Z1] obtained the first results about the distribution of resonances for the Schrödinger operator with compactly supported potentials on the real line. The author obtained the characterization (plus uniqueness and recovering) of S-matrix for the Schrödinger operator with a compactly supported potential on the real line [K2] and the half-line [K1], see also [Z2], [BKW] about uniqueness.

For the Schrödinger operator on the half line the author [K3] obtained the following stability results.

(i) Let $z_n, n \geqslant 1$ be the sequence of all zeros (all states) of the Jost function for some real compactly supported potential q. Assume that $\sum_{n \geqslant 1} n^3 |z_n - \widetilde{z}_n|^2 < \infty$ for some sequence $\widetilde{z}_n \in \mathbb{C}, n \geqslant 1$. Then $\widetilde{z}_n$ is the sequence of all zeroes of the Jost function for some unique real compactly supported potential.

(ii) The measure associated with the zeros of the Jost function is the Carleson measure, and the sum $\sum(1+|z_n|)^{-\alpha}, \alpha>1$ is estimated in terms of the L^1-norm of the potential q.

Brown and Weikard [BW] considered the Schrödinger operator $-y''+(p_A+q)y$ on the half-line, where p_A is an algebro-geometric potentials and q is a compactly supported potential. They proved that the zeros of the Jost function determine q uniquely.

Christiansen [Ch] considered resonances associated to the Schrödinger operator $-y''+(p_S+q)y$ on the real line, where p_S is a step potential. She determined asymptotics of the resonance-counting function. Moreover, she proved that the resonances determine q uniquely.

The plan of the paper is as follows. In Section 2 we describe the preliminary results about fundamental solutions for the operator H_0. In Section 3 we describe the scattering for H, H_0. In Sections 4 we study the function ξ and the entire function $F=\zeta(z)\zeta(-z)$. Here it is important that ξ has a finite number zeros in $\mathbb{C}_+$. That makes possible to reformulate the problem for the differential operator theory as a problem of the entire function theory and the conformal mapping theory. Thus we should study the function F using various "geometric properties" of conformal mappings from [KK], [MO]. The properties of F are the key of the proof of main Theorems 1.1-1.4, given in Section 5.

2. The unperturbed operator H_0

2.1. Fundamental solutions. In order to describe the spectral properties of the operator H_0, we start from the properties of the canonical fundamental system ϑ, φ of the equation $-y''+py=z^2y, z\in\mathbb{C}$ with the initial conditions $\varphi'(0,z)=\vartheta(0,z)=1$ and $\varphi(0,z)=\vartheta'(0,z)=0$, where $u'=\partial_x u$. They satisfy the integral equations

$$\vartheta(x,z)=\cos zx+\int_0^x \frac{\sin z(x-s)}{z}p(s)\vartheta(s,z)ds,$$

$$\varphi(x,z)=\frac{\sin zx}{z}+\int_0^x \frac{\sin z(x-s)}{z}p(s)\varphi(s,z)ds. \quad (2.1)$$

For each $x\in\mathbb{R}$ the functions $\vartheta(x,z),\varphi(x,z)$ are entire in $z\in\mathbb{C}$ and satisfy

$$\max\{|z|_1|\varphi(x,z)|,\ |\varphi'(x,z)|, |\vartheta(x,z)|, \frac{1}{|z|_1}|\vartheta'(x,z)|\}\leqslant X,$$

$$|\varphi(x,z)-\frac{\sin zx}{z}|\leqslant \frac{X}{|z|_1^2}\|p\|_x, \quad |\vartheta(x,z)-\cos zx|\leqslant \frac{X}{|z|_1}\|p\|_x, \quad (2.2)$$

for all $(p,x,z)\in L^1_{loc}(\mathbb{R})\times\mathbb{R}\times\mathbb{C}$, see p. 13 in [PT], where

$$X=e^{|\operatorname{Im} z|x+\|p\|_x}, \qquad \|p\|_t=\int_0^t |p(s)|ds, \qquad |z|_1=\max\{1,|z|\}.$$

Let $\varphi(x,z,\tau)$ be the solutions of the equation with the parameter $\tau\in\mathbb{R}$

$$-\varphi''+p(x+\tau)\varphi=z^2\varphi, \quad \varphi(0,z,\tau)=0, \quad \varphi'(0,z,\tau)=1, \quad z\in\mathbb{C}. \quad (2.3)$$

This solution $\varphi(x,z,\tau)$ is expressed in terms of $\vartheta(x,\cdot),\varphi(x,\cdot)$ by

$$\varphi(x,\cdot,\tau)=\vartheta(\tau,\cdot)\varphi(x+\tau,\cdot)-\varphi(\tau,\cdot)\vartheta(x+\tau,\cdot). \quad (2.4)$$

The function $\varphi(1,z,x)$ for all $(x,z)\in\mathbb{R}\times\mathbb{C}$ satisfies the following identity (see [Tr])

$$\varphi(1,\cdot,\cdot)=\varphi(1,\cdot)\vartheta^2-\vartheta'(1,\cdot)\varphi^2+2\beta\varphi\vartheta=\varphi(1,\cdot)\psi_-\psi_+. \quad (2.5)$$

Below we need a solution of the equation $-y''+(p-z^2)y=f,\ y(0)=y'(0)=0$ given by

$$y=\int_0^x \varphi(x-\tau,z,\tau)f(\tau)d\tau. \tag{2.6}$$

Recall that μ_n^2 is the Dirichlet eigenvalue, defined by (1.6). The following asymptotics hold true as $n\to\infty$ (see [PT], [K5]):

$$\mu_n=\pi n+\varepsilon_n(p_{c0}-p_{cn}+O(\varepsilon_n)),\qquad \varepsilon_n=\frac{1}{2\pi n}, \tag{2.7}$$

$$e_n^\pm=\pi n+\varepsilon_n(p_0\pm|p_n|+O(\varepsilon_n)),\qquad p_n=\int_0^1 p(x)e^{-i2\pi nx}dx=p_{cn}-ip_{sn}. \tag{2.8}$$

2.2. The quasimomentum. In order to describe the quasimomentum from [F3], [MO] we start from the properties of the Lyapunov function defined by $\Delta(z)=\frac12(\varphi'(1,z)+\vartheta(1,z))$. We shortly describe the properties the Lyapunov function:

1) The function $\Delta(z)$ is entire and satisfies (due to the symmetry principle, since Δ is real on $\mathbb{R}$ and $i\mathbb{R}$)

$$\Delta(z)=\Delta(-z)=\overline{\Delta}(-\overline{z})=\overline{\Delta}(\overline{z}),\qquad z\in\mathbb{C}. \tag{2.9}$$

2) For each $n\in\mathbb{Z}$ there exists an unique point $e_n\in[e_n^-,e_n^+]$ such that

$$\Delta'(e_n)=0,\qquad (-1)^n\Delta(e_n)=\max_{\lambda\in[e_n^-,e_n^+]}|\Delta(z)|=\cosh h_n\geqslant 1,\qquad \text{for some } h_n\geqslant 0. \tag{2.10}$$

Recall that $g_n=(e_n^-,e_n^+)$. Note that if $g_n=\emptyset$, then $e_n=e_n^\pm$ and if $g_n\neq\emptyset$, then $e_n\in g_n$ and the point e_n is very close to the centrum of the gap g_n for n large enough, see (2.40).

3) $\Delta(e_n^\pm)=(-1)^n$ and $\Delta([e_n^+,e_{n+1}^-])=[-1,1]$ for all n.

We introduce the quasimomentum $k(\cdot)$ for H_0 by $k(z)=\arccos\Delta(z),\ z\in\mathcal{Z}=\mathbb{C}\setminus\cup\overline{g}_n$ and by the asymptotics:

$$k(z)=z+O(1/z)\quad as\quad |z|\to\infty. \tag{2.11}$$

The function $k(z)$ is analytic in $z\in\mathcal{Z}$. Moreover, the quasimomentum $k(\cdot)$ is a conformal mapping from $\mathcal{Z}$ onto the quasimomentum domain $\mathcal{K}$ (see Fig. 5 and [F3], [MO]) given by

$$\mathcal{K}=\mathbb{C}\setminus\cup\Gamma_n,\qquad \Gamma_n=(\pi n-ih_n,\pi n+ih_n).$$

Here Γ_n is the vertical cut with the height h_n, which is defined by (2.10). Moreover, we have $(nh_n)_{n\geqslant1}\in\ell^2$ iff $p\in L^2(0,1)$, see [MO], [K1], [K2]. The function $\sin k(\cdot)$ is analytic on $\mathcal{M}$.

We shortly recall properties of the quasimomentum $k(\cdot)$ from [MO] or [KK]:

Properties of the quasimomentum:

Here and below we rewrite the quasimomentum $k(\cdot)$ in terms of real functions $u(\cdot),v(\cdot)$ by

$$k(z)=u(z)+iv(z),\qquad z\in\mathcal{Z},$$

where u,v are harmonic in $\mathcal{Z}$. Moreover, $\pm v$ is positive in $\mathbb{C}_\pm$ and satisfies:

$$v(z)=\operatorname{Im}z\biggl(1+\frac1\pi\int_{\cup_{n\neq0}g_n}\frac{v(\tau)}{|\tau-z|^2}d\tau\biggr),\quad z\in\mathbb{C}_\pm. \tag{2.12}$$

1) $v(z)\geqslant\operatorname{Im}z>0$ and $v(z)=-v(\overline{z})$ for all $z\in\mathbb{C}_+$ and

$$k(z)=-k(-z)=\overline{k}(\overline{z})=-\overline{k}(-\overline{z}),\quad \forall\ z\in\mathcal{Z}, \tag{2.13}$$

$$(-1)^{n+1}i\sin k(z)=\sinh v(z)=\pm|\Delta^2(z)-1|^{\frac12}>0\quad all\quad z\in g_n^\pm. \tag{2.14}$$

2) $v(z)=0$ for all $z\in\sigma_n=[e_{n-1}^+,e_n^-],n\geqslant1$.

