SOURCES OF LOG CANONICAL CENTERS

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1. Introduction

Let X be a smooth variety and $S \subset X$ a smooth hypersurface. The *Poincaré residue map* is an isomorphism

$$\mathcal{R} : \omega_X(S)|_S \cong \omega_S.$$

In additive form it gives the *adjunction formula* $(K_X + S)|_S \sim K_S$, but this variant does not show that $\mathcal{R}$ is a *canonical* isomorphism.

Its generalization to log canonical pairs $(X, S+\Delta)$ has been an important tool in birational geometry; see, for instance, [K$^+$92, KM98]. One defines a twisted version of the restriction of Δ to S, called the *different* and, for $m > 0$ sufficiently divisible, one gets a Poincaré residue map

$$\mathcal{R}^m : \bigl(\omega_X^{[m]}(mS + m\Delta)\bigr)|_S \cong \omega_S^{[m]}\bigl(m \operatorname{Diff}_S \Delta\bigr),$$

where the exponent $[m]$ denotes the double dual of the mth tensor power. As before, it is frequently written as a $\mathbb{Q}$-linear equivalence of divisors

$$\bigl(K_X + S + \Delta)|_S \sim_{\mathbb{Q}} K_S + \operatorname{Diff}_S \Delta.$$

There have been several attempts to extend these formulas to the case when S is replaced by a higher codimension log canonical center of a pair (X, Δ) [Kaw97, Kaw98, Kol07]. None of these have been completely successful; the main difficulty is understanding what kind of object the different should be.

Let $Z \subset X$ be a log canonical center of a pair (X, Δ). We can choose a resolution $f : X' \to X$ such that if we write $f^*\bigl(K_X + \Delta\bigr) \sim_{\mathbb{Q}} K_{X'} + \Delta'$ then there is a divisor $S \subset X'$ that dominates Z and appears in Δ' with coefficient 1. The usual adjunction formula now gives

$$\bigl(K_{X'} + \Delta'\bigr)|_S \sim_{\mathbb{Q}} K_S + \operatorname{Diff}_S(\Delta' - S) =: K_S + \Delta_S.$$

Note further that $K_{X'} + \Delta'$ is trivial on the fibers of f, hence so is $K_S + \Delta_S$. Thus

$$f|_S : (S, \Delta_S) \to Z$$

is a fiber space whose (possibly disconnected) fibers have (numerically) trivial (log) canonical class. The aim of previous attempts was to generalize Kodaira's canonical bundle formula for elliptic surfaces (cf. [BPVdV84, Sec.V.12]) to this setting. The difficulty is to make sure that we do not lose information in the summand that corresponds to the j-invariant of the fibers in the classical case. (For families of elliptic curves this could be achieved by keeping the corresponding variation of Hodge structures as part of our data.)

This suggests that it could be better to view the pair (S, Δ_S) as the answer to the problem. However, in general there are many divisors $S_j \subset X'$ that satisfy our requirements and they do not seem to be related to each other in any nice way.

Our aim is to remedy this problem, essentially by looking at the smallest possible intersections of the various divisors S_j on a dlt model of (X, Δ). There can be many of these models and intersections, but they turn out to be birational to each other and have several unexpectedly nice properties. These are summarized in the next theorem. For the rest of this note we work over a field of characteristic 0.

Theorem 1. *Let (X, Δ) be an lc pair, $Z \subset X$ an lc center and $n : Z^n \to Z$ its normalization. Let $f : \big(X^m, \Delta^m\big) \to (X, \Delta)$ be a dlt model (5) and $S \subset X^m$ a minimal (with respect to inclusion) lc center of $\big(X^m, \Delta^m\big)$ that dominates Z. Set $\Delta_S := \operatorname{Diff}^*_S \Delta^m$ (4) and $f_S := f|_S$. Let $f^n_S : S \to \tilde{Z}_S \to Z^n$ denote the Stein factorization.*

(1) *(Uniqueness of sources) The birational equivalence class of (S, Δ_S) does not depend on the choice of X^m and S. It is called the* source *of Z and denoted by* $\operatorname{Src}(Z, X, \Delta)$.
(2) *(Uniqueness of springs) The isomorphism class of $\tilde{Z}_S$ does not depend on the choice of X^m and S. It is called the* spring *of Z and denoted by* $\operatorname{Spr}(Z, X, \Delta)$.
(3) *(Crepant log structure) (S, Δ_S) is dlt, $K_S + \Delta_S \sim_{\mathbb{Q}} f^*_S\big(K_X + \Delta\big)$ and (S, Δ_S) is klt on the generic fiber of f_S.*
(4) *(Poincaré residue map) For $m > 0$ sufficiently divisible, there are well defined isomorphisms*
$$\begin{aligned} f^*\big(\omega_X^{[m]}(m\Delta)\big)|_S &\cong \omega_S^{[m]}(m\Delta_S) \quad \textit{and} \\ n^*\big(\omega_X^{[m]}(m\Delta)\big) &\cong \Big(\big(f^n_S\big)_* \omega_S^{[m]}(m\Delta_S)\Big)^{\mathrm{inv}} \end{aligned}$$
where the exponent inv *denotes the invariants under the action of the group of birational self-maps* $\operatorname{Bir}_Z(S, \Delta_S)$.
(5) *(Galois property) The extension $\tilde{Z}_S \to Z$ is Galois and $\operatorname{Bir}_Z(S, \Delta_S) \twoheadrightarrow \operatorname{Gal}\big(\tilde{Z}_S/Z\big)$ is surjective.*
(6) *(Adjunction) Assume $\Delta = D + \Delta_1$. Let $n_D : D^n \to D$ be the normalization and $Z_D \subset D^n$ an lc center of $\big(D^n, \operatorname{Diff}_{D^n} \Delta_1\big)$ such that $n_D(Z_D) = Z$. Then there is a commutative diagram*
$$\begin{array}{ccc} \operatorname{Src}\big(Z_D, D^n, \operatorname{Diff}_{D^n} \Delta_1\big) & \stackrel{bir}{\sim} & \operatorname{Src}\big(Z, X, D + \Delta_1\big) \\ \downarrow & & \downarrow \\ Z_D & \stackrel{n_D}{\to} & Z. \end{array}$$

Crepant log structures are defined in Section 2. Theorem 10 shows that minimal lc centers are birational to each other; this proves (1.1) and it also establishes (1.6). Its consequences for the Poincaré residue map are derived in Section 3. Sources and springs are formally defined in Section 4 and (1.5) is proved in (20).

Section 5 contains the main application, Theorems 25–26. We show that normalization gives a one-to-one correspondence:
$$\left\{ \begin{array}{c} \text{slc pairs } (X, \Delta) \\ \text{such that} \\ K_X + \Delta \text{ is ample} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{lc pairs } \big(\bar{X}, \bar{D} + \bar{\Delta}\big) \text{ plus an} \\ \text{involution } \tau \text{ of } \big(\bar{D}^n, \operatorname{Diff}_{\bar{D}^n} \bar{\Delta}\big) \\ \text{such that } K_{\bar{X}} + \bar{D} + \bar{\Delta} \text{ is ample} \end{array} \right\}.$$
The papers [Oda11, OX11] contain further applications to K-stability and to slc models of deminormal schemes.

Shokurov informed me that his forthcoming paper [Sho11] contains another approach to Theorem 1.

2. Crepant log structures

Definition 2. Let Z be a normal variety. A *crepant log structure* on Z is a proper, surjective morphism $f:(X,\Delta)\to Z$ such that

(1) f has connected fibers,
(2) (X,Δ) is lc and
(3) $K_X+\Delta\sim_{f,\mathbb{Q}}0$.

A proper morphism $f:(X,\Delta)\to Z$ is called a *weak crepant log structure* on Z if it satisfies (2–3) but the fibers of f are allowed to be reducible and Δ is allowed to be a non-effective sub-boundary. (This variant is probably too far from a crepant log structure to be of practical use, but it is convenient to have a notion that is birationally invariant and applies to all restrictions to lc centers. One would need to pose further restrictions as in the notion of quasi-log varieties [Amb03] to get useful applications.)

Any lc pair $\big(Z,\Delta_Z\big)$ has a trivial crepant log structure where $(X,\Delta)=\big(Z,\Delta_Z\big)$. Conversely, if f is birational then $\big(Z,\Delta_Z:=f_*\Delta\big)$ is lc.

An irreducible subvariety $W\subset Z$ is a *log canonical center* or *lc center* of a weak crepant log structure $f:(X,\Delta)\to Z$ iff it is the image of an lc center $W_X\subset X$ of (X,Δ). A weak crepant log structure has only finitely many lc centers.

Let (Z,Δ_Z) be an lc pair and $f:X\to Z$ a proper, birational morphism. Write $K_X+\Delta_X\sim_{\mathbb{Q}}f^*(K_Z+\Delta_Z)$. Then $f:(X,\Delta_X)\to Z$ is a weak crepant log structure. The lc centers of $f:(X,\Delta_X)\to Z$ are the same as the lc centers of (Z,Δ_Z).

By (5), we can choose f such that $f:(X,\Delta_X)\to(Z,\Delta_Z)$ is a crepant log structure, X is $\mathbb{Q}$-factorial and (X,Δ_X) is dlt.