3) If some $g_n \neq \emptyset, n \geqslant 1$, then the function $v(z+i0) > 0$ and $v''(z+i0) < 0$ for all $z \in g_n$, and $v(z+i0)$ has a maximum at $e_n \in g_n$ (see (2.10) and Fig. 4) such that $v'(e_n) = 0$, and

$$v(z+i0) = -v(z-i0) > v_n(z) = |(z-e_n^-)(z-e_n^+)|^{\frac{1}{2}} > 0, \qquad \forall\ z \in g_n \neq \emptyset, \quad (2.15)$$

$$v(z+i0) = v_n(z)\biggl(1 + \frac{1}{\pi}\int_{\mathbb{R}\setminus g_n} \frac{v(t+i0)dt}{v_n(t)|t-z|}\biggr), \qquad z \in g_n, \quad (2.16)$$

$$|g_n| \leqslant 2h_n. \quad (2.17)$$

4) $u'(z) > 0$ on all (e_{n-1}^+, e_n^-) and $u(z) = \pi n$ for all $z \in g_n \neq \emptyset, n \in \mathbb{Z}$.
5) The function $k(z)$ maps a horizontal cut (a "gap") $[e_n^-, e_n^+]$ onto the vertical cut Γ_n and the momentum band $\sigma_n = [e_{n-1}^+, e_n^-]$ onto the segment $[\pi(n-1), \pi n]$ for all $n \in \mathbb{Z}$, i.e.,

$$k([e_n^-, e_n^+]) = \Gamma_n, \qquad k(\sigma_n) = [\pi(n-1), \pi n], \qquad n \in \mathbb{Z}. \quad (2.18)$$

6) The following identities hold true:

$$k(z) = z + \frac{1}{\pi}\int_{\cup g_n} \frac{v(t+i0)dt}{t-z}, \qquad z \in \overline{\mathbb{C}}_+ \setminus \cup \overline{g}_n. \quad (2.19)$$

2.3. The momentum Riemann surface $\mathcal{M}$. Recall that we will work with the momentum $z = \sqrt{\lambda}$, where $\lambda \in \Lambda$ is an energy. The function $\lambda \to z = \sqrt{\lambda}$ maps the cut Riemann surface $\Lambda \setminus \cup \gamma_n^c$ onto the cut momentum domain $\mathcal{Z}$ given by

$$\mathcal{Z} = \mathbb{C} \setminus \cup_{n\neq 0} \overline{g}_n, \quad g_n = (e_n^-, e_n^+) = -g_{-n}, \quad e_n^\pm = -e_{-n}^\mp = \sqrt{E_n^\pm} > 0, \quad n \geqslant 1. \quad (2.20)$$

Slitting the n-th nontrivial momentum gap g_n, we obtain a cut g_n^c with an upper g_n^+ and lower rim g_n^-. Below we will identify this cut g_n^c and the union of the upper rim (gap) $\overline{g}_n^+$ and the lower rim (gap) $\overline{g}_n{}^-$, i.e.,

$$g_n^c = \overline{g}_n^+ \cup \overline{g}_n^-, \quad \textit{where} \quad g_n^\pm = g_n \pm i0. \quad (2.21)$$

In order to construct the Riemann surface $\mathcal{M}$ we take the cut domain $\mathcal{Z}$ and identify (i.e. we glue) the upper rim g_n^+ of the cut g_n^c with the upper rim g_{-n}^+ of the cut g_{-n}^c and correspondingly the lower rim g_n^- of the cut g_n^c with the lower rim g_{-n}^- of the cut g_{-n}^c for all nontrivial gaps. The mapping $z = \sqrt{\lambda} : \Lambda \to \mathcal{M}$ is one-to-one and onto. The bounded physical gap $\gamma_n^{(1)} \subset \Lambda_1$ is mapped onto $g_n^+ \subset \mathcal{M}_1$ and the bounded nonphysical gap $\gamma_n^{(2)} \subset \Lambda_2$ is mapped onto $g_n^- \subset \mathcal{M}_2$. Moreover,

1) $\mathbb{C}_+$ plus all physical gaps g_n^+ is a so-called physical "sheet" $\mathcal{M}_1$,
2) $\mathbb{C}_-$ plus all nonphysical gaps g_n^- is a so-called non physical "sheet" $\mathcal{M}_2$.
3) The momentum bands $\sigma_n = [e_{n-1}^+, e_n^-], n \in \mathbb{Z}$ joint the first and second sheets.

For the construction of the Riemann surface $\mathcal{M}$ we need to write few simple remarks:

1) $\mathcal{M}$ is the Riemann surface of the function $\sin k(z) = \sqrt{1-\Delta^2(z)}$ and Λ is the Riemann surface of the function $\sin k(\sqrt{\lambda}) = \sqrt{1-\Delta^2(\sqrt{\lambda})}$.

2) It is important that (2.13) gives

$$\sin k(z+i0) = \sin k(-z+i0) = \sin(\pi n + iv) = (-1)^n i \sinh v, \quad v = \operatorname{Im} k(z) > 0, \quad \forall\ z \in g_n \neq \emptyset.$$

Due to this identity the upper rim g_n^+ of the cut g_n^c is glued with the upper rim g_{-n}^+ of the cut g_{-n}^c. Correspondingly the lower rim g_n^- of the cut g_n^c is glued with the lower rim g_{-n}^- of the cut g_{-n}^c. Due to these facts the function $\sin k(z), z \in \mathcal{Z}$ is analytic in $\mathcal{M}$ and $\mathcal{M}$ is the Riemann surface of $\sin k(z)$.

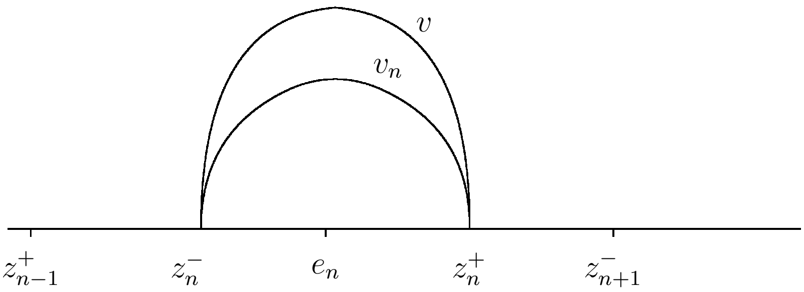

FIGURE 4. The graph of $v(z+i0),\ z \in [z_{n-1}^+, z_{n+1}^-]$ and $h_n = v(e_n + i0) > 0$

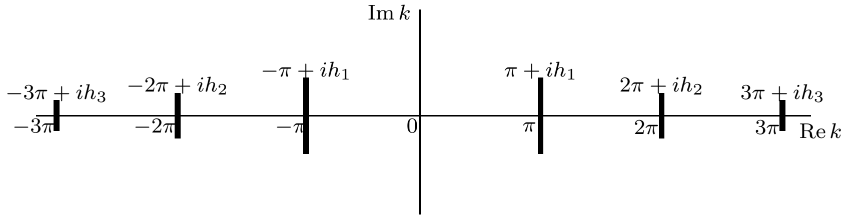

FIGURE 5. The domain $\mathcal{K} = \mathbb{C} \setminus \cup \Gamma_n$ with the cuts $\Gamma_n = (\pi n - ih_n, \pi n + ih_n)$

3) Let f be entire. The function $f(z), z \in \mathbb{C}$ is even, i.e., $f(z) = f(-z), z \in \mathbb{C}$, iff f is analytic on the Riemann surface $\mathcal{M} \neq \mathbb{C}$.

2.4. The Floquet solutions. The Floquet solutions $\psi_\pm(x, z), z \in \mathcal{Z}$ of H_0 are given by

$$\psi_\pm(x, z) = \vartheta(x, z) + m_\pm(z)\varphi(x, z), \quad m_\pm = \frac{\beta \pm i \sin k}{\varphi(1, \cdot)}, \qquad \beta = \frac{\varphi'(1, \cdot) - \vartheta(1, \cdot)}{2}, \tag{2.22}$$

where $\varphi(1, z)\psi_+(\cdot, z) \in L^2(\mathbb{R}_+)$ for all $z \in \mathbb{C}_+ \cup \cup_{n \neq 0} g_n^+$. If $p = 0$, then $k = z$ and $\psi_\pm(x, z) = e^{\pm izx}$. Substituting estimates (2.2) into (2.1) we obtain the standard asymptotics

$$\beta(z) = \int_0^1 \frac{\sin z(2x-1)}{z} p(x) dx + \frac{O(e^{|\operatorname{Im} z|})}{z^2},$$
$$\dot\beta(z) = \int_0^1 \frac{\cos z(2x-1)}{z} p(x)(2x-1) dx + \frac{O(e^{|\operatorname{Im} z|})}{z^2} \quad as \quad |z| \to \infty, \ here \ \ \dot\beta = \partial_z \beta. \tag{2.23}$$

The function $\sin k$ and each function $\varphi(1, \cdot)\psi_\pm(x, \cdot), x \in \mathbb{R}$ are analytic on the Riemann surface $\mathcal{M}$. Recall that the Floquet solutions $\psi_\pm(x, z), (x, z) \in \mathbb{R} \times \mathcal{M}$ satisfy (see [T])

$$\psi_\pm(0, z) = 1, \quad \psi_\pm(0, z)' = m_\pm(z), \quad \psi_\pm(1, z) = e^{\pm ik(z)}, \quad \psi_\pm(1, z)' = e^{\pm ik(z)} m_\pm(z), \tag{2.24}$$

$$\psi_\pm(x, z) = e^{\pm ik(z)x}(1 + O(1/z)) \quad as \quad |z| \to \infty, \quad z \in \mathcal{Z}_\varepsilon, \tag{2.25}$$

uniformly in $x \in \mathbb{R}$, where the set $\mathcal{Z}_\varepsilon$ is given by

$$\mathcal{Z}_\varepsilon = \{z \in \mathcal{Z} : \mathrm{dist}\{z, g_n\} > \varepsilon, g_n \neq \emptyset, n \in \mathbb{Z}\}, \quad \varepsilon > 0.$$

Below we need the simple identities

$$\beta^2 + 1 - \Delta^2 = 1 - \varphi'(1,\cdot)\vartheta(1,\cdot) = -\varphi(1,\cdot)\vartheta'(1,\cdot). \tag{2.26}$$

This yields

$$m_+(z)m_-(z) = -\frac{\vartheta'(1,z)}{\varphi(1,z)}, \qquad z \neq \mu_n. \tag{2.27}$$

Let $\mathbb{D}_r(z_0) = \{|z - z_0| < r\}$ be a disk for some $r > 0$. We need the following result (see [Zh3]), where $\mathscr{A}(z_0), z_0 \in \mathcal{M}$, denotes the set of functions analytic in some disc $\mathbb{D}_r(z_0), r > 0$.