Let $f:(X,\Delta_X)\to Z$ be a dlt crepant log structure and $Y\subset X$ an lc center. Consider the Stein factorization

$$f|_Y:Y\stackrel{f_Y}{\longrightarrow}Z_Y\stackrel{\pi}{\longrightarrow}Z$$

and set $\Delta_Y:=\operatorname{Diff}^*_Y\Delta_X$ (cf. (4)). Then $\big(Y,\Delta_Y\big)$ is dlt, $f_Y:\big(Y,\Delta_Y\big)\to Z_Y$ is a crepant log structure and $f|_Y:\big(Y,\Delta_Y\big)\to Z$ is a weak crepant log structure.

Definition 3 (Divisorial log terminal)**.** A pair $(X,\sum a_iD_i)$ is called *simple normal crossing* (abbreviated as *snc*) if X is smooth and for every $p\in X$ one can choose an open neighborhood $p\in U$ and local coordinates x_i such that for every i there is an index $a(i)$ such that $D_i\cap U=(x_{a(i)}=0)$.

As key examples, I emphasize that the pair $\big(\mathbb{A}^2_k,(x^2=y^2+y^3)\big)$ is not snc and $\big(\mathbb{A}^2_k,(x^2+y^2=0)\big)$ is snc iff $\sqrt{-1}\in k$. Thus being snc is a Zariski local but not an étale local property.

Given any pair (X,Δ), there is a largest open subset $X^{snc}\subset X$ such that $\big(X^{snc},\Delta|_{X^{snc}}\big)$ is snc.

A pair (X,Δ) is called *divisorial log terminal* (abbreviated as *dlt*) if the discrepancy $a(E,X,\Delta)$ is >-1 for every divisor whose center is contained in $X\setminus X^{snc}$.

Definition 4 (Different)**.** Let (X,Δ) be a dlt pair and $Y\subset X$ an lc center. Generalizing the usual notion of the different [K+92, Sec.16], there is a naturally defined $\mathbb{Q}$-divisor $\operatorname{Diff}^*_Y\Delta$, called the *different* of Δ on Y such that

$$\big(K_X+\Delta\big)|_Y\sim_{\mathbb{Q}}K_Y+\operatorname{Diff}^*_Y\Delta.$$

The traditional different [K+92, Sec.16] is defined such that if $Y = D$ is a divisor then

$$\big(K_X + D + \Delta\big)|_D \sim_{\mathbb{Q}} K_D + \operatorname{Diff}_D \Delta.$$

Thus, in this case, $\operatorname{Diff}^*_D(D + \Delta) = \operatorname{Diff}_D \Delta$. This inductively defines $\operatorname{Diff}^*_Y \Delta$ whenever Y is an irreducible component of a complete intersection of divisors in $\lfloor \Delta \rfloor$. In the dlt case, this takes care of every lc center; see [Kol10, Chap.2] for details.

The following result was proved by Hacon (and published in [KK10]). A simplified proof is in [Fuj10].

Proposition 5. *Let (Z, Δ_Z) be an lc pair. Then it has a $\mathbb{Q}$-factorial, crepant, dlt model $p : (X, \Delta_X) \to (Z, \Delta_Z)$. That is, X is $\mathbb{Q}$-factorial, (X, Δ_X) is dlt, $K_X + \Delta_X$ is p-nef and $\Delta_X = E + p_*^{-1}\Delta_Z$ where E contains all p-exceptional divisors with multiplicity 1.* □

6 (Birational weak crepant log structures)**.**

Let $f : (X, \Delta) \to Z$ be a weak crepant log structure. If f factors as $X \stackrel{g}{\to} X' \stackrel{f'}{\to} Z$ where g is birational, then $f' : (X', \Delta' := g_*\Delta) \to Z$ also a weak crepant log structure. We say that $f : (X, \Delta) \to Z$ *birationally dominates* $f' : (X', \Delta') \to Z$.

Conversely, assume that $f' : (X', \Delta') \to Z$ is a weak crepant log structure and $g : X \to X'$ is a proper birational morphism. Write $K_X + \Delta \sim_{\mathbb{Q}} g^*\big(K_{X'} + \Delta'\big)$. Then $f := f' \circ g : (X, \Delta) \to Z$ is also a weak crepant log structure.

By (5), every (weak) crepant log structure $f : (X, \Delta) \to Z$ is dominated by another (weak) crepant log structure $f^* : (X^*, \Delta^*) \to Z$ such that (X^*, Δ^*) is dlt and $\mathbb{Q}$-factorial. If Δ is effective then we can choose Δ^* to be effective.

Two weak crepant log structures $f_i : (X_i, \Delta_i) \to Z$ are called *birational* if there is a third weak crepant log structure $h : (Y, \Delta_Y) \to Z$ which birationally dominates both of them. If the Δ_i are effective and Y is the normalization of the closure of the graph of the birational map then Δ_Y is also effective. Thus if the $f_i : (X_i, \Delta_i) \to Z$ are crepant log structures then we can choose $h : (Y, \Delta_Y) \to Z$ to be a crepant log structure.

The group of birational self-maps of a weak crepant log structure $f : (X, \Delta) \to Z$ is denoted by $\operatorname{Bir}_Z(X, \Delta)$. By also allowing k-automorphisms, we get the larger group $\operatorname{Bir}_k(X, \Delta)$.

Let $f : (X, \Delta) \to Z$ be a weak crepant log structure and $f' : X' \to Z$ a proper morphism. Assume that there is a birational map $\phi : X \dashrightarrow X'$ such that $f' \circ \phi = f$. By the above, there is a unique $\mathbb{Q}$-divisor Δ' such that $f' : (X', \Delta') \to Z$ is a weak crepant log structure that is birational to $f : (X, \Delta) \to Z$. If ϕ^{-1} has no exceptional divisors, then $\Delta' = \phi_*\Delta$ and hence Δ' is effective if Δ is.

Let $f_i : (X_i, \Delta_i) \to S$ be weak crepant log structures and $\phi : X_1 \dashrightarrow X_2$ a birational map. Let $Z_1 \subset X_1$ an lc center such that, at the generic point of Z_1, the pair (X_1, Δ_1) is dlt and ϕ is a local isomorphism. Then $Z_2 := \phi_* Z_1$ is also an lc center and

$$\phi|_{Z_1} : \big(Z_1, \operatorname{Diff}^*_{Z_1} \Delta_1\big) \dashrightarrow \big(Z_2, \operatorname{Diff}^*_{Z_2} \Delta_1\big) \quad \text{is birational.}$$

Theorem 7. [NU73, Uen75, Gon10, FG10] *Let $f : (X, \Delta_X) \to Z$ be a crepant log structure. Then:*

(1) *The $\operatorname{Bir}_Z(X, \Delta_X)$ action on $\omega_X^{[m]}(m\Delta_X)$ is finite for every $m \geq 0$.*

(2) *If Z is projective and $K_X+\Delta_X \sim_{\mathbb{Q}} f^*$(ample $\mathbb{Q}$-divisor) then the* $\operatorname{Bir}_k(X,\Delta_X)$ *action on Z is finite.* □

8 (Minimal dominating lc centers)**.** Let $f:(X,\Delta)\to S$ be a dlt, weak crepant log structure. Let $W\subset S$ be an lc center and $\{W_i : i\in I(W)\}$ the minimal (with respect to inclusion) lc centers of (X,Δ) that dominate W. We claim that the set of their birational isomorphism classes

$$\big\{\big(W_i, \operatorname{Diff}^*_{W_i}\Delta\big) : i\in I(W)\big\} \tag{8.1}$$

is a birational invariant of $f:(X,\Delta)\to S$.

To see this note that by [Sza94] we can assume that (X,Δ) is snc. Then it is enough to check birational invariance for one smooth blow up. If we blow up $V\subset X$ that is not an lc center, then the set of lc centers is unchanged.

If V is an lc center that is the complete intersection of say $D_1,\dots,D_r\subset\lfloor\Delta\rfloor$, then we get an exceptional divisor E_V that is a $\mathbb{P}^{r-1}$-bundle over V. Locally on V, we get a direct product

$$\big(E_V, \operatorname{Diff}^*_{E_V}\Delta_{B_VX}\big)\cong\big(V,\operatorname{Diff}^*_V\Delta\big)\times\big(\mathbb{P}^{r-1},(x_1\cdots x_r=0)\big),$$

thus every minimal lc center of $\big(V,\operatorname{Diff}^*_V\Delta\big)$ corresponds to r isomorphic copies of itself among the minimal lc centers of $\big(E_V,\operatorname{Diff}^*_{E_V}\Delta_{B_VX}\big)$, hence among the minimal lc centers of $\big(B_VX,\Delta_{B_VX}\big)$. □

Our next aim is to prove that for crepant log structures, the invariant defined in (8.1) consist of a single birational equivalence class.

$\mathbb{P}^1$-linking of minimal lc centers.

Definition 9 ($\mathbb{P}^1$-linking)**.** A *standard* $\mathbb{P}^1$*-link* is a dlt, $\mathbb{Q}$-factorial, pair $\big(X, D_1+D_2+\Delta\big)$ whose sole lc centers are D_1,D_2 (hence D_1 and D_2 are disjoint) plus a proper morphism $\pi:X\to S$ such that $K_X+D_1+D_2+\Delta\sim_{\mathbb{Q},\pi}0$, $\pi:D_i\to S$ are both isomorphisms and every reduced fiber $\operatorname{red}X_s$ is isomorphic to $\mathbb{P}^1$.