Lemma 2.1. *i) The following asymptotics hold true:*

$$m_\pm(z) = \pm iz + O(1) \quad as \quad |z| \to \infty, \quad z \in \mathcal{Z}_\varepsilon, \varepsilon > 0. \tag{2.28}$$

ii) If $g_n = \emptyset$ for some $n \in \mathbb{Z}$, then the functions $\sin k, m_\pm$ are analytic in some disk $\mathbb{D}(\mu_n, \varepsilon) \subset \mathcal{Z}, \varepsilon > 0$. The functions $\sin k(z)$ and $\varphi(1,z)$ have the simple zero at μ_n and satisfies

$$m_\pm(\mu_n) = \frac{\dot\beta(\mu_n) \pm i(-1)^n \dot k(\mu_n)}{\dot\varphi(1,\mu_n)}, \quad \operatorname{Im} m_\pm(\mu_n) \neq 0. \tag{2.29}$$

iii) If the function m_+ has a pole at $\mu_n + i0$ for some $n \geqslant 1$, then $k(\mu_n + i0) = \pi n + ih_{sn}$ and

$$h_{sn} > 0, \quad \beta(\mu_n) = i \sin k(\mu_n + i0) = -(-1)^n \sinh h_{sn}, \quad m_+ \in \mathscr{A}(\mu_n - i0),$$

$$m_+(\mu_n + z) = \frac{\rho_n + O(z)}{z} \quad as\ z \to 0,\ z \in \mathbb{C}_+, \quad \rho_n = \frac{-2\sinh|h_{sn}|}{(-1)^n\dot\varphi(1,\mu_n)} < 0. \tag{2.30}$$

iv) If the function m_+ has a pole at $\mu_n - i0$ for some $n \geqslant 1$, then $k(\mu_n - i0) = \pi n + ih_{sn}$ and

$$h_{sn} < 0, \quad \beta(\mu_n) = -i \sin k(\mu_n - i0) = (-1)^n \sinh h_{sn},\ m_+ \in \mathscr{A}(\mu_n + i0),$$

$$m_+(\mu_n + z) = \frac{\rho_n + O(z)}{z} \quad as\ z \to 0,\ z \in \mathbb{C}_-. \tag{2.31}$$

v) Let $e_n^- < e_n^+$ for some $n \neq 0$. Then $\mu_n = e_n^-$ or $\mu_n = e_n^+$ iff

$$m_+(\mu_n + z) = \frac{\rho_n^\pm + O(z)}{\sqrt{z}} \quad as\ z \to 0,\ z \in \mathbb{C}_+, \quad \text{for some const } \rho_n^\pm \neq 0. \tag{2.32}$$

Proof. The results of this lemma is well-known. We will give a sketch.

i) The asymptotics follows from (2.23), (2.2), (2.11), see [T].

ii) If $g_n = (e_n^-, e_n^+) = \emptyset$, then due to (2.18) we have $k(e_n^\pm) = \pi n$ and the function $k(\cdot)$ is analytic at $e_n^- = e_n^+$. Moreover, (2.19) gives

$$k'(z) = 1 + \frac{1}{\pi}\int_{\cup g_n} \frac{v(t+i0)dt}{(t-z)^2} \geqslant 1, \qquad \text{at} \qquad z = e_n^\pm.$$

Thus this yields that the function $\sin k(z)$ is analytic at $z = e_n^\pm$ and $z = e_n^\pm$ is a simple zero of $\sin k(z)$. The point $\mu_n = e_n^\pm$ is a simple zero of $\varphi(1,z)$, since the point μ_n^2 is the Dirichlet eigenvalue, see (1.6). This implies the proof of ii).

iii) Let $g_n = (e_n^-, e_n^+) \neq \emptyset$ and let the function m_+ have a pole at $\mu_n + i0$. The point $\mu_n \in g_n$, then $k(\mu_n + i0) = \pi n + ih_{sn} \in \Gamma_n$ for some $h_{sn} > 0$, since k is the conformal mapping and $k(g_n) = \Gamma_n$ and $\mu_n \neq e_n^\pm$.

Due to (2.27), the function m_- is analytic at $\mu_n + i0$ and we get $m_-(\mu_n + i0) \neq 0$. Then $\beta(\mu_n) - i \sin k(\mu_n + i0) = 0$, which yields

$$\beta(\mu_n) = i \sin k(\mu_n + i0) = i \sin(\pi n + i h_{sn}) = \frac{(-1)^n}{2}(e^{-h_{sn}} - e^{h_{sn}}) = -(-1)^n \sinh h_{sn}.$$

Moreover, using similar arguments (analytic properties of $k(\cdot)$), we obtain $m_+(\mu - i0) = m_-(\mu + i0)$ for all $\mu \in (e_n^-, e_n^+)$, this gives $m_+ \in \mathscr{A}(\mu_n - i0)$.

Let $z \to 0,\ z \in \mathbb{C}_+$. Then

$$m_+(\mu_n + z) = \frac{\beta(\mu_n + z) + i \sin k(\mu_n + z)}{\varphi(1, \mu_n + z)} = \frac{-2(-1)^n \sinh h_{sn} + O(z)}{z(\dot\varphi(1, \mu_n) + O(z))} = \frac{\rho_n + O(z)}{z}.$$

Thus iii) has been proved.

The proof of iv) and v) is similar to the case of iii). ■

2.5. Properties of fundamental solutions. Let $\nu_n^2, n \geqslant 1$ be the Neumann spectrum of the equation $-y'' + py = \nu^2 y$ on the interval $[0, 1]$ with the boundary condition $y'(0) = y'(1) = 0$. It is well known that each $\nu_n^2 \in [E_n^-, E_n^+], n \geqslant 1$.

Lemma 2.2. *Let $p \in L^1(0, 1)$. Then*
i) The following asymptotics hold true uniformly in $z \in [e_n^-, e_n^+]$ as $n \to \infty$:

$$\varphi(1, z) = (-1)^n \frac{(z - \mu_n)}{\pi n}(1 + O(1/n)), \tag{2.33}$$

$$-\frac{\vartheta'(1, z)}{z^2} = (-1)^n \frac{(z - \nu_n)}{\pi n}(1 + O(1/n)). \tag{2.34}$$

ii) Let $z \in g_n \subset (8e^{\|p\|_1}, \infty)$. Then the following estimates hold true (here $\dot y = \partial_z y$):

$$|g_n|^2 \leqslant \frac{8e^{\|p\|_1}}{|z_n|} < 1, \tag{2.35}$$

$$|\dot\varphi(1, z)| \leqslant \frac{3e^{\|p\|_1}}{2|z|}, \qquad |\varphi(1, z)|_{z \in g_n} \leqslant |g_n| \frac{3e^{\|p\|_1}}{2|z|}, \tag{2.36}$$

$$|\dot\vartheta'(1, z)| \leqslant |z| \frac{3e^{\|p\|_1}}{2}, \qquad |\vartheta'(1, z)| \leqslant |g_n||z| \frac{3e^{\|p\|_1}}{2}, \tag{2.37}$$

$$|\dot\beta(z)| \leqslant \frac{3e^{\|p\|_1}}{2|z|}, \qquad |\beta(e_n^{\pm})| \leqslant |g_n| \frac{3e^{\|p\|_1}}{2}, \qquad |\beta(z)| \leqslant |g_n| \frac{9e^{\|p\|_1}}{4|z|}. \tag{2.38}$$

iii) In each disk $\mathbb{D}_{\frac{\pi}{4}}(\pi n) \subset \mathbb{D} = \{|z| > 32e^{2\|p\|_1}\}$ there exists exactly one momentum gap g_n. Moreover, if $g_n, g_{n+1} \subset \mathbb{D}$, then $e_{n+1}^- - e_n^+ \geqslant \pi$.

Proof. i) We have the Taylor formula $\varphi(1, z) = \dot\varphi(1, \mu_n)\tau + \ddot\varphi(1, \mu_n + \alpha\tau)\frac{\tau^2}{2}$ for any $z \in [e_n^-, e_n^+]$ and some $\alpha \in [0, 1]$, where $\tau = z - \mu_n$. Asymptotics (2.2) give $\dot\varphi(1, \mu_n) = 2(-1)^n \frac{(1+O(1/n))}{2\pi n}$ and $\ddot\varphi(1, \mu_n + \alpha\tau)\tau = O(n^{-2})$, which yields (2.33). Similar arguments imply (2.34).

ii) Using $|\Delta(z) - \cos z| \leqslant \frac{e^{\|p\|_1}}{|z|}$, for all $|z| \geqslant 2$, we obtain

$$\frac{h_n^2}{2} \leqslant \cosh h_n - 1 = |\Delta(z_n)| - 1 \leqslant \frac{e^{\|p\|_1}}{|z_n|}.$$

Then the estimate $|g_n| \leqslant 2h_n$ (see (2.17)) gives (2.35).

Due to (2.2), the function $f = z\varphi(1,z)$ has the estimate $|f(z)| \leqslant C_1 = e^{\|p\|_1}, z \in \mathbb{R}$. Then the Bernstein inequality gives $|\dot f(z)| = |\varphi(1,z) + z\dot\varphi(1,z)| \leqslant C_1, z \in \mathbb{R}$, which yields $|\dot\varphi(1,z)| \leqslant \frac{C_1}{|z|}(1+\frac{1}{|z|}), z \in \mathbb{R}$. Moreover, we obtain $|\varphi(1,z)| \leqslant |g_n| \max_{z\in g_n} |\dot\varphi(1,z)| \leqslant |g_n|\frac{3C_1}{2|z|}$.

The proof of (2.37) and the estimate $|\dot\beta(z)| \leqslant \frac{3e^{\|p\|_1}}{2|z|}$ is similar. Identity (2.26) gives $\beta^2(e_n^\pm) = -\varphi(1,e_n^\pm)\vartheta'(1,e_n^\pm)$. Then (2.36), (2.37) imply $|\beta(e_n^\pm)| \leqslant |g_n|\frac{3e^{\|p\|_1}}{2}$. Using these estimates and $\beta(z) = \beta(e_n^-) + \dot\beta(z_*)(z-e_n^-)$ for some $z_* \in g_n$, we obtain (2.38).

iii) Using (2.2) we obtain

$$|(\Delta^2(z)-1) - (\cos^2 z - 1)| \leqslant 2X|\Delta(z) - \cos z| \leqslant 2X^2/|z|, \quad X = e^{|\operatorname{Im}|+\|p\|_1}.$$

After this the standard arguments (due to Rouche's theorem) give the proof of iii). ■

Lemma 2.3. *i) Let $q \in L^1(0,t)$ for some $t > 0$ and let $z - e_n^\pm = O(|g_n|/n)$ as $n \to \infty$. Then*

$$\int_{\mathbb{R}} \varphi(1,z,\tau)q(\tau)d\tau = \frac{(-1)^n|g_n|}{2\pi n}(\mp\widehat{q}_0 + c_n\widehat{q}_{cn} - s_n\widehat{q}_{sn} + O(1/n)), \tag{2.39}$$

where c_n, s_n are defined in (1.8) and $\widehat{q}_0 = \int_{\mathbb{R}} q(\tau)d\tau$.

ii) Let v_n be defined by (2.15). The following asymptotics holds true:

$$e_n = \frac{e_n^- + e_n^+}{2} + O(|g_n|^2/n^2), \tag{2.40}$$

$$v(z) = \operatorname{Im} k(z) = \pm v_n(z)(1 + O(n^{-2})), \qquad z \in g_n \pm i0 \quad \text{as } n \to \infty. \tag{2.41}$$

Proof. i) We consider the case $z - e_n^- = O(|g_n|/n)$, the proof for $z - e_n^+ = O(|g_n|/n)$ is similar.

We need the following facts from Theorem 2 in [K5]: Let $\mu_n^2(\tau), \tau \in \mathbb{R}$ be the Dirichlet eigenvalue for the problem $-y'' + q(x+\tau)y = z^2 y, y(0) = y(1) = 0$. Then there exists a real function $\phi_n(\tau), \tau \in \mathbb{R}$ such that $\phi_n', \phi_n'' \in L^2_{loc}(\mathbb{R})$ and the following identity and asymptotics

$$\frac{E_n^- + E_n^+}{2} - \mu_n^2(\tau) = \frac{|\gamma_n|}{2}\cos\phi_n(\tau), \qquad \forall\, \tau \in \mathbb{R}, \quad \cos\phi_n(0) = c_n, \quad \sin\phi_n(0) = s_n, \tag{2.42}$$

$$\phi_n(\tau) = \phi_n(0) + 2\pi n\tau + O(\varepsilon_n) \qquad as \qquad n \to \infty, \qquad \varepsilon_n = \frac{1}{2\pi n}, \tag{2.43}$$

hold true, uniformly with respect to $\tau \in [0,1]$.