Let F denote a general smooth fiber. Then $\big((K_X+D_1+D_2)\cdot F\big)=0$, hence $(\Delta\cdot F)=0$. That is, Δ is a vertical divisor, the projection gives an isomorphism $\big(D_1,\operatorname{Diff}_{D_1}\Delta\big)\cong\big(D_2,\operatorname{Diff}_{D_2}\Delta\big)$ and these pairs are klt.

The simplest example of a standard $\mathbb{P}^1$-link is a product

$$\big(S\times\mathbb{P}^1, S\times\{0\}+S\times\{\infty\}+\Delta_S\times\mathbb{P}^1\big)$$

for some $\mathbb{Q}$-divisor Δ_S.

It turns out that every standard $\mathbb{P}^1$-link is locally the quotient of a product. To see this note that $\big((D_1-D_2)\cdot F\big)=0$, thus every point $s\in S$ has an open neighborhood U such that $D_1-D_2\sim_{\mathbb{Q}}0$ on $\pi^{-1}(U)$. Taking the corresponding cyclic cover we get another standard $\mathbb{P}^1$-link

$$\tilde\pi:\big(\tilde X_U,\tilde D_1+\tilde D_2+\tilde\Delta\big)\to\tilde U$$

where the $\tilde D_i$ are now Cartier divisors and $\tilde\Delta=\tilde\pi^*\tilde\Delta_U$ for some $\mathbb{Q}$-divisor $\tilde\Delta_U$. Here $\tilde D_1\sim\tilde D_2$, hence the linear system $|\tilde D_1,\tilde D_2|$ maps $\tilde X_U$ to $\mathbb{P}^1$. Together with $\tilde\pi$ this gives an isomorphism

$$\big(\tilde U\times\mathbb{P}^1,\tilde U\times\{0\}+\tilde U\times\{\infty\}+\tilde\Delta_U\times\mathbb{P}^1\big)\cong\big(\tilde X_U,\tilde D_1+\tilde D_2+\tilde\Delta\big).$$

Let $g:(X,\Delta)\to S$ be a crepant, dlt log structure and $Z_1,Z_2\subset X$ two lc centers. We say that Z_1,Z_2 are *directly* $\mathbb{P}^1$*-linked* if there is an lc center $W\subset X$

containing the Z_i such that $g(W) = g(Z_1) = g(Z_2)$ and $\bigl(W, \operatorname{Diff}^*_W \Delta\bigr)$ is birational to a standard $\mathbb{P}^1$-link with Z_i mapping to D_i.

We say that $Z_1, Z_2 \subset X$ are $\mathbb{P}^1$-*linked* if there is a sequence of lc centers $Z'_1, \dots, Z'_m$ such that $Z'_1 = Z_1, Z'_m = Z_2$ and Z'_i is directly $\mathbb{P}^1$-linked to Z'_{i+1} for $i = 1, \dots, m-1$ (or $Z_1 = Z_2$).

The following strengthening of [KK10, 1.7] was the reason to introduce the notion of $\mathbb{P}^1$-linking.

Theorem 10. *Let k be a field and S essentially of finite type over k. Let $f : (X, \Delta) \to S$ be a proper morphism such that $K_X + \Delta \sim_{\mathbb{Q},f} 0$ and (X, Δ) is dlt. Let $s \in S$ be a point such that $f^{-1}(s)$ is connected (as a $k(s)$-scheme). Let $Z \subset X$ be minimal (with respect to inclusion) among the lc centers of (X, Δ) such that $s \in f(Z)$. Let $W \subset X$ be an lc center of (X, Δ) such that $s \in f(W)$.*

Then there is an lc center $Z_W \subset W$ such that Z and Z_W are $\mathbb{P}^1$-linked.

In particular, all the minimal (with respect to inclusion) lc centers $Z_i \subset X$ such that $s \in f(Z_i)$ are $\mathbb{P}^1$-linked to each other.

Remarks. For the applications it is crucial to understand the case when $k(s)$ is not algebraically closed. See (12) for some relevant examples.

Each $\mathbb{P}^1$-linking defines a birational map $Z \dashrightarrow Z_W$, but different $\mathbb{P}^1$-linkings can give different birational maps, see (13).

Proof. We use induction on $\dim X$ and on $\dim Z$.

Write $\lfloor \Delta \rfloor = \sum D_i$. By passing to a strict étale neighborhood of $s \in S$ we may assume that each $D_i \to Y$ has connected fiber over s and every lc center of (X, Δ) intersects $f^{-1}(s)$. (We need a *strict* étale neighborhood, that is, the residue field at s is unchanged, to make sure that $f^{-1}(s)$ stays connected, cf. [Mil80, I.4.2].)

Assume first that $f^{-1}(s) \cap \sum D_i$ is connected. By suitable indexing, we may assume that $Z \subset D_1$, $W \subset D_r$ and $f^{-1}(s) \cap D_i \cap D_{i+1} \neq \emptyset$ for $i = 1, \dots, r-1$.

By induction, we can apply (10) to $D_1 \to S$ with Z as Z and $D_1 \cap D_2$ as W. We get that there is an lc center $Z_2 \subset W$ such that Z and Z_2 are $\mathbb{P}^1$-linked. As we noted in (9), Z_2 is also minimal (with respect to inclusion) among the lc centers of (X, Δ) such that $s \in f(Z_2)$. Note that Z_2 is an lc center of $\bigl(D_1, \operatorname{Diff}^*_{D_1}(\Delta)\bigr)$. By adjunction, it is an lc center of (X, Δ) and also an lc center of $\bigl(D_2, \operatorname{Diff}^*_{D_2}(\Delta)\bigr)$.

Next we apply (10) to $D_2 \to S$ with Z_2 as Z and $D_2 \cap D_3$ as W, and so on. At the end we work on $D_r \to S$ with Z_r as Z and W as W to get an lc center $Z_W \subset W$ such that Z and Z_W are $\mathbb{P}^1$-linked. This proves the first claim if $f^{-1}(s) \cap \sum D_i$ is connected.

If $f^{-1}(s) \cap \sum D_i$ is disconnected, then write $\Delta = \sum_{i=1}^m D_i + \Delta_1$. We claim that in this case $m = 2$ and $D_1, D_2 \subset X$ are directly $\mathbb{P}^1$-linked (by $W = X$). We may assume that X is $\mathbb{Q}$-factorial.

First we show that $\sum D_i$ dominates S. Indeed, consider the exact sequence

$$0 \to \mathcal{O}_X(-\textstyle\sum D_i) \to \mathcal{O}_X \to \mathcal{O}_{\sum D_i} \to 0$$

and its push-forward

$$\mathcal{O}_S \cong f_*\mathcal{O}_X \to f_*\mathcal{O}_{\sum D_i} \to R^1 f_* \mathcal{O}_X(-\textstyle\sum D_i).$$

Since $-\sum D_i \sim_{\mathbb{Q},f} K_X + \Delta_1$, the sheaf $R^1 f_* \mathcal{O}_X(-\sum D_i)$ is torsion free by [Kol86] (see [KK10] for the extension to the klt case that we use). Thus $\mathcal{O}_S \twoheadrightarrow f_*\mathcal{O}_{\sum D_i}$ is surjective hence $\sum D_i \to S$ has connected fibers, a contradiction.

This implies that $K_X+\Delta_1$ is not f-pseudo-effective and so by [BCHM10, 1.3.2] one can run the (X,Δ_1)-MMP over S.

Every step is numerically $K_X+\sum D_i+\Delta_1$-trivial, hence $\sum D_i$ is ample on every extremal ray. Therefore a connected component of $\sum D_i$ can never be contracted by a birational contraction. By the Connectedness Theorem [K$^+$92, 17.4], the connected components of $\sum D_i$ are unchanged for birational contractions and flips. Thus, at some point, we must encounter a Fano contraction $p:(X^*,\Delta_1^*)\to V$ where $\sum D_i^*$ is p-ample. So there is an irreducible component, say D_1^* that has positive intersection with the contracted ray. Therefore D_1^* is p-ample. By assumption, there is another irreducible component, say D_2^* that is disjoint from D_1^*. Let $F_v\subset X^*$ be any fiber that intersects D_2^*. Since D_2^* is disjoint from D_1^*, we see that D_2^* does not contain F_v. Thus D_2^* also has positive intersection with the contracted ray, hence D_2^* is also p-ample.

Thus D_1^* and D_2^* are both relatively ample (possibly multi-) sections of p and they are disjoint. This is only possible if p has fiber dimension 1, the generic fiber is a smooth rational curve and D_1^* and D_2^* are sections of p.