Using (2.42) we rewrite $e_n^- - \mu_n(\tau)$ in the form

$$e_n^- - \mu_n(\tau) = \frac{\frac{E_n^- + E_n^+ - |\gamma_n|}{2} - \mu_n^2(\tau)}{e_n^- + \mu_n(\tau)} = \frac{|\gamma_n|}{2}\frac{(-1+\cos\phi_n(\tau))}{e_n^- + \mu_n(\tau)}$$
$$= \frac{|g_n|}{2}(-1 + \cos\phi_n(\tau) + O(|g_n|\varepsilon_n)), \tag{2.44}$$

where the following asymptotics have been used:

$$\frac{|\gamma_n|}{e_n^- + \mu_n(\tau)} = |g_n|\frac{e_n^- + e_n^+}{e_n^- + \mu_n(\tau)} = |g_n|\left(1 + \frac{e_n^+ - \mu_n(\tau)}{e_n^- + \mu_n(\tau)}\right) = |g_n|(1 + O(|g_n|\varepsilon_n)) \tag{2.45}$$

as $n \to \infty$. Asymptotics (2.33) and (2.44) yield

$$\varphi(1,z,\tau) = \frac{(-1)^n}{\pi n}(1+O(\varepsilon_n))(z - \mu_n(\tau)) = \frac{(-1)^n}{\pi n}(1+O(\varepsilon))(e_n^- - \mu_n(\tau) + \varepsilon_n O(|g_n|)) \tag{2.46}$$
$$= \frac{(-1)^n|g_n|}{2\pi n}(-1 + \cos\phi_n(\tau) + O(\varepsilon_n)).$$

Combine (2.46), (2.43) we obtain

$$\int_{\mathbb{R}} \varphi(1,z,\tau)q(\tau)d\tau = \frac{(-1)^n|g_n|}{2\pi n}\int_{\mathbb{R}}\Big[-1+\cos\phi_n(\tau)+O(\varepsilon_n)\Big]q(\tau)d\tau$$
$$= \frac{(-1)^n|g_n|}{2\pi n}(-\widehat{q}_0 + c_n\widehat{q}_{cn} - s_n\widehat{q}_{sn} + O(\varepsilon_n)),$$

where $c_n = \cos\phi_n(0), s_n = \sin\phi_n(0)$, and this yields (2.39).

ii) We need the estimate from Lemma 2.1 in [K4] for all $n \geqslant 1$:

$$\Big|e_n - \frac{e_n^- + e_n^+}{2}\Big| \leqslant \frac{|g_n|^2}{4}M_n, \qquad M_n = \max_{z\in\overline{g}_n} G_n(z),$$
$$G_n(z) = \frac{1}{\pi}\int_{\mathbb{R}\setminus g_n}\frac{v(\tau+i0)d\tau}{v_n(\tau)|\tau - z|}, \quad z \in g_n. \quad (2.47)$$

We need the estimate from Lemma 2.4 in [K4]:

$$G_n(z) \leqslant \frac{1}{|z|\sqrt{|e_n^+e_n^-|}}\bigg(\widehat{p}_0 + \frac{\int_0^1 p^2(x)dx}{4r^2}\bigg), \qquad z\in\overline{g}_n, \qquad r = \min_n |e_n^+ - e_{n+1}^-| > 0. \quad (2.48)$$

This gives $M_n = O(1/n^2)$ as $n\to\infty$, since $e_n^{\pm} = \pi n + o(1)$ as $n \to \infty$. Thus substituting (2.48) into (2.47) and into (2.16), we obtain (2.40) and (2.41). ■

3. The perturbed operator H

We recall the well-known results about the scattering for H, H_0, see e.g. [F3], [F1]. The equation $-f'' + (p+q)f = z^2 f$ has unique Jost solutions $f_\pm(x,z)$ such that

$$f_+(x,z) = \psi_+(x,z),\ x \geqslant t, \ \text{ and } \ f_-(x,z) = \psi_-(x,z),\ x\leqslant 0, \quad z\in\sigma_M = \mathbb{R}\setminus\cup[e_n^-, e_n^+]. \quad (3.1)$$

The Jost solutions satisfy

$$f_+(x,z) = \overline{f}_+(x,-z), \qquad \forall z\in\sigma_M.$$

This yields the basic identity

$$f_+(x,z) = b(z)f_-(x,z) + a(z)f_-(x,-z), \qquad \forall z \in \sigma_M, \quad (3.2)$$

where

$$b = \frac{s}{w_0}, \qquad a = \frac{w}{w_0}, \qquad s = \{f_+(x,z), f_-(x,-z)\},$$
$$w = \{f_-, f_+\}, \qquad w_0 = \{\psi_-, \psi_+\} = \frac{2i\sin k}{\varphi(1,\cdot)}, \quad (3.3)$$

and $\{f,g\} = fg' - f'g$ is the Wronskian. The scattering matrix $\mathcal{S}_M$ for H, H_0 is given by

$$\mathcal{S}_M(z) \equiv \begin{pmatrix} a(z)^{-1} & r_-(z) \\ r_+(z) & a(z)^{-1}\end{pmatrix}, \quad r_\pm = \frac{s(\mp z)}{w(z)} = \mp\frac{b(\mp z)}{a(z)}, \quad z\in\sigma_M, \quad (3.4)$$

where $1/a$ is the transmission coefficient and $r_\pm$ is the reflection coefficient. We have the following identities from [F1], [F3]:

$$|a(z)|^2 = 1 + |b(z)|^2, \qquad z\in\sigma_M = \mathbb{R}\setminus\cup[e_n^-, e_n^+]. \quad (3.5)$$

We will represent the Jost solutions $f_\pm(x,z)$ in the form $f_+ = \widetilde{\vartheta} + m_+\widetilde{\varphi}$ (see (3.9)) and recall that $m_\pm$ is the Weyl-Titchmarsh function given by (2.22). Here $\widetilde{\vartheta}, \widetilde{\varphi}$ are the solutions of the equations $-y'' + (p+q)y = z^2 y, z \in \mathbb{C}$ and satisfying

$$\widetilde{\varphi}(x,z) = \varphi(x,z), \qquad \widetilde{\vartheta}(x,z) = \vartheta(x,z) \quad for \ \ all \ x \geqslant t. \tag{3.6}$$

Due to (2.6), the solutions $\widetilde{\vartheta}, \widetilde{\varphi}$ and f_+ of the equation $-y'' + (p+q)y = z^2 y$ satisfy the equation

$$y(x,z) = y_0(x,z) - \int_x^t \varphi(x-\tau, z, \tau) q(\tau) y(\tau, z) d\tau, \qquad x \leqslant t, \tag{3.7}$$

where y is one from $\widetilde{\vartheta}, \widetilde{\varphi}$ and f_+; y_0 is the corresponding function from ϑ, φ and ψ_+. For each $x \in \mathbb{R}$ the functions $\widetilde{\vartheta}(x,z), \widetilde{\varphi}(x,z)$ are entire in $z \in \mathbb{C}$ and satisfy

$$\max\left\{ |z|_1 |\widetilde{\varphi}(x,z)|, \ |\widetilde{\varphi}'(x,z)|, |\widetilde{\vartheta}(x,z)|, \frac{|\widetilde{\vartheta}'(x,z)|}{|z|_1} \right\} \leqslant X_1,$$

$$|\widetilde{\vartheta}(x,z) - \vartheta(x,z)| \leqslant \frac{\|q\|_t}{|z|_1} X_1, \quad |\widetilde{\varphi}(x,z) - \varphi(x,z)| \leqslant \frac{\|q\|_t}{|z|_1^2} X_1,$$

$$\|p\|_t = \int_0^t |p(s)| ds, \quad |z|_1 = \max\{1, |z|\}, \qquad X_1 = e^{|\operatorname{Im} z||2t-x| + \|q\|_t + \|p\|_t + \int_x^t |p(\tau)| d\tau}, \tag{3.8}$$

for all $(p,x,z) \in L^1_{loc}(\mathbb{R}) \times \mathbb{R} \times \mathbb{C}$. The proof of (3.8) repeats the standard arguments (see [PT]) proving (2.2).

Lemma 3.1. *Each function $f_+(x,z), x \in \mathbb{R}$ has an analytic continuation from $z \in \sigma_M$ into $z \in \mathcal{Z}$. Moreover, for all $z \in \mathcal{Z}$ the following identities and asymptotics hold true:*

$$f_+(\cdot, z) = \widetilde{\vartheta}(\cdot, z) + m_+(z)\widetilde{\varphi}(\cdot, z), \tag{3.9}$$

$$f_+(0,z) = 1 + \int_0^t \varphi(x,z) q(x) f_+(x,z) dx, \quad f_+'(0,z) = m_+(z) - \int_0^t \vartheta(x,z) q(x) f_+(x,z) dx, \tag{3.10}$$

$$|f_+(x,z) - \psi_+(x,z)| \leqslant e^{-vx + B(t,x)} \frac{\eta(z)}{|z|_1} \int_x^t |q(r)| dr, \quad \forall x \in [0,t], \tag{3.11}$$

where $v = \operatorname{Im} k(z)$ and

$$\eta(z) = \sup_{x \in [0,t]} |e^{-ik(z)x} \psi_+(x,z)|, \quad B(t,x) = 2(t-x)\mathfrak{J} + \int_x^t (|p(r)| + |q(r)|) dr, \quad \mathfrak{J} = \frac{|v| - v}{2},$$

$$f_+(x,z) = e^{\pm ik(z)x}(1 + e^{\pm(t-x)2\mathfrak{J}} O(1/z)), \qquad x \in [0,t], \tag{3.12}$$

$$\eta(z) \to 1, \qquad f_+(0,z) = 1 + \frac{O(e^{2t\mathfrak{J}})}{z}, \qquad f_+'(0,z) = iz + O(1) + o(e^{2t\mathfrak{J}}) \tag{3.13}$$

as $|z| \to \infty$, $z \in \mathcal{Z}_\varepsilon$, $\varepsilon > 0$, where $\widehat{q}(z) = \int_0^t q(x) e^{2izx} dx$, $z \in \mathbb{C}$.

Proof. Using (3.6), (3.1) we obtain (3.9). Then each function $f_+(x,z), x \in \mathbb{R}$ is analytic in $z \in \mathcal{Z}$, since m_+ is analytic in $z \in \mathcal{Z}$.

Using the identity (2.4), we obtain $\varphi(-s, \cdot, s) = -\varphi(s, \cdot)$ and $\varphi'(-s, \cdot, s) = \vartheta(s, \cdot)$. Substituting the last identities into (3.7) we get (3.10).

We will show (3.11) for the case $z \in \overline{\mathbb{C}}_-$, the proof for $z \in \mathbb{C}_+$ is similar. Let

$$f(x) = e^{-ik(z)x} f_+(x,z), \quad f_0(x) = e^{-ik(z)x}\psi_+(x,z), \quad K(x,s) = \varphi(x-s,z,s)e^{ik(z)(s-x)}.$$

The standard iteration of (3.7) yields

$$f = f_0 + \sum_{n\geqslant 1} f_n, \qquad f_n(x,z) = -\int_x^t K(x,s)q(s)f_{n-1}(s,z)ds. \tag{3.14}$$

Using (2.2) and estimate $|\operatorname{Im} z| \leqslant |v(z)|, z \in \mathbb{C}$ (see (2.12)) we obtain

$$|e^{ik(z)(s-x)}| \leqslant e^{|v|(s-x)}, \qquad |\varphi(x-s,z,s)| \leqslant \frac{e^{|\operatorname{Im} z|(s-x)+\int_x^s |p(r)|dr}}{|z|_1} \leqslant \frac{e^{|v|(s-x)+\int_x^s |p(r)|dr}}{|z|_1},$$

$$|K(x,s)| \leqslant \frac{e^{2|v|(s-x)+\int_x^s |p(r)|dr}}{|z|_1}, \qquad \forall s > x, z \in \overline{\mathbb{C}}_-. \tag{3.15}$$

Substituting (3.15) into (3.14) we obtain

$$|f_n(x)| \leqslant \int_x^t |K(x,s_1)q(s_1)f_{n-1}(s_1)|ds_1 \leqslant$$

$$\leqslant \eta \int_x^t |K(x,s_1)q(s_1)|ds_1 \int_{s_1}^t |K(s_1,s_2)q(s_2)|ds_2 .... \int_{s_{n-1}}^t |K(s_{n-1},s_n)q(s_n)|ds_n$$

$$\leqslant \frac{\eta e^{\int_s^t |p(r)|dr}}{|z|_1^n} \int_x^t e^{2|v(z)|(s_1-x)+\int_x^{s_1}|p|dr}|q(s_1)|ds_1 \int_{s_1}^t e^{2|v(z)|(s_2-s_1)+\int_{s_1}^{s_2}|p(r)|dr}|q(s_2)|ds_2....$$

$$\int_{s_{n-1}}^t e^{2|v(z)|(s_n-s_{n-1})+\int_{s_{n-1}}^{s_n}|p(r)|dr}|q(s_n)|ds_n$$

$$\leqslant \frac{\eta e^{2|v(z)|(t-x)+\int_s^t |p(r)|dr}}{|z|_1^n} \int_x^t |q(s_1)|ds_1 \int_{s_1}^t |q(s_2)|ds_2.... \int_{s_{n-1}}^t |q(s_n)|ds_n$$

$$= \eta(z)e^{2|v(z)|(t-x)+\int_s^t |p(r)|dr}\frac{(\int_x^t |q(r)|dr)^n}{n!|z|_1^n}.$$

Thus summing we deduce that

$$|f(x,z) - f_0(x,z)| \leqslant \eta e^{2(t-x)\Im+\int_x^t |p(r)|dr} \sum_{n\geqslant 1} \frac{(\int_x^t |q(r)|dr)^n}{|z|_1^n n!} \leqslant \frac{\eta}{|z|_1} e^{B(t,x)} \int_x^t |q(r)|dr,$$

which yields (3.11). Substituting (2.25) into (3.11) we obtain (3.12). The proof of (3.13) is similar. ■

Firsova [F4], [F1] obtained the following results: the function $a(z)$ has an analytic continuations from $z \in \sigma_M$ into $z \in \mathbb{C}_+$ and the following identity holds true:

$$a(z) = D(z) = \det(I + q(H - z^2)^{-1}), \qquad z \in \mathbb{C}_+. \tag{3.16}$$

We prove the main result of this section.

Lemma 3.2. *i) The functions $\xi(z), s(z), w(z), a(z)$ have analytic continuations from $z \in \sigma_M$ into $z \in \mathcal{Z}$ and satisfy*

$$a(-z) = \overline{a}(\overline{z}), \quad w(-z) = \overline{w}(\overline{z}), \quad s(-z) = \overline{s}(\overline{z}), \quad w_0(-z) = \overline{w}_0(\overline{z}), \quad \forall\ z \in \mathcal{Z}. \tag{3.17}$$

Moreover, for each $z \in \mathcal{Z}$ the following identities

$$w(z) = f_+'(0,z) - m_-(z) f_+(0,z) = \frac{2i \sin k(z)}{\varphi(1,z)} - \int_0^t q(x)\psi_-(x,z) f_+(x,z)dx, \tag{3.18}$$

$$\xi(z) = 2ia(z)\sin k(z) = \varphi(1,z)w(z), \tag{3.19}$$

$$s(z) = m_+(z) f_+(0,z) - f_+'(0,z) = \int_0^t q(x)\psi_+(x,z) f_+(x,z)dx, \tag{3.20}$$

and the following asymptotics

$$\xi(z) = 2i \sin z(1 + O(e^{2t\Im}/z)), \qquad s(z) = O(e^{2t\Im}), \tag{3.21}$$

hold true as $|z| \to \infty, z \in \mathcal{Z}_\varepsilon, \varepsilon > 0$.
ii) The function $s(\cdot)$ has exponential type $\rho_\pm$ in the half plane $\mathbb{C}_\pm$, where $\rho_+ = 0, \rho_- = 2t$.

Proof. We have $w = \{f_-, f_+\} = \psi_- f_+{}' - m_- f_+|_{x=0}$, which yields the identity $w = f_+'(0,\cdot) - m_- f_+(0,\cdot)$ in (3.18). Substituting (3.10) into $w = f_+'(0,\cdot) - m_- f_+(0,\cdot)$ we get (3.18).

Definitions of s and f_- give $s = \{f_+, \psi_+\} = f_+\psi_+{}' - f_+{}'|_{x=0}$, which yields the identity $s = f_+(0,\cdot)m_+ - f_+'(0,\cdot)$ in (3.20). Substituting (3.10) into the last identity we obtain (3.20).

These identities and analyticity of the functions $f_+'(0,z), f_+(0,z), m_\pm(z)$ in the domain $\mathcal{Z}$ imply that the functions $\xi(z), s(z), w(z), a(z)$ have analytic continuations from $z \in \mathbb{R} \setminus \cup \overline{g}_n$ into $z \in \mathcal{Z}$. The functions a, b, s, w are analytic in $\mathcal{Z}$ and are real on $i\mathbb{R}$. Then the symmetry principle yields (3.17).

Asymptotics from Lemma 3.1, (2.2) and (2) and identity (3.18), (3.20) and $\xi = \varphi(1,\cdot)w$ (see (3.19)) imply (3.21).

ii) We show $\rho_- = 2t$. Due to (3.21), s has exponential type $\rho_- \leqslant 2t$. The decompositions $f_+ = e^{ixz}(1+f)$ and $\psi_+ = e^{ixz}(1+\psi)$ give $(1+f)(1+\psi) = 1+T,\ T = f + \psi + \psi f$ and

$$s(z) = \int_0^t q(x)\psi_+(x,z) f_+(x,z)dx = \int_0^t q(x)e^{i2xz}(1 + T(x,z))dx, \qquad z \in \mathcal{Z}_\varepsilon. \tag{3.22}$$

Asymptotics (2.25), (2.2), (3.12) and $k(z) = z + O(1/z)$ as $|z| \to \infty$ (see (2.11)) yield

$$\psi(x,z) = O(1/z), \qquad f(x,z) = e^{2(t-x)|\operatorname{Im} z|}O(1/z) \quad \text{as } |z| \to \infty, \quad z \in \mathcal{Z}_\varepsilon. \tag{3.23}$$

We need the following variant of the Paley-Wiener type Theorem from [Fr]:
Let $q \in \mathcal{Q}_t^2$ and let each $G(x,z), x \in [0,t]$ be analytic in $z \in \mathbb{C}_-$ and $G \in L^2((0,t)dx, \mathbb{R}dz)$. Then $\int_0^t e^{2izx}q(x)(1 + G(x,z))dx$ has exponential type at least $2t$ in $\mathbb{C}_-$.

We can not apply this result to the function $T(x,z), z \in \mathbb{C}_-$, since $m_+(z)$ may have a singularity at $\mu_n - i0 \in \overline{g}_n + i0$ if $g_n \neq \emptyset$. But we can use this result for the function $T(x, z-i), z \in \mathbb{C}_-$, since (3.22), (3.23) imply $\sup_{x\in[0,1]} |T(x, -i+\tau)| = O(1/\tau)$ as $\tau \to \pm\infty$. Then the function $s(z)$ has exponential type $2t$ in the half plane $\mathbb{C}_-$. The proof for $\rho_+ = 0$ is similar. ■

4. Properties of the function ξ

We start with the basic properties of the function ξ.

Lemma 4.1. *i) The function $\xi = 2ia(z)\sin k(z), z \in \mathcal{Z}$ is analytic on the Riemann surface $\mathcal{M}$ and satisfies:*

$$\xi(z) = 2i\sin k(z)(1 + A(z)) + J(z), \quad z \in \mathcal{M},$$

$$A(z) = \int_{\mathbb{R}} q(x)Y_2(x,z)dx, \qquad Y_2 = \frac{1}{2}(\varphi\widetilde{\vartheta} - \vartheta\widetilde{\varphi}),$$

$$J(z) = -\int_{\mathbb{R}} q(x)Y_1(x,z)dx, \qquad Y_1 = \varphi(1,\cdot)\vartheta\widetilde{\vartheta} - \vartheta'(1,\cdot)\varphi\widetilde{\varphi} + \beta(\varphi\widetilde{\vartheta} + \vartheta\widetilde{\varphi}) = \varphi(1,\cdot,\cdot) + Y_{11},$$

$$Y_{11} = \varphi(1,\cdot)\vartheta\widetilde{\vartheta}_* - \vartheta'(1,\cdot)\varphi\widetilde{\varphi}_* + \beta(\varphi\widetilde{\vartheta}_* + \vartheta\widetilde{\varphi}_*), \quad \widetilde{\vartheta}_* = \widetilde{\vartheta} - \vartheta, \widetilde{\varphi}_* = \widetilde{\varphi} - \varphi, \tag{4.1}$$

$$\xi(z) = 2(-1)^{n+1}(1 + A(z))\sinh v(z) + J(z), \qquad z \in g_n^{\pm} \neq \emptyset,$$

$$Y_{11}(z,x) = O(|g_n|/n^2) \quad as \quad z \in g_n, n \to \infty. \tag{4.2}$$

where $v = \operatorname{Im} k$ and $\pm v(z) > 0$ for $z \in g_n^{\pm}$. The functions J, A are entire and satisfy

$$\xi(z) = \overline{\xi}(-\overline{z}), \qquad \forall\ z \in \mathcal{Z}, \tag{4.3}$$

$$J(z) = J(-z) = \overline{J}(\overline{z}) = \overline{J}(-\overline{z}), \qquad A(z) = A(-z) = \overline{A}(-\overline{z}) = \overline{A}(\overline{z}), \quad \forall\ z \in \mathbb{C}. \tag{4.4}$$

ii) The following estimates hold true

$$|J(z)| \leqslant C_{p,q}\|q\|_t\Big(|\varphi(1,z)| + \frac{|\vartheta'(1,z)|}{|z|^2} + \frac{|\beta(z)|}{|z|}\Big)e^{2t|\operatorname{Im} z|} \leqslant \frac{C_0}{4|z|}e^{(2t+1)|\operatorname{Im} z|},$$

$$|A(z)| \leqslant \frac{\|q\|_t^2 C_{p,q}}{|z|^2}e^{2t|\operatorname{Im} z|}, \quad where \quad C_0 = 12\|q\|_t e^{\|p\|_1 + \|q\|_t + 2\|p\|_t}, \quad C_{p,q} = e^{2\|p\|_t + \|q\|_t}, \tag{4.5}$$

$$|J(e_n)| \leqslant \frac{C_0}{|z_n|}\sinh h_n, \qquad if \quad e_n^- \geqslant 8e^{\|p\|_1}. \tag{4.6}$$