Since p is an extremal contraction, $R^1p_*\mathcal{O}_{X^*}=0$, which implies that every fiber of p is a tree of smooth rational curves. Both D_1^* and D_2^* intersects every fiber in a single point and they both intersect every contracted curve. Thus every fiber is irreducible and so $p:(X^*,\Delta^*)\to V$ is a standard $\mathbb{P}^1$-link with D_1^*, D_2^* as sections. As we noted in (9), the rest of Δ^* consists of vertical divisors. Thus any other D_i^* would make $f^{-1}(s)\cap\sum D_i$ connected. Therefore D_1^*, D_2^* are the only lc centers of $(X^*, D_1^*+D_2^*+\Delta_1^*)$ and so D_1, D_2 are the only lc centers of (X,Δ). As noted at the end of (6), $D_1, D_2\subset X$ are directly $\mathbb{P}^1$-linked (by $W=X$). □

Corollary 11. *Let $f:(X,\Delta_X)\to S$ be a dlt, crepant log structure. Let $Y\subset X$ be an lc center. Consider the Stein factorization $f|_Y: Y\xrightarrow{f_Y} S_Y\xrightarrow{\pi} S$ and set $\Delta_Y:=\operatorname{Diff}^*_Y\Delta_X$. Then*

1. (1) *$f_Y:(Y,\Delta_Y)\to S_Y$ is a dlt, crepant log structure.*
2. (2) *Let $W_Y\subset S_Y$ be an lc center of $f_Y:(Y,\Delta_Y)\to S_Y$. Then $\pi(W_Y)\subset S$ is an lc center of $f:(X,\Delta_X)\to S$ and every minimal lc center of (Y,Δ_Y) dominating W_Y is also a minimal lc center of (X,Δ_X) dominating $\pi(W_Y)$.*
3. (3) *Let $W\subset S$ be an lc center of $f:(X,\Delta_X)\to S$. Then every irreducible component of $\pi^{-1}(W)$ is an lc center of $f_Y:(Y,\Delta_Y)\to S_Y$.*

Proof. (1) is clear. To see (2), note that W_Y is dominated by an lc center V_Y of $(Y,\operatorname{Diff}^*_Y\Delta)$. Thus, by adjunction, V_Y is also an lc center of (X,Δ), hence $\pi(W_Y)=f(V_Y)$ is an lc center of S. By (10), a minimal lc center of Y that dominates W_Y is also a minimal lc center of X that dominates $\pi(W_Y)$. Thus $\operatorname{Src}(W_Y,Y,\Delta_Y)\sim\operatorname{Src}(\pi(W_Y),X,\Delta_X)$.

Finally let $W\subset S$ be an lc center of $f:(X,\Delta_X)\to S$ and $w\in W$ the generic point. Let $V_X\subset X$ be a minimal lc center that dominates W. By (10), there is an lc center $V_Y\subset Y$ that is $\mathbb{P}^1$-linked to V_X. By adjunction, V_Y is also an lc center of $(Y,\operatorname{Diff}^*_Y\Delta)$. Thus $f_Y(V_Y)\subset S_Y$ is an lc center of $f_Y:(Y,\Delta_Y)\to S_Y$ and it is also one of the irreducible components of $\pi^{-1}(W)$.

In order to get (3), after replacing S by an étale neighborhood of w, we may assume that $Y=\cup Y_j$ such that each $f^{-1}(w)\cap Y_j$ is connected. By the previous argument, each Y_j yields an lc center $f_{Y_j}(V_{Y_j})\subset S_{Y_j}$ and together these show that every irreducible component of $\pi^{-1}(W)$ is an lc center of $f_Y:(Y,\Delta_Y)\to S_Y$. □

The following example illustrates some of the subtler aspects of the dlt condition in (10).

Example 12. Set $X = \mathbb{A}^3$ and D_1, D_2, D_3 planes intersecting only at the origin. Let $\pi : B_0X \to X$ denote the blow-up of the origin with exceptional divisor E. Then $K_{B_0X} + E + \sum D_i' \sim \pi^*(K_X + \sum D_i)$ where $D_i' := \pi_*^{-1} D_i$. There are 3 minimal lc centers over 0, given by $p_i := E \cap D_{i-1}' \cap D_{i+1}'$ (indexing modulo 3).

Assume now that we are over $\mathbb{Q}$, D_1 is defined over $\mathbb{Q}$ and $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ interchanges D_2, D_3. Now there are 2 minimal lc centers. One is p_1 the other is the irreducible $\mathbb{Q}$-scheme p_2+p_3. Thus p_1 and p_2+p_3 can not be $\mathbb{P}^1$-linked. This is not a contradiction since $(B_0X, E + \sum D_i')$ is not dlt; the divisor $D_2' + D_3'$ (which is irreducible over $\mathbb{Q}$) is not normal. We get a dlt model by blowing up the curve $D_2' \cap D_3'$. Now there are 2 minimal lc centers over 0, both isomorphic to $p_2 + p_3$.

Similarly, if $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ permutes the 3 planes, then we need to blow up all 3 intersections $D_i' \cap D_j'$ to get a dlt model. Over $\bar{\mathbb{Q}}$, there are 6 minimal lc centers over $\bar{\mathbb{Q}}$. Over $\mathbb{Q}$ there is either only one (if $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the planes as the symmetric group S_3) or two, both consisting of 3 conjugate points and isomorphic as $\mathbb{Q}$-schemes to each other (if $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ permutes cyclically).

Example 13. Fix $m \geq 3$ and ϵ a primitive mth root of unity. On $\mathbb{P}^{m-1}$ consider the μ_m-action generated by

$$\tau_1 : (x_0 : x_1 : \cdots : x_{m-1}) \mapsto (x_0 : \epsilon x_1 : \cdots : \epsilon^{m-1} x_{m-1}).$$

The action moves the divisor $D_0 := (x_0 + x_1 + \cdots + x_{m-1} = 0)$ into m different divisors $D_0, \dots, D_{m-1}$. One easily checks that $\left(\mathbb{P}^{m-1}, D_0 + \cdots + D_{m-1}\right)$ is snc (if ϵ is in our base field) and has trivial log canonical class.

Let A be an abelian variety with a μ_m-action τ_2. On

$$\left(\mathbb{P}^{m-1} \times A, \Delta := D_0 \times A + \cdots + D_{m-1} \times A\right)$$

we have a μ_m-action generated by $\tau := (\tau_1, \tau_2)$.

Let $X_1 := \left(\mathbb{P}^{m-1} \times A\right)/\langle \tau \rangle$. The quotient of the boundary Δ has only 1 component but it has complicated self-intersections, hence it is not dlt. Let (X, Δ_X) be a dlt model.

We see that the minimal lc centers are isomorphic to $(A, 0)$ and the different $\mathbb{P}^1$-linkings between them differ from each other by a power of τ_2.

3. Poincaré residue map

Definition 14. Let (X, Δ) be a dlt pair and $Z \subset X$ an lc center. As in (4), if $\omega_X^{[m]}(m\Delta)$ is locally free, then, by iterating the usual Poincaré residue maps for divisors, we get a *Poincaré residue map*

$$\mathcal{R}^m_{X \to Z} : \omega_X^{[m]}(m\Delta)|_Z \xrightarrow{\cong} \omega_Z^{[m]}(m \cdot \mathrm{Diff}_Z^* \Delta). \tag{14.1}$$

(This is well defined in m is even, defined only up to sign if m is odd.)

Let $f : (X, \Delta) \to Y$ be a dlt, weak crepant log structure. Choose $m > 0$ even such that $\omega_X^{[m]}(m\Delta) \sim f^*L$ for some line bundle L on Y. Let $Z \subset X$ be an lc center of (X, Δ). For $m > 0$ and even, we can view the Poincaré residue map as

$$\mathcal{R}^m_{X \to Z} : f^*L|_Z \cong \omega_X^{[m]}(m\Delta)|_Z \xrightarrow{\cong} \omega_Z^{[m]}(m \cdot \mathrm{Diff}_Z^* \Delta). \tag{14.2}$$

The following result shows, that, for minimal lc centers, (14.2) is essentially independent of the choice of Z.

Proposition 15. *Let $f : (X, \Delta) \to Y$ be a dlt crepant log structure (2). Choose $m > 0$ even such that $\omega_X^{[m]}(m\Delta) \cong f^*L$ for some line bundle L on Y. Let Z_1, Z_2 be minimal lc centers of (X, Δ) such that $f(Z_1) = f(Z_2)$. Then there is a birational map $\phi : Z_2 \dashrightarrow Z_1$ such that the following diagram commutes*

$$\begin{array}{ccc} \omega_X^{[m]}(m\Delta) & \cong f^*L \cong & \omega_X^{[m]}(m\Delta) \\ \mathcal{R}^m_{X\to Z_1} \downarrow & & \downarrow \mathcal{R}^m_{X\to Z_2} \\ \omega_{Z_1}^{[m]}\big(m\,\mathrm{Diff}^*_{Z_1}\Delta\big) & \stackrel{\phi^*}{\longrightarrow} & \omega_{Z_2}^{[m]}\big(m\,\mathrm{Diff}^*_{Z_2}\Delta\big) \end{array} \qquad (15.1)$$

Proof. By (10) it is sufficient to prove this in case there is an lc center W that is birational to a $\mathbb{P}^1$-bundle $\mathbb{P}^1 \times U$ with Z_1, Z_2 as sections. Thus projection to U provides a birational isomorphism $\phi : Z_2 \dashrightarrow Z_1$.

Since $\mathcal{R}^m_{X\to Z_i} = \mathcal{R}^m_{W\to Z_i} \circ \mathcal{R}^m_{X\to W}$, we may assume that $X = W$. The sheaves in (15.1) are torsion free, hence it is enough to check commutativity after localizing at the generic point of U. This reduces us to the case when $W = \mathbb{P}^1_L$ with coordinates $(x{:}y)$, $Z_1 = (0{:}1)$ and $Z_2 = (1{:}0)$. A generator of $H^0\big(\mathbb{P}^1, \omega_{\mathbb{P}^1}(Z_1 + Z_2)\big)$ is dx/x which has residue 1 at Z_1 and -1 at Z_2. Thus (15.1) commutes for m even and anti-commutes for m odd. □

Remark 16. By (15), we get a Poincaré residue map as stated in (1.4) but it is not yet completely canonical. We think of (Z, Δ_Z) as an element of a birational equivalence class thus so far $\mathcal{R}^m$ is defined only up to the action of $\mathrm{Bir}_Y(Z, \Delta_Z)$. However, by (7), the image of this action is a finite group of rth roots of unity for some r. Thus the $\mathrm{Bir}_Y(Z, \Delta_Z)$ action is trivial on $\omega_Z^{[mr]}(mr\Delta_Z)$ hence

$$\mathcal{R}^{mr} : \omega_X^{[mr]}(mr\Delta)|_Z \cong \omega_Z^{[mr]}(mr\Delta_Z) \qquad (16.1)$$

is completely canonical. Assume next that $\omega_X^{[mr]}(mr\Delta) \sim f^*L$. Let us factor $f|_Z : Z \to f(Z)$ using $g : Z \to W$ and the normalization $n : W \to f(Z)$. Then we can push forward (16.1) to get an isomorphism

$$n^*L \cong \big(g_*\omega_Z^{[m]}(m\Delta_Z)\big)^{\mathrm{inv}} \qquad (16.2)$$

where the exponent inv denotes the invariants under the action of the group of birational self-maps $\mathrm{Bir}_Y(Z, \Delta_Z)$. This shows the second isomorphism in (1.4).

Notation 17. Let (Y, Δ_Y) be lc and $(X, \Delta_X) \to (Y, \Delta_Y)$ a crepant, dlt model. Let $W \subset Y$ be an lc center of (Y, Δ_Y) and $Z \subset X$ minimal (with respect to inclusion) among the lc centers of (X, Δ_X) that dominate W. By (14), we obtain a Poincaré residue map $\mathcal{R}_{X\to Z}$.

Let $D \subset \lfloor \Delta_Y \rfloor$ be a divisor with normalization $\pi : D^n \to D$. Let $D_X \subset X$ be its birational transform on X and set $\Delta_{D_X} := \mathrm{Diff}^*_{D_X}\Delta_X$. Let $W_D \subset D^n$ be an lc center of $\big(D^n, \mathrm{Diff}^*_{D^n}\Delta_Y\big)$. Then $W_X := \pi(W_D)$ is an lc center of $\big(Y, \Delta_Y\big)$. Choose minimal lc centers $Z_X \subset X$ (resp. $Z_D \subset D_X$) dominating W_X (resp. W_D).

Theorem 18. *Notation and assumptions as in (17). Then there is a birational map $\phi : Z_D \dashrightarrow Z_X$ such that for m sufficiently divisible, the following diagram* *commutes*

$$\begin{array}{ccc} \omega_X^{[m]}(m\Delta_X) & \stackrel{\mathcal{R}^m_{X\to D_X}}{\longrightarrow} & \omega_{D_X}^{[m]}(m\Delta_{D_X}) \\ \mathcal{R}^m_{X\to Z_X}\downarrow & & \downarrow \mathcal{R}^m_{D_X\to Z_D} \\ \omega_{Z_X}^{[m]}\bigl(m\operatorname{Diff}^*_{Z_X}\Delta_X\bigr) & \stackrel{\phi^*}{\longrightarrow} & \omega_{Z_D}^{[m]}\bigl(m\operatorname{Diff}^*_{Z_D}\Delta_{D_X}\bigr) \end{array}$$

Proof. If we choose Z_X as the image of Z_D, this holds by the definition of the higher codimension residue maps. This and (15) proves the claim for every other choice of Z_X. □

4. Sources and Springs

Definition 19. Let $f : (X, \Delta) \to S$ be a crepant, dlt log structure and $Z \subset S$ an lc center. An lc center Z' of (X, Δ) is called a *source of* Z if $f(Z') = Z$ and Z' is minimal (with respect to inclusion) among the lc centers that dominate Z.

By restriction we have $f|_{Z'} : \bigl(Z', \operatorname{Diff}^*_{Z'}\Delta\bigr) \to Z$ and $K_{Z'} + \operatorname{Diff}^*_{Z'}\Delta \sim_{f,\mathbb{Q}} 0$. By adjunction, there is a one-to-one correspondence between lc centers of $\bigl(Z', \operatorname{Diff}^*_{Z'}\Delta\bigr)$ and lc centers of (X, Δ) that are contained in Z'. Thus Z' is a source of Z iff the general fiber of $\bigl(Z', \operatorname{Diff}^*_{Z'}\Delta\bigr) \to Z$ is klt.

By (10), all sources of Z are birational to each other (as weak crepant log structures over Z). Any representative of their birational equivalence class will be denoted by $\operatorname{Src}(Z, X, \Delta)$. One can choose a representative $(S^t, \Delta^t) \to Z$ whose generic fiber is terminal. Such models are still not unique, but their generic fibers are isomorphic outside codimension 2 sets. However, if there is an irreducible component of Δ^t whose coefficient is 1 (these can not dominate Z) then it does not seem possible to choose a sensible subclass of models that are isomorphic to each other outside codimension 2 sets.

Note further that by (8), if two crepant log structures $f_i : (X_i, \Delta_i) \to Y$ are birational over Y, then $\operatorname{Src}(Z, X_1, \Delta_1)$ is birational to $\operatorname{Src}(Z, X_2, \Delta_2)$.

One can uniquely factor $f|_{Z'}$ as

$$f|_{Z'} : \bigl(Z', \operatorname{Diff}^*_{Z'}\Delta'\bigr) = \operatorname{Src}(Z, X, \Delta) \stackrel{c_Z}{\longrightarrow} \tilde{Z}' \stackrel{p_Z}{\longrightarrow} Z \tag{19.1}$$

where $\tilde{Z}'$ is normal, p_Z is finite and c_Z has connected fibers.

Thus in (19.1), $\tilde{Z}'$ is uniquely defined up to isomorphism over Z. Any representative of its isomorphism class will be denoted by $\operatorname{Spr}(Z, X, \Delta)$ and called the *spring* of Z.

Define the group of *source-automorphisms* of $\operatorname{Spr}(Z, X, \Delta)$ as

$$\operatorname{Aut}^s \operatorname{Spr}(Z, X, \Delta) := \operatorname{im}\bigl[\operatorname{Bir}_k \operatorname{Src}(Z, X, \Delta) \to \operatorname{Aut}_k \operatorname{Spr}(Z, X, \Delta)\bigr]. \tag{19.2}$$

By (7), if $K_X + \Delta$ is ample then $\operatorname{Aut}^s \operatorname{Spr}(Z, X, \Delta)$ is finite for every lc center $Z \subset X$.

Let (Y, Δ) be lc and $f : (X, \Delta_X) \to (Y, \Delta)$ a dlt model (5). Let $Z \subset Y$ be an lc center of (Y, Δ). As noted above, the source $\operatorname{Src}(Z, X, \Delta_X)$ of Z depends only on (Y, Δ) but not on the choice of (X, Δ_X). Thus we also use $\operatorname{Src}(Z, Y, \Delta)$ (resp. $\operatorname{Spr}(Z, Y, \Delta)$) to denote the source (resp. spring) of Z.

Next we prove (1.5).

Proposition 20. *Let $f : (X, \Delta) \to Y$ be a crepant log structure and $Z \subset Y$ an lc center. Then the field extension $k\big(\operatorname{Spr}(Z, X, \Delta)\big)/k(Z)$ is Galois and*

$$\operatorname{Gal}\big(\operatorname{Spr}(Z, X, \Delta)/Z\big) \subset \operatorname{Aut}^s \operatorname{Spr}(Z, X, \Delta).$$

Proof. We may localize at the general point of Z. Thus we may assume that Z is a point and then prove the following more precise result.

Lemma 21. *Let $g : (X, \Delta) \to Y$ be a weak crepant log structure over a field k. Assume that (X, Δ) is dlt and X is $\mathbb{Q}$-factorial. Let $z \in Y$ be an lc center such that $g^{-1}(z)$ is connected (as a $k(z)$-scheme). Then there is a unique smallest finite field extension $K(z) \supset k(z)$ such that*

1. *Every lc center of $(X_{\bar{k}}, \Delta_{\bar{k}})$ that intersects $g^{-1}(z)$ is defined over $K(z)$.*
2. *Let $W_{\bar{z}} \subset Y_{\bar{k}}$ be a minimal lc center contained in $g^{-1}(z)$. Then $K(z) = k_{ch}(W_{\bar{z}})$, the field of definition of $W_{\bar{z}}$.*
3. *$K(z) \supset k(z)$ is a Galois extension.*
4. *Let W_z be a minimal lc center contained in $g^{-1}(z)$. Then*

$$\operatorname{Bir}_{k(z)}\big(W_z, \operatorname{Diff}^*_{W_z} \Delta\big) \to \operatorname{Gal}\big(K(z))/k(z)\big) \quad \textit{is surjective.}$$

Proof. There are only finitely many lc centers and a conjugate of an lc center is also an lc center. Thus the field of definition of any lc center is a finite extension of k. Since $K(z)$ is the composite of some of them, it is finite over $k(z)$.