Proof. i) Using (3.3), we rewrite the identity (3.18) in the form

$$\xi(z) = 2i\sin k(z) - \int_{\mathbb{R}} q(x)Y(x,z)dx, \quad Y = \varphi(1,\cdot)\psi_-(x,\cdot)f_+(x,\cdot), \tag{4.7}$$

for $z \in \mathbb{C}_+$. Let $\varphi_1 = \varphi(1,\cdot), \vartheta_1' = \vartheta'(1,\cdot)$. Using (4.7), (3.9) we rewrite Y in the form

$$Y = \varphi_1(\vartheta + m_-\varphi)(\widetilde{\vartheta} + m_+\widetilde{\varphi}) = \varphi_1\Big(\vartheta\widetilde{\vartheta} + m_+\vartheta\widetilde{\varphi} + m_-\varphi\widetilde{\vartheta} - \frac{\vartheta_1'}{\varphi_1}\varphi\widetilde{\varphi}\Big) = Y_1 - i2Y_2\sin k.$$

Substituting the last identity into $\int_{\mathbb{R}} q(x)Y(x,z)dx$ and using (2.5) we obtain $Y_1 = \varphi(1,\cdot,\cdot) + Y_{11}$, which gives (4.1).

(2.14) implies the identity in (4.2). Substituting asymptotics from (3.8), Lemma 2.2 into Y_{11} we obtain $Y_{11}(z,x) = O(|g_n|/n^2)$ as $z \in g_n, n \to \infty$.

The function ξ is real on $i\mathbb{R}$, then the symmetry principle yields (4.3).

The functions A, J are entire and are real on $i\mathbb{R}, \mathbb{R}$. Then the symmetry principle yields (4.4). Then $\xi(z)$ is analytic in $\mathcal{M}$, since $\sin k(z)$ is analytic in $\mathcal{M}$

ii) Using (3.8), (2.2) and (4.1) and Lemma 2.2, we obtain (4.5). Estimates (4.5) and Lemma 2.2 give $|J(e_n)| \leqslant \frac{C_0}{2|z_n|}|g_n|$; and the estimate $|g_n| \leqslant 2h_n \leqslant 2\sinh h_n$ (see (2.17)) yields (4.6). ∎

Define the functions F, S by

$$F(z) = |\xi(z)|^2 = 4|\sin k(z)|^2|a(z)|^2 > 0, \qquad S(z) = |\varphi(1,z)s(z)|^2, \quad z \in \sigma_M = \mathbb{R} \setminus \cup[e_n^-, e_n^+].$$

Lemma 4.2. *i) The functions $F(z), S(z), z \in \sigma_M$ have analytic continuations into the whole complex plane $\mathbb{C}$ and satisfy*

$$F(z) = \xi(z)\xi(-z) = \xi(z)\overline{\xi}(\overline{z}), \qquad S(z) = \varphi^2(1,z)s(z)s(-z), \qquad z \in \mathcal{Z}, \tag{4.8}$$

$$F = 4(1-\Delta^2)(1+A)^2 + J^2 = 4(1-\Delta^2) + S. \tag{4.9}$$

Moreover, $F(z) > 0$ and $S(z) \geqslant 0$ on each interval $(e_{n-1}^+, e_n^-), n \geqslant 1$ and F has even number of zeros on each interval $[e_n^-, e_n^+], n \geqslant 1$.
ii) The function F has only simple zeros at $e_n^\pm, g_n \neq \emptyset$. Furthermore,
if $e_n^- = e_n^+$ for some $n \neq 0$, then e_n^- is a double zero of F and e_n^- is not a state of H,
if $F(0) = 0$, then $\zeta = 0$ is a double zero of F and $\zeta = 0$ is a virtual state of H.
iii) Let $\zeta \in \mathbb{C}_- \setminus i\mathbb{R}$. The point ζ is a zero of F iff ζ is a zero of ξ (with the same multiplicity).
iv) $\zeta \in i\mathbb{R}_-$ is a zero of F iff $\zeta \in i\mathbb{R}_-$ or $-\zeta \in i\mathbb{R}_+$ is a zero of ξ.
v) Let $g_n \neq \emptyset, n \geqslant 1$. The point $\zeta \in g_n$ is a zero of F iff $\zeta + i0 \in g_n^+$ or $\zeta - i0 \in g_n^-$ is a zero of ξ (with the same multiplicity).

Proof. i) Using (4.3) we deduce that $F = \xi(z)\xi(-z), z \in \sigma_M$. Then by (4.1), F satisfies $F = (1-\Delta^2)(2-A)^2 + J^2$ and then F is entire. Using (3.5), (3.17), we obtain that F, S satisfy (4.8), (4.9) and then S is entire.

Recall that $F > 0$ and $S \geqslant 0$ inside each (e_{n-1}^+, e_n^-), since $|\sin k(z)| > 0$ and $|a(z)| \geqslant 1$ on $\mathbb{R} \setminus \cup_{n\neq 0}\overline{g}_n$, see (3.5). Due to $F(e_n^\pm) \geqslant 0$, we get that F has even number of zeros on each interval $[e_n^-, e_n^+]$.

ii) Consider the case $\zeta = e_n^+, g_n \neq \emptyset$, the proof for $\zeta = e_n^-$ is similar. We have $F = F_0 + S$, where $F_0 = 4(1-\Delta^2)$. Thus if $F(\zeta) = 0$, then we get $S(\zeta) = 0$, since $\Delta^2(e_n^+) = 1$. Moreover, $F_0'(\zeta) = -2\Delta(\zeta)\Delta'(\zeta) > 0$ and $S'(\zeta) \geqslant 0$, which gives that $\zeta = e_n^+$ is a simple zero of F.

Let $g_n = \emptyset$. Then the functions φ_1, ϑ_1' have zero at $e_n = e_n^\pm$ and then identity (2.26) gives $\beta(e_n) = 0$. Then identity (4.1) gives $J(e_n) = 0$ and identity (4.9) implies that e_n is a zero of S with the multiplicity $\geqslant 2$. Differentiating $F = 4(1-\Delta^2) + S$ we obtain $\ddot{F}(e_n) = -8\ddot{\Delta}\Delta + \ddot{S}|_{e_n} \geqslant -8\ddot{\Delta}\Delta|_{e_n} > 0$, since $\ddot{S}(e_n) \geqslant 0$. Then e_n is a second order zero of F.

The proof for the case $F(0) = 0$ is similar.

iii) The function ξ has not zeros in $\mathbb{C}_+ \setminus i\mathbb{R}$, see [F1]. This and the identity $F = \xi(z)\xi(-z)$ yields v).

iv) The definition $F(z) = \xi(z)\xi(-z), z \in \mathcal{Z}$ gives iv).

v) The statement v) follows from iii). ■

Due to Lemma 4.2 we study the entire function F instead of the function ξ on the Riemann surface $\mathcal{M}$. Now we describe the forbidden domain for the resonances.

Lemma 4.3. *F has not zeros in $\mathcal{D}_0 \setminus \cup[e_n^-, e_n^+]$, where*

$$\mathcal{D}_0 = \Big\{ z \in \mathbb{C} : |z| > \max\{180e^{2\|p\|_1}, C_0 e^{2t|\operatorname{Im} z|}\} \Big\}, \quad C_0 = 12\|q\|_t e^{\|p\|_1 + \|q\|_t + 2\|p\|_t}. \tag{4.10}$$

Moreover, if $[e_n^-, e_n^+] \subset \mathcal{D}_0$, then F has exactly two zeros $z_n^\pm \in [e_n^-, e_n^+]$ such that:
if $e_n^- < e_n^+$, then z_n^-, z_n^+ are the simple zeros such that $e_n^- \leqslant z_n^- < e_n < z_n^+ \leqslant e_n^+$,
if $e_n^- = e_n^+$, then $z_n^- = z_n^+$ is a zero of order two,
There are no other zeros of F in $\mathcal{D}_0$.

Proof. Using Lemma 4.1, (3.8) and (2.2) we obtain

$$|A(z)| \leqslant \frac{\|q\|_t^2}{|z|_1^2} X_2, \qquad |J(z)| \leqslant \frac{3\|q\|_t}{|z|_1} X_2 X \qquad |\Delta(z) - \cos z| \leqslant \frac{X}{|z|_1}, \quad z \in \mathbb{C},$$

where $X_2 = e^{|\operatorname{Im} z|2t + \|q\|_t + 2\|p\|_t}$, $X = e^{|\operatorname{Im} z| + \|p\|_1}$ and $|z|_1 = \max\{1, |z|\}$. Substituting these estimates into the identity

$$F - 4\sin^2 z = 4(\cos^2 z - \Delta^2) + J^2 + 4(1 - \Delta^2)A(A + 2),$$

which follows from (4.8), we obtain

$$|F(z) - 4\sin^2 z| \leqslant 9X^2 C_1, \qquad C_1 = \frac{1}{|z|} + \frac{\|q\|_t^2 X_2^2}{|z|^2} + \frac{\|q\|_t^2 X_2}{|z|^2}\left(2 + \frac{\|q\|_t^2 X_2}{|z|^2}\right), \quad |z| \geqslant 1.$$

Using the simple estimate $e^{|\operatorname{Im} z|} \leqslant 4|\sin z|$ for all $|z - \pi n| \geqslant \frac{\pi}{4}, n \in \mathbb{Z}$, (see p. 27 [PT]), we obtain

$$9X^2 = 9e^{2|\operatorname{Im} z| + 2\|p\|_1} \leqslant |4\sin^2 z| r_0^2 \quad \text{all } |z - \pi n| \geqslant \frac{\pi}{4}, \quad n \in \mathbb{Z}, \qquad r_0 = 6e^{\|p\|_1},$$

which yields

$$|F(z) - 4\sin^2 z| \leqslant 4|\sin^2 z| r_0^2 C_1 < 4|\sin^2 z|, \quad \forall \quad z \in \mathcal{D}_0 \setminus \bigcup \mathbb{D}_{\frac{\pi}{4}}(\pi n),$$

since for $z \in \mathcal{D}_0$ the following estimates hold true

$$\frac{r_0^2}{|z|} < \frac{1}{5}, \qquad \frac{r_0 \|q\|_t X_2}{|z|} < \frac{1}{2},$$

$$r_0^2 C_1 \leqslant \frac{r_0^2}{|z|} + \frac{r_0^2 \|q\|_t^2 X_2^2}{|z|^2} + \frac{r_0^2 \|q\|_t^2 X_2}{|z|^2}\left(2 + \frac{\|q\|_t^2 X_2}{|z|^2}\right) < \frac{1}{5} + \frac{1}{4} + \frac{1}{2} + \frac{1}{(24)^2} < \frac{19}{20}.$$

Thus by Rouche's theorem, F has as many roots, counted with multiplicities, as $\sin^2 z$ in each disk $\mathbb{D}_{\frac{\pi}{4}}(\pi n) \subset \mathcal{D}_0$. Since $\sin z$ has only the roots $\pi n, n \geqslant 1$, then F has two zeros in each disk $\mathbb{D}_{\frac{\pi}{4}}(\pi n) \subset \mathcal{D}_0$ and F has not zeros in $\mathcal{D}_0 \setminus \cup \mathbb{D}_{\frac{\pi}{4}}(\pi n)$.

Let $e_n^- < e_n^+$. Estimate (4.5) gives $|A(e_n)| \leqslant \frac{1}{2}$ and (4.6) imply $|J(e_n)| \leqslant \sinh h_n$. The substitution of these estimates into (4.8) yields

$$F(e_n) \leqslant -\left(1 - \frac{C_0^2}{|e_n|^2}\right) \sinh^2 h_n < 0, \tag{4.11}$$

and e_n, h_n are defined by (2.10). The function $F(e_n^\pm) \geqslant 0$ and due to (4.11), we deduce that F has exactly two zeros $z_n^\pm$ on the segment $[e_n^-, e_n^+]$ such that $e_n^+ \leqslant z_n^- < e_n < z_n^+ \leqslant e_n^+$.

If $e_n^- = e_n^+$, then by Lemma 4.2 ii), $z_n^- = z_n^+$ is a second order zero of F. ∎

We discuss the relationship of states from the Definition S and poles of the resolvent $(H - z^2)^{-1}$. The kernel of the resolvent $R = (H - z^2)^{-1}, z \in \mathbb{C}_+$, has the form

$$R(x, x', z) = \frac{f_-(x, z) f_+(x', z)}{-w(z)} = \frac{R_1(x, x', z)}{-\xi(z)}, \quad x < x', \quad R_1 = \varphi(1, z) f_-(x, z) f_+(x', z),$$

and $R(x, x', z) = R(x', x, z),\ x > x'$. Identity (3.9) yields $f_\pm = \widetilde{\vartheta} + m_\pm \widetilde{\varphi}$, $\widetilde{\vartheta} = \widetilde{\vartheta}(x, z)$, $\widetilde{\varphi} = \widetilde{\varphi}(x, z)$. Let $\widetilde{\vartheta}_* = \widetilde{\vartheta}(x', z)$, $\widetilde{\varphi}_* = \widetilde{\varphi}(x', z)$. Then using (2.26) $m_\pm = \frac{\beta \pm i \sin k}{\varphi(1, \cdot)}$, we obtain

$$R_1(x, x', z) = \varphi(1, \cdot)\widetilde{\vartheta}\widetilde{\vartheta}_* + (\beta - i\sin k)\widetilde{\varphi}\widetilde{\vartheta}_* + (\beta + i\sin k)\widetilde{\vartheta}\widetilde{\varphi}_* - \vartheta'(1, \cdot)\widetilde{\varphi}_*\widetilde{\varphi}.$$

Then for fixed $x, x' \in \mathbb{R}$ the function $R_1(x, x', z)$ is analytic on $\mathcal{M}$ and R_1 is locally bounded on $\mathbb{R}^2 \times \mathcal{M}$. The zeros of ξ create the singularities of the kernel $R(x, x', z)$. Thus if $\xi(z) = \varphi(1, z)w(z) = 0$ at some $z \in \mathcal{M}$, then $R(x, x', z)$ has singularity at z. The poles of $R(x, x', z)$ define the bound states and resonances. The zeros of ξ define the bound states and resonances, since the function R_1 is locally bounded.

Consider the unperturbed case $q = 0$. Recall that $\sin k(z)$ is analytic in $\mathcal{M}$ and $\sin k(z) = 0$ for some $z \in \mathcal{M}$ iff $z = e_n^-$ or $z = e_n^+$ for some $n \geqslant 0$. The function ξ is analytic on $\mathcal{M}$ and has branch points $e_n^\pm, g_n \neq \emptyset$. Then $R_0 = (H_0 - z^2)^{-1}$ has the form

$$R_0(x, x', z) = \frac{R_{10}(x, x', z)}{-\xi_0(z)}, \quad \xi_0 = \varphi(1, z)w_0(z) = 2i \sin k(z),$$

$$R_{10} = \varphi(1, \cdot)\psi_-(x, \cdot)\psi_+(x', \cdot) = \varphi(1, \cdot)\vartheta\vartheta_* + (\beta - i \sin k)\varphi\vartheta_* + (\beta + i \sin k)\vartheta\varphi_* - \vartheta'(1, \cdot)\varphi\varphi_*,$$

where $\varphi = \varphi(x, \cdot), ..$ and $\varphi_* = \varphi(x', \cdot), ..$ Thus $R_0(x, x', z)$ has singularity at some $z \in \mathcal{M}$ iff $\sin k(z) = 0$, i.e., $k(z) = \pi n$ and then $z = e_n^\pm$.

Remark that if $\zeta = e_n^- = e_n^+$ (i.e.. the gap $g_n = \emptyset$), then ξ is analytic at ζ, and $\xi(\zeta) = 0$, but such point ζ is not the state. The function $a(z)$ is analytic at ζ and (3.5) yields $|a(\zeta)| \geqslant 1$

5. Proof of Theorems 1.1-1.4

Proof of Theorem 1.1. i) By Lemma 4.1, ξ is analytic on $\mathcal{M}$ and the function J is entire.

ii) Recall that (4.8) gives

$$F(z) = \xi(\zeta)\xi(-\zeta) = \xi(\zeta)\overline{\xi}(\overline{\zeta}), \qquad \zeta = z + i0 \in g_n + i0.$$

Then the zeros $\zeta \in g_n + i0$ of $\xi(\zeta)$ give the bound states and the zeros $\zeta \in g_n + i0$ of $\xi(\overline{\zeta})$ give the antibound states. Their global number on g_n^c plus the possible virtual states at $e_n^\pm$ is even $\geqslant 0$, see Lemma 4.2, i).

The similar arguments and Lemma 4.3 yield iii).

Moreover, if an open gap $g_n = (e_n^-, e_n^+) \subset \mathcal{D}_0$ ($\mathcal{D}_0$ is defined by (4.3)), then there exist exactly two simple zeros $z_n^\pm \in [e_n^-, e_n^+]$ such that

$$e_n^- \leqslant z_n^- < e_n < z_n^+ \leqslant e_n^+. \tag{5.1}$$

The asymptotics (2.40) yields $\delta_n^\pm \leqslant \frac{2}{3}|g_n|$ as $n \to \infty$. Note that if $g_n = \emptyset$, then F has a double zero $e_n^\pm = z_n^\pm = e_n$. There are no other zeros of F in $\mathcal{D}_0$.

iv) Due to (5.1) we have $z_n^\pm = e_n^\pm \mp \delta_n^\pm, \delta_n^\pm \geqslant 0$. Let $\zeta = z_n^\pm, \delta = \delta_n^\pm$. Then the equation $0 = \xi(\zeta) = (-1)^{n+1}2(1 + A(\zeta)) \sinh v(\zeta) + J(\zeta), \ \ \zeta \in \overline{g}_n^\pm \neq \emptyset$ and Lemma 2.2, (4.5) imply

$$\sinh |v(\zeta)| = O(J(\zeta)) = O\bigg(|\varphi(1, \zeta)| + \frac{|\vartheta'(1, \zeta)|}{|\zeta|^2} + \frac{|\beta(\zeta)|}{\zeta}\bigg) = \varepsilon_n O(|g_n|), \quad \varepsilon_n = \frac{1}{2\pi n}$$

as $n \to \infty$. Moreover, using the estimate $|(z - e_n^-)(z - e_n^+)|^{\frac{1}{2}} \leqslant |v(z)|$ for each $z \in g_n$ (see (2.15)) we obtain $|\delta(|g_n| - \delta)|^{\frac{1}{2}} \leqslant |v(\zeta)| = \varepsilon_n O(|g_n|)$, which yields $\delta = \varepsilon_n^2 O(|g_n|)$. Thus the points $z_n^\pm$ are close to $e_n^\pm$ and satisfy:

$$|v(z_n^\pm)| = \varepsilon_n O(|g_n|), \qquad \delta_n^\pm = z_n^\pm - e_n^\pm = \varepsilon_n^2 O(|g_n|). \tag{5.2}$$

Consider the case $\delta = \delta_n^- = \varepsilon_n^2 O(|g_n|)$, the proof for $\delta = \delta_n^+$ is similar. Using (4.2) we obtain

$$J = J_0 + O(\varepsilon_n^2 |g_n|), \qquad J_0(z) = -\int_{\mathbb{R}} \varphi(1, z, x)q(x)dx.$$

Asymptotics (2.39) gives

$$J_0(\zeta) = (-1)^n |g_n| \varepsilon_n (\widehat{q}_0 - c_n \widehat{q}_{cn} + s_n \widehat{q}_{sn} + O(\varepsilon_n)), \tag{5.3}$$

where $c_n = \cos \phi_n(0), s_n = \sin \phi_n(0)$, and (4.2) yields

$$(-1)^n J(\zeta) = \varepsilon_n |g_n| I_n^-, \qquad I_n^- = \widehat{q}_0 - c_n \widehat{q}_{cn} + s_n \widehat{q}_{sn} + O(\varepsilon_n). \tag{5.4}$$

Using (4.5) we obtain

$$\sinh v(\zeta) = \frac{(-1)^n J(\zeta)}{2 + 2A(\zeta)} = \frac{\varepsilon_n |g_n| I_n^-}{2 + O(\varepsilon_n^2)}, \quad \operatorname{sign} v(\zeta) = \operatorname{sign}(-1)^n J(\zeta) = \operatorname{sign} I_n^-. \tag{5.5}$$

Note that if $v(\zeta) > 0$, then $\zeta \in g_n^+$ is a bound state,
if $v(\zeta) < 0$, then $\zeta \in g_n^-$ is a resonance,
if $v(\zeta) = 0$, then $\zeta = e_n^-$ or $\zeta = e_n^+$ is a virtual state.
Then (5.2) gives $\sinh v(\zeta) = v(\zeta)(1 + O(|g_n|^2 \varepsilon_n^2))$ and using asymptotics (2.41), we obtain

$$v(\zeta) = \sqrt{\delta(|g_n| - \delta)}(1 + O(\varepsilon_n^2)) = \sqrt{\delta |g_n|}(1 + O(\varepsilon_n^2)).$$

This and (5.5) yield $\delta_n^- = \frac{|g_n|\varepsilon_n^2}{4}(I_n^-)^2$ and (2.8) gives $|\gamma_n| = (2\pi n)|g_n|(1 + O(\varepsilon_n^2))$, which yields $\delta_n^- = \frac{2|\gamma_n|}{(4\pi n)^3}(I_n^-)^2(1 + O(\varepsilon_n^2))$. This and (5.4) give (1.10).