Let $W_{\bar{z}} \subset X_{\bar{k}}$ be a minimal lc center contained in $g^{-1}(z)$ and $k_{ch}(W_{\bar{z}})$ its field of definition. Let $D_i \subset \lfloor \Delta \rfloor$ be the irreducible components that contain $W_{\bar{z}}$. Each D_i is smooth at the generic point of $W_{\bar{z}}$, hence the $\bar{k}$-irreducible component of D_i that contains $W_{\bar{z}}$ is also defined over $k_{ch}(W_{\bar{z}})$. Thus every lc center of $(X_{\bar{k}}, \Delta_{\bar{k}})$ containing $W_{\bar{z}}$ is also defined over $k_{ch}(W_{\bar{z}})$. Therefore, any lc center that is $\mathbb{P}^1$-linked to $W_{\bar{z}}$ is defined over $k_{ch}(W_{\bar{z}})$. By (10) this implies that every lc center of $(X_{\bar{k}}, \Delta_{\bar{k}})$ that intersects $g^{-1}(z)$ is defined over $k_{ch}(W_{\bar{z}})$, hence $k_{ch}(W_{\bar{z}}) \supset K(z)$. By construction, $k_{ch}(W_{\bar{z}}) \subset K(z)$, thus $k_{ch}(W_{\bar{z}}) = K(z)$.

A conjugate of $W_{\bar{z}}$ over $k(z)$ is defined over the corresponding conjugate field of $k_{ch}(W_{\bar{z}})$. By the above, every conjugate of the field of $k_{ch}(W_{\bar{z}})$ over $k(z)$ is itself, hence $k_{ch}(W_{\bar{z}}) = K(z)$ is Galois over $k(z)$.

Finally, in order to see (4), fix $\sigma \in \operatorname{Gal}\big(K(z)/k(z)\big)$ and let $W_{\bar{z}}^{\sigma}$ be the corresponding conjugate of $W_{\bar{z}}$. By (10), $W_{\bar{z}}^{\sigma}$ and $W_{\bar{z}}$ are $\mathbb{P}^1$-linked over $K(z)$; fix one such $\mathbb{P}^1$-link. The union of the conjugates of this $\mathbb{P}^1$-link over $k(z)$ define an element of $\operatorname{Bir}_{k(z)}\big(W_z, \operatorname{Diff}^*_{W_z} \Delta\big)$ which induces σ on $K(z)/k(z)$. (The $\mathbb{P}^1$-link is not unique, hence the lift is not unique. Thus in (4) we only claim surjectivity, not a splitting.) □

Example 22. The Galois extension $K(z)/k(z)$ can be arbitrary. To see this pick a Galois extension $K = k(\alpha)/k$ of degree n. In $\mathbb{A}^n_k$ consider the subspace $(\sum_i \alpha^{i-1} x_i = 0)$ and its n conjugates $D_1, \dots, D_n$. Then $\big(\mathbb{A}^n_k, \sum D_i\big)$ is lc, the origin is an lc center and its spring gives the Galois extension K/k.

From the classification of 2-dimensional lc pairs we see that if $\operatorname{codim}_X Z = 2$ then $\operatorname{Gal}\big(\operatorname{Spr}(Z, X, \Delta)/Z\big)$ is cyclic or dihedral.

The examples in [Kol11b] show that if $\operatorname{codim}_X Z = 3$ then $\operatorname{Gal}\big(\operatorname{Spr}(Z, X, \Delta)/Z\big)$ can be arbitrary.

We also note the following direct consequence of (11).

Corollary 23 (Adjunction for sources)**.** *Let $(X, D+\Delta)$ be lc and $n : D^n \to D$ the normalization. Let $Z_D \subset D^n$ be an lc center of $\bigl(D^n, \operatorname{Diff}_{D^n}\Delta\bigr)$ and $Z_X := n(Z_D)$ its image in X. Then*

1. $\operatorname{Src}\bigl(Z_D, D^n, \operatorname{Diff}_{D^n}\Delta\bigr) \overset{bir}{\sim} \operatorname{Src}\bigl(Z_X, X, D+\Delta\bigr)$ *and*
2. $\operatorname{Spr}\bigl(Z_D, D^n, \operatorname{Diff}_{D^n}\Delta\bigr) \cong \operatorname{Spr}\bigl(Z_X, X, D+\Delta\bigr)$. □

5. Applications to slc pairs

24 (Normalization of slc pairs)**.** Let (X, Δ) be a semi log canonical pair. Let $\pi : \bar{X} \to X$ denote the normalization of X, $\bar{\Delta}$ the divisorial part of $\pi^{-1}(\Delta)$ and $\bar{D} \subset \bar{X}$ the conductor of π. Since X is seminormal, $\bar{D}$ is reduced. X has an ordinary node at a codimension 1 singular point, thus interchanging the two preimages of the node gives an involution τ of the normalization $n : \bar{D}^n \to \bar{D}$. This gives an injection

$$\bigl\{ \text{ slc pairs } (X, \Delta) \bigr\} \quad \hookrightarrow \quad \left\{ \begin{array}{c} \text{lc pairs } \bigl(\bar{X}, \bar{D}+\bar{\Delta}\bigr) \\ \text{plus an involution } \tau \text{ of } \bar{D}^n \end{array} \right\}. \tag{24.1}$$

For many purposes, it is important to understand the image of this map. That is, we would like to know which quadruples $\bigl(\bar{X}, \bar{D}+\bar{\Delta}, \tau\bigr)$ correspond to an slc pair (X, Δ). An easy condition to derive is that τ is an involution not just of the variety $\bar{D}^n$ but of the lc pair $\bigl(\bar{D}^n, \operatorname{Diff}_{\bar{D}^n}\bar{\Delta}\bigr)$. Thus we obtain a refined version of the map

$$\bigl\{ \text{ slc pairs } (X, \Delta) \bigr\} \quad \hookrightarrow \quad \left\{ \begin{array}{c} \text{lc pairs } \bigl(\bar{X}, \bar{D}+\bar{\Delta}\bigr) \text{ plus an} \\ \text{involution } \tau \text{ of } \bigl(\bar{D}^n, \operatorname{Diff}_{\bar{D}^n}\bar{\Delta}\bigr) \end{array} \right\}. \tag{24.2}$$

For surfaces, the above constructions are discussed in [K+92, Sec.12]. The higher dimensional generalizations are straightforward; see [Kol11a, Chap3].

There are three major issues involved in trying to prove that the map (24.2) is surjective.

24.3.1. *Does τ generate a finite equivalence relation?*

The normalization $n : \bar{D}^n \to \bar{D} \to \bar{X}$ and τ generate an equivalence relation $R(\tau)$, called the *gluing relation*, on the points of $\bar{X}$ by declaring $n(p) \sim n(\tau(p))$ for every $p \in \bar{D}^n$. It is clear that if X exists, then every equivalence class of $R(\tau)$ is contained in a fiber of $\pi : \bar{X} \to X$. In particular, if X exists then the $R(\tau)$-equivalence classes are finite.

Example 35 shows that in general the $R(\tau)$-equivalence classes need not be finite. Moreover, non-finiteness can appear in high codimension.

This is the question that we will study here using the sources of lc centers, especially their Galois property (1.5).

24.3.2. *Constructing (X, Δ) from $\bigl(\bar{X}, \bar{D}+\bar{\Delta}, \tau\bigr)$.*

Assume that the $R(\tau)$-equivalence classes are finite. Following the method of [Kol08], it is proved in [Kol11a, Chap.3], that if the $R(\tau)$-equivalence classes are finite, then (X, Δ) exists.

24.3.3. *Is $K_X + \Delta$ a $\mathbb{Q}$-Cartier divisor?* The answer turns out to be yes, see [Kol11a, Chap.3], but my proof, using Poincaré residue maps and (7), is somewhat indirect. Example 36 shows that this result depends on delicate properties of lc pairs.

As a consequence we obtain that (24.2) is one-to-one for pairs with ample log canonical class.

Theorem 25. *Taking the normalization gives a one-to-one correspondence between the following two sets, where $X, \bar{X}$ are projective schemes over a field.*

$$\left\{\begin{array}{c}\textit{slc pairs } (X,\Delta) \\ \textit{such that} \\ K_X+\Delta \textit{ is ample}\end{array}\right\} \cong \left\{\begin{array}{c}\textit{lc pairs } \big(\bar{X},\bar{D}+\bar{\Delta}\big) \textit{ plus an} \\ \textit{involution } \tau \textit{ of } \big(\bar{D}^n, \operatorname{Diff}_{\bar{D}^n}\bar{\Delta}\big) \\ \textit{such that } K_{\bar{X}}+\bar{D}+\bar{\Delta} \textit{ is ample}\end{array}\right\}.$$

This can be extended to the relative case as follows.

Theorem 26. *Let S be a scheme which is essentially of finite type over a field. Taking the normalization gives a one-to-one correspondence between the following two sets.*

(1) *Slc pairs (X, Δ) such that X/S is proper and $K_X + \Delta$ is ample on the generic fiber of $W \to S$ for every lc center $W \subset X$.*
(2) *Lc pairs $\big(\bar{X}, \bar{D}+\bar{\Delta}\big)$ such that $\bar{X}/S$ is proper and $K_{\bar{X}}+\bar{D}+\bar{\Delta}$ is ample on the generic fiber of $\bar{W} \to S$ for every lc center $\bar{W} \subset \bar{X}$, plus an involution τ of $\big(\bar{D}^n, \operatorname{Diff}_{\bar{D}^n}\bar{\Delta}\big)$.*

Furthermore, the cases when $K_X + \Delta$ is ample on X/S correspond to the cases when $K_{\bar{X}}+\bar{D}+\bar{\Delta}$ is ample on $\bar{X}/S$.

As we noted in (24.3), the following result implies (25).

Proposition 27. *Let $\big(\bar{X}, \bar{D}+\bar{\Delta}\big)$ be an lc pair and τ an involution of $\big(\bar{D}^n, \operatorname{Diff}_{\bar{D}^n}\bar{\Delta}\big)$.*

Assume that X is proper over a base scheme S that is essentially of finite type over a field. Assume furthermore that $K_{\bar{X}}+\bar{D}+\bar{\Delta}$ is ample on the generic fiber of $\bar{W} \to S$ for every lc center $\bar{W} \subset \bar{X}$.

Then the gluing relation $R(\tau)$, defined in (24.3.1), is finite.

This in turn will be derived from a structure theorem (33) on the gluing relation $R(\tau)$ which applies whether $K_{\bar{X}}+\bar{D}+\bar{\Delta}$ is ample or not.

Roughly speaking, we prove that for every lc center $\bar{W} \subset \bar{X}$ there is a "canonically" defined finite cover $p : \tilde{W} \to \bar{W}$ such that the following hold outside the union of lower dimensional lc centers.

(27.1) $(p\times p)^{-1}\big(R\cap(\bar{W}\times\bar{W})\big)$ is the union of graphs $\{\Gamma(g) : g \in G\}$ where G is a subgroup of $\operatorname{Aut}(\tilde{W})$.
(27.2) G is compatible with $p^*\big(K_{\bar{X}}+\bar{D}+\bar{\Delta}\big)$.

The compatibility condition (27.2) is somewhat delicate to state. Thus I give the actual construction of $\tilde{W}$ and then specify the compatibility condition for that particular case.

Almost group actions.

Definition 28. Let Y be an irreducible variety and G a countable (discrete) group acting on Y. For $g \in G$, let $\Gamma(g) \subset Y \times Y$ be the graph of g. As a set, $\Gamma(g) = \{(y, g(y)) : y \in Y\}$. Their union $\Gamma(G) := \cup_g \Gamma(g)$ is a pro-finite set-theoretic equivalence relation on Y. Note that $\Gamma(G)$ is finitely generated (that is, it is the equivalence closure of finite relation) iff G is a finitely generated group.

Somewhat imprecisely, we say that a pro-finite equivalence relation $R \subset Y \times Y$ is a *group action* if $R = \Gamma(G)$ for some group G.

Let X be an irreducible variety and $R \subset X \times X$ a pro-finite set-theoretic equivalence relation. We say that R is *almost a group action* if there is an irreducible variety Y, a finite surjection $p : Y \to X$ and an (at most) countable group G acting on Y such that $\operatorname{red}(p \times p)^{-1}R = \Gamma(G)$.

Similarly, if X, Y are reducible, one can define the notion of R being *almost a groupoid action.* This holds if every irreducible component of $\operatorname{red}(p \times p)^{-1}R$ is the graph of an isomorphism between two irreducible components of Y.

Note that not every pro-finite equivalence relation is almost a group action.

First of all, $\Gamma(G)$ is pure dimensional of dimension $\dim Y$, thus if R is almost a group action then R is pure dimensional of dimension $\dim X$. Every finite and pure dimensional equivalence relation is almost a group action; see, for instance [Kol97, 21]. This fails for pro-finite equivalence relations, see (30), but the problem seems to be entirely field-theoretic.

Proposition 29. *Let X be an irreducible, normal variety and $R = \cup_{i\in I} R_i$ a set-theoretic equivalence relation of pure dimension $\dim X$. Assume that R is generated by the sub-relation $R_J := \cup_{i\in J} R_i$ for some subset $J \subset I$. The following are equivalent.*

1. *R is almost a group action.*
2. *There is a field K and embeddings $j_i : k(R_i) \hookrightarrow K$ such that*

$$k(X) \xrightarrow{\pi_1^*} k(R_i) \xrightarrow{j_i} K \quad \text{and} \quad k(X) \xrightarrow{\pi_2^*} k(R_i) \xrightarrow{j_i} K \tag{29.2.i}$$

   are both finite degree Galois extensions for every $i \in I$, where π_1, π_2 are the coordinate projections.
3. *The maps (29.2.i) are both finite degree Galois extensions for every $i \in J$.*

Proof. Assume that $p : Y \to X$ and the group G show that R is almost a group action. Then $\operatorname{red}(p \times p)^{-1}(\text{diagonal of } X \times X) \subset \Gamma(G)$ corresponds to a finite subgroup $H \subset G$ and $H = \operatorname{Gal}(Y/X)$. Then $K := k(Y)$ shows (2) and the latter clearly implies (3).

Assume (3) and let Y be the normalization of X in K. Since $K/k(X)$ is Galois, $K \otimes_{k(X)} k(R_i)$ is a direct sum of copies of K for both inclusions $\pi_j^* : k(X) \hookrightarrow k(R_i)$. Thus the irreducible components of $\operatorname{red}(p \times p)^{-1}(R_i)$ have degree 1 over Y for both projections. They are also finite, hence graphs of automorphisms. Thus the equivalence relation they generate is a group action. □

The next example shows that, even on $\mathbb{P}^1$, not every purely 1-dimensional equivalence relation is an almost group action.

Example 30. Let $R \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the equivalence relation generated by the graph of $(x : y) \mapsto (x^2 : y^2)$ and by any curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ of geometric genus at least 2.

We claim that R is not almost a group action; it is not even a sub-relation of an almost group action.

Assume to the contrary that there is a finite morphism $p : D \to \mathbb{P}^1$ and a group G acting on D such that $(p \times p)^{-1}R \subset \Gamma(G)$. Note first that $(x : y) \mapsto (x^2 : y^2)$ generates a pro-algebraic equivalence relation most of whose equivalence classes are infinite. Thus the group G has to be infinite. On the other hand, one of the components of $\Gamma(G)$ dominates C, hence $g(D) \geq g(C) \geq 2$. Thus $\operatorname{Aut}(D)$ is finite and so is G, a contradiction.

A more complicated, but theoretically much more significant example is the following.

Example 31. [BT09] There is a smooth curve D of genus ≥ 2 and a finite relation $R_0 \subset D \times D$ such that both projections $R_0 \rightrightarrows D$ are étale but R_0 generates a non-finite pro-algebraic equivalence relation.

The structure of gluing relations.

We are now ready to formulate and prove a structure theorem for gluing relations: they are almost groupoids on every stratum of the log canonical stratification.

Notation 32. Let (X,Δ) be lc. Let $S_i^*(X,\Delta)$ be the union of all $\leq i$-dimensional lc centers of $\bigl(X,\Delta\bigr)$ and set $S_i(X,\Delta) := S_i^*(X,\Delta)\setminus S_{i-1}^*(X,\Delta)$. Let $Z_{ij}^0 \subset S_i(X,\Delta)$ be the irreducible components. The closure Z_{ij} of Z_{ij}^0 is an lc center of $\bigl(X,\Delta\bigr)$, hence it has a spring $p_{ij} : \operatorname{Spr}(Z_{ij},X,\Delta) \to Z_{ij}$ (19). Set $\operatorname{Spr}(Z_{ij}^0,X,\Delta) := p_{ij}^{-1}Z_{ij}^0$ and

$$\operatorname{Spr}_i\bigl(X,\Delta\bigr) := \amalg_j \operatorname{Spr}(Z_{ij}^0,X,\Delta).$$

Let $p_i : \operatorname{Spr}_i\bigl(X,\Delta\bigr) \to S_i(X,\Delta)$ be the induced morphism. Then p_i is finite, surjective and universally open since $S_i(X,\Delta)$ is normal. Furthermore, p_i is Galois over every Z_{ij} by (20)

Theorem 33. *Let $\bigl(X,D+\Delta\bigr)$ be lc, τ an involution of $\bigl(D^n,\operatorname{Diff}_{D^n}\Delta\bigr)$ and $R(\tau) \subset X\times X$ the corresponding pro-finite equivalence relation (24.3.1). Let $p_i : \operatorname{Spr}_i\bigl(X,D+\Delta\bigr) \to S_i$ be as in (32). Then*

1. *$(p_i\times p_i)^{-1}\bigl(R(\tau)\cap(S_i(X,\Delta)\times S_i(X,\Delta))\bigr)$ is a groupoid on $\operatorname{Spr}_i\bigl(X,D+\Delta\bigr)$.*
2. *For every irreducible component $Z_{ij}^0 \subset S_i(X,\Delta)$, the stabilizer of its spring $\operatorname{Spr}(Z_{ij}^0,X,D+\Delta) \subset \operatorname{Spr}_i\bigl(X,D+\Delta\bigr)$ is a subgroup of the source-automorphism group $\operatorname{Aut}^s\operatorname{Spr}(Z_{ij},X,D+\Delta)$ (19.2).*

Proof. We need to describe how the generators of $R(\tau)$ pull back to the spring $\operatorname{Spr}_i\bigl(X,D+\Delta\bigr)$.

First, the preimage of the diagonal of $Z_{ij}^0 \times Z_{ij}^0$ is a group $\Gamma(G_{ij})$ and $G_{ij} = \operatorname{Gal}\bigl(\operatorname{Spr}(Z_{ij},X,D+\Delta)/Z_{ij}\bigr)$ is a subgroup of $\operatorname{Aut}^s\operatorname{Spr}(Z_{ij},X,D+\Delta)$ by (20).

Second, let $Z_{ijk} \subset D^n$ be an irreducible component of the preimage of Z_{ij}. Then Z_{ijk} is an lc center of $\bigl(D^n,\operatorname{Diff}_{D^n}\Delta\bigr)$ and

$$\operatorname{Src}\bigl(Z_{ijk},D^n,\operatorname{Diff}_{D^n}\Delta\bigr) \overset{bir}{\sim} \operatorname{Src}\bigl(Z_{ij},X,D+\Delta\bigr)$$

by (23). Thus, for each ijk, the isomorphism $\tau : D^n \cong D^n$ lifts to isomorphisms

$$\tau_{ijkl} : \operatorname{Spr}(Z_{ij}^0,X,D+\Delta) \cong \operatorname{Spr}(Z_{il}^0,X,D+\Delta).$$

Given ijk, the value of l is determined by $Z_{il} := n\bigl(\tau(Z_{ijk})\bigr)$, but the lifting is defined only up to left and right multiplication by elements of G_{ij} and G_{il}.

Thus $(p_i\times p_i)^{-1}\bigl(R(\tau)\cap(S_i(X,\Delta)\times S_i(X,\Delta))\bigr)$ is the groupoid generated by the G_{ij} and the τ_{ijkl}, hence the stabilizer of $\operatorname{Spr}(Z_{ij}^0,X,D+\Delta)$ is generated by the groups $\tau_{ijkl}^{-1}G_{il}\tau_{ijkl}$. The latter are all subgroups of $\operatorname{Aut}^s\operatorname{Spr}(Z_{ij},X,D+\Delta)$. □

34 (Proof of (27))**.** We apply (33) to $\bigl(\bar X,\bar D+\bar\Delta\bigr)$.

Since $\operatorname{Spr}_i(X,D+\Delta)$ has finitely many irreducible components, the groupoid is finite iff the stabilizer of each $\operatorname{Spr}(Z_{ij}^0,X,D+\Delta)$ is finite. By (33) this holds if the groups $\operatorname{Aut}^s\operatorname{Spr}(Z_{ij},X,D+\Delta)$ are finite.

The automorphism group of a variety $\tilde Z$ over a base scheme S injects into the automorphism group of the generic fiber $\tilde Z_{gen}$.

By assumption, $K_{\bar X}+\bar D+\bar\Delta$ is ample on the generic fiber of $Z_{ij}\to S$, thus (7) implies that each $\operatorname{Aut}^s\operatorname{Spr}(Z_{ij},X,D+\Delta)$ is finite. □

Examples.

The next example shows that the stabilizer groups of the strata can change drastically as we go to higher codimensions.

Example 35. Pick involutions $r_1,r_2,r_3\in PGL(2,\mathbb{C})$ such that any 2 of them generate a finite subgroup but the 3 together generate an infinite subgroup.

Consider $X=\mathbb{A}^3\times\mathbb{P}^1$. Let x_i be the coordinates on $\mathbb{A}^3$ and $D_i:=(x_i=0)\times\mathbb{P}^1$. On D_i consider the involution τ_i which is the identity on D_i and r_i on the $\mathbb{P}^1$-factor. Let $R\rightrightarrows X$ be the pro-finite set theoretic equivalence relation generated by the $\tau_i: i=1,2,3$.

Note that

$$\pi_1:\bigl(X\setminus D_1\bigr)\times\mathbb{P}^1\to\bigl(X\setminus D_1\bigr)\times\bigl(\mathbb{P}^1/\langle r_2r_3\rangle\bigr)$$

is finite, thus $R|_{X\setminus D_1}$ is a finite set theoretic equivalence relation. Similarly, $\bigl(X\setminus D_i\bigr)/\bigl(R|_{X\setminus D_1}\bigr)$ exists for $i=2,3$. Set $\mathbb{P}^1_0:=\{0\}\times\mathbb{P}^1$. Then the geometric quotient

$$\bigl(X\setminus\mathbb{P}^1_0\bigr)/\bigl(R|_{X\setminus\mathbb{P}^1_0}\bigr)$$

exists, but the restriction of R to $\mathbb{P}^1_0$ is not a finite equivalence relation since the subgroup generated by r_1,r_2,r_3 is infinite. Thus R is not a finite relation and there is no geometric quotient of X by R.

In order to find such r_1,r_2,r_3, its is easier to work with $SO(3,\mathbb{R})\cong SU(2,\mathbb{C})$. Let $L_i\subset\mathbb{R}^3$ be 3 lines such that the angles between them are rational multiples of π. Let r_i denote the reflections determined by the lines L_i. By assumption, the angle between any 2 lines is a rational multiple of π, hence any 2 rotations generate a finite dihedral group.

The finite subgroups of $G\subset SO(3,\mathbb{R})$ are all known. If G is not cyclic or dihedral, then any rotation in G has order ≤ 6. Thus, as soon as the denominator of the angle between L_i,L_j is large enough, the subgroup generated by r_1,r_2,r_3 is infinite. □

The following example shows that for a seminormal surface T with normalization $(\bar T,\bar C,\tau)$ the $\mathbb{Q}$-Cartier property of K_T depends very subtly on τ.

Example 36. We describe a flat family of seminormal surfaces $\bigl\{T(\lambda,\mu):(\lambda,\mu)\in\mathbb{C}^*\times\mathbb{C}^*\bigr\}$ such that the canonical class of $T(\lambda,\mu)$ is $\mathbb{Q}$-Cartier for a Zariski dense set of pairs $(\lambda,\mu)\in\mathbb{C}^*\times\mathbb{C}^*$ and not $\mathbb{Q}$-Cartier for another Zariski dense set of pairs.

In these examples the normalizations $\bigl(\bar T(\lambda,\mu),\bar C(\lambda,\mu)\bigr)$ are all isomorphic to each other, $\bar C(\lambda,\mu)$ is 2 copies of $(xy=0)\subset\mathbb{A}^2$ and $\tau(\lambda,\mu)$ is multiplication by λ on the x-axis and by μ on the x-axis.

Start with a cone S over a hyperelliptic curve and two rulings $C_x,C_y\subset S$. Take two copies of S and glue them together by the isomorphisms $C^1_x\to C^2_x$ and $C^1_y\to C^2_y$ which are multiplication by $\lambda\in\mathbb{C}^*$ (resp. $\mu\in\mathbb{C}^*$) to get a non-normal surface $T(\lambda,\mu)$. We show that its canonical class is $\mathbb{Q}$-Cartier iff λ/μ is a root of unity.

To get concrete examples, fix an integer $a\ge 0$ and set

$$S:=\bigl(z^2=xy(x^{2a}+y^{2a})\bigr)\subset\mathbb{A}^3\quad\text{and}\quad C:=C_x+C_y$$

where $C_x = (y = z = 0)$ and $C_y = (x = z = 0)$. Note that C is not Cartier but $2C = (xy = 0)$ is. Furthermore, ω_S is locally free with generator $z^{-1}dx \wedge dy$ and so $\omega_S^2(2C)$ is locally free with generator

$$\frac{1}{xyz^2}\bigl(dx \wedge dy\bigr)^{\otimes 2} = \frac{1}{x^2y^2(x^{2a}+y^{2a})}\bigl(dx \wedge dy\bigr)^{\otimes 2}.$$

The restriction of $\omega_S^2(2C)$ to C_x is thus locally free with generator

$$\frac{1}{x^2(x^{2a}+y^{2a})}\Bigl(dx \wedge \frac{dy}{y}\Bigr)^{\otimes 2}\Big|_{C_x} = \frac{1}{x^{2+2a}}\bigl(dx\bigr)^{\otimes 2}.$$

Hence the different on C_x is the origin with coefficient $1+a$. Similarly, the restriction of $\omega_S^2(2C)$ to C_y is locally free with generator $y^{-2-2a}\bigl(dy\bigr)^{\otimes 2}$.

Take now 2 copies S_i with coordinates (x_i, y_i, z_i) for $i \in \{1, 2\}$. Let $\tau(\lambda, \mu) : C_1 \to C_2$ be an isomorphism such that $\tau(\lambda, \mu)^* x_2 = \lambda x_1$ and $\tau(\lambda, \mu)^* y_2 = \mu y_1$. Let $T(\lambda, \mu)$ be obtained by gluing $C_1 \subset S_1$ to $C_2 \subset S_2$ using $\tau(\lambda, \mu)$.

Assume that $\omega^{2m}_{T(\lambda,\mu)}$ is locally