If $\widehat{q}_0 > 0$, then $I_n^- > 0$ and above arguments yield that z_n^- is a bound state and z_n^+ is an antibound state. Conversely, if $\widehat{q}_0 < 0$, then $I_n^- < 0$ and we deduce that z_n^- is an antibound state and z_n^+ is a bound state. This yields (1.11). ■

Note that due to (1.10), the high energy real states of H and H_0 are very close. This gives

$$\#(H, r, \cup_{n \geqslant 1} g_n^c) = \#(H_0, r, \cup_{n \geqslant 1} g_n^c) + 2N_* \quad as \quad r \to \infty, \quad r \notin \cup_{n \geqslant 1} \overline{g}_n. \tag{5.6}$$

for some $N_* \in \mathbb{Z}$.

Proof of Corolarry 1.2. We have $-c_n \widehat{q}_{cn} + s_n \widehat{q}_{sn} = -|\widehat{q}_n| \cos(\phi_n + \tau_n)$, where c_n, s_n given by (1.8). Then Theorem 1.1 iv) yields the Statement i) and ii).

If $p \in L^2_{even}(0, 1)$, then the coefficient $s_n = 0$ for all $n \geqslant 1$ (see remark before Corolarry 1.2). Thus Statement i) yields Statement iii). ■

Proof of Theorem 1.3. An entire function $f(z)$ is said to be of *exponential type* if there is a constant α such that $|f(z)| \leqslant$ const. $e^{\alpha|z|}$ everywhere. The function f is said to belong to the Cartwright class $Cart_\rho$, if f is entire, of exponential type, and the following conditions hold true:

$$\int_{\mathbb{R}} \frac{\log(1 + |f(x)|)dx}{1 + x^2} < \infty, \quad \rho_\pm(f) = \rho, \quad \text{where} \quad \rho_\pm(f) \equiv \lim\sup_{y \to \infty} \frac{\log |f(\pm iy)|}{y}.$$

Let $\mathcal{N}(r, f)$ be the total number of zeros of f with modulus $\leqslant r$, each zero being counted according to its multiplicity. We recall the well known result (see [Koo]).

Levinson Theorem. *Let the entire function $f \in Cart_\rho, \rho > 0$. Then $\mathcal{N}(r, f) = \frac{2r}{\pi}(\rho + o(1))$ as $r \to \infty$, and for each $\delta > 0$ the number of zeros of f with modulus $\leqslant r$ lying outside both of the two sectors $|\arg z|, |\arg z - \pi| < \delta$ is $o(r)$ for $r \to \infty$.*

Consider the functions F, S.

By Lemma Lemma 4.2, the functions F, S are entire and by (4.5), the function $F \in L^\infty(\mathbb{R})$ and then the function $S \in L^\infty(\mathbb{R})$. Using Lemma 3.2 ii) and (3.8), we deduce the function S

has the exponential type $\rho_\pm(S) = 2 + 2t$ in the half plane $\mathbb{C}_\pm$. Thus the function S belongs to $Cart_{2+2t}$ and the identity (4.9) gives that $F \in Cart_{2+2t}$ and the Levinson Theorem implies

$$\mathcal{N}(r, F) = 2r\frac{2 + 2t + o(1)}{\pi} \quad as \quad r \to \infty. \tag{5.7}$$

Let $\pm\zeta_n > 0, n \geqslant 1$ be all real zeros $\neq 0$ of F and let the zero $\zeta_0 = 0$ have the multiplicity $n_0 \leqslant 2$. Define the function $F_1 = z^{n_0} \lim_{r\to\infty} \prod_{|\zeta_n|\leqslant r}(1 - \frac{z}{\zeta_n})$. Recall that by Lemma 4.2, $F(z) > 0$ on the set $\mathbb{R} \setminus \cup_{n\neq 0} g_n$ and by Lemma 4.3, the function F has exactly two zeros on each set $[e_n^-, e_n^+]$ for n large enough. Then

$$\mathcal{N}(r, F_1) = 2r\frac{2 + o(1)}{\pi} \quad \text{as } r \to \infty. \tag{5.8}$$

Combining (5.7) and (5.8), we obtain

$$\mathcal{N}(r, F/F_1) = \mathcal{N}(r, F) - \mathcal{N}(r, F_1) = 2r\frac{2 + 2t + o(1)}{\pi} - 2r\frac{2 + o(1)}{\pi} = 4r\frac{t + o(1)}{\pi}.$$

Denote by $\mathcal{N}_+(r, f)$ (or $\mathcal{N}_-(r, f)$) the number of zeros of f with imaginary part > 0 (or < 0) having modulus $\leqslant r$, each zero being counted according to its multiplicity. The function F is real on the real line, then

$$\mathcal{N}_+(r, F) = \mathcal{N}_-(r, F) = \frac{1}{2}\mathcal{N}(r, F/F_1) = 2r\frac{t + o(1)}{\pi}.$$

Then Lemma 4.2 gives the identities $\mathcal{N}_-(r, F) = \mathcal{N}_+(r, F) = \mathcal{N}_-(r, \xi) + N_*$ for some integer $N_* \geqslant 0$, which yields $\mathcal{N}_-(r, \xi) = 2r\frac{t+o(r)}{\pi}$ as $r \to \infty$ and (1.13). ■

Proof of Theorem 1.4. i) Let the operator H_0 have infinitely many gaps $\gamma_n \neq \emptyset$ for some $p \in L^2(0, 1)$ and let $\varkappa = (\varkappa_n)_1^\infty$ be any sequence, where $\varkappa_n \in \{0, 2\}$. For this p there exist a unique sequence of angles $\phi_n \in [0, 2\pi), n \geqslant 1$, defined by (1.8).

We take a real potential $q \in L^2(0, 1), \operatorname{supp} q \subset (0, 1)$ given by

$$q(x) = \sum_{n\geqslant 1} \frac{1}{|n|^\alpha}(e^{i\tau_n + i2\pi nx} + e^{-i\tau_n - i2\pi nx}), \quad x \in (0, 1), \quad \frac{1}{2} < \alpha < 1. \tag{5.9}$$

Let $\varepsilon \in (0, 1)$. We take $\tau_n \in [0, 2\pi)$ such that

if $\varkappa_n = 2$, then we choose τ_n such that $\cos(\phi_n + \tau_n) < -\varepsilon$,

if $\varkappa_n = 0$, then we choose τ_n such that $\cos(\phi_n + \tau_n) > \varepsilon$.

Then due to Corolarry 1.2 i), the operator H has $\varkappa_n = 1 - \operatorname{sign}\cos(\phi_n + \tau_n)$ bound states in the physical gap $g_n^+ \neq \emptyset$ and $2 - \varkappa_n$ resonances inside the nonphysical gap $g_n^- \neq \emptyset$ for n large enough.

ii) Let $q \in \mathcal{Q}_t, t > 0$ satisfy $\widehat{q}_0 = 0$ and let $\widehat{q}_n = |\widehat{q}_n|e^{i\tau_n}$, where $|\widehat{q}_n| > n^{-\alpha}$ and $\tau_n \in [0, 2\pi)$ for all n large enough and some $\alpha \in (0, 1)$. Let $\varkappa = (\varkappa_n)_1^\infty$ be any sequence, where $\varkappa_n \in \{0, 2\}$.

Let $\delta = (\delta_n)_1^\infty \in \ell^2$ be a sequence of nonnegative numbers $\delta_n \geqslant 0, n \geqslant 1$ and infinitely many $\delta_n > 0$ and let $(\widetilde{\phi}_n)_1^\infty$ be a sequence of angles $\widetilde{\phi}_n \in [0, 2\pi), n \geqslant 1$. Let $\varepsilon \in (0, 1)$. We take $\widetilde{\phi}_n \in [0, 2\pi)$ such that

if $\varkappa_n = 2$, then we choose $\widetilde{\phi}_n$ such that $\cos(\widetilde{\phi}_n + \tau_n) < -\varepsilon$,

if $\varkappa_n = 0$, then we choose $\widetilde{\phi}_n$ such that $\cos(\widetilde{\phi}_n + \tau_n) > \varepsilon$.

Recall the result from [K5]:

The mapping $\Psi : \mathcal{H} \to \ell^2 \oplus \ell^2$ *given by* $\Psi = ((\Psi_{cn})_1^\infty, (\Psi_{sn})_1^\infty)$ *where*

$$\Psi_{cn} = \frac{E_n^- + E_n^+}{2} - \mu_n^2, \quad \Psi_{sn} = \left| \frac{|\gamma_n|^2}{4} - \Psi_{cn}^2 \right|^{\frac{1}{2}} \operatorname{sign}(|\varphi'(1, \mu_n)| - 1), \tag{5.10}$$

is a real analytic isomorphism between real Hilbert spaces $\mathcal{H} = \{p \in L^2(0,1) : \int_0^1 p(x)dx = 0\}$ *and* $\ell^2 \oplus \ell^2$. Note that (5.10) and (1.8) give $\Psi_{cn} = \frac{|\gamma_n|}{2} c_n, \Psi_{sn} = \frac{|\gamma_n|}{2} s_n$, since $\varphi(\cdot, \mu_n) = y_n(\cdot)$.

Then for any sequence $\delta = (\delta_n)_1^\infty \in \ell^2$ and any sequence of angles $(\widetilde{\phi}_n)_1^\infty$ there exists a unique potential $p \in \mathcal{H} \subset L^2(0,1)$ such that each gap length $|\gamma_n| = \delta_n$ and the corresponding angle $\phi_n = \widetilde{\phi}_n$ for all $n \geqslant 1$. We consider the operator $H = -\frac{d^2}{dx^2} + p + C + q$, where the constant C is such that $E_0^+ = 0$.

Then due to Corolarry 1.2 i), the operator H has $\varkappa_n = 1 - \operatorname{sign}\cos(\phi_n + \tau_n)$ bound states in the physical gap $g_n^+ \neq \emptyset$ and $2 - \varkappa_n$ resonances inside the nonphysical gap $g_n^- \neq \emptyset$ for n large enough. ■

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Saint-Petersburg University, Russia korotyaev@gmail.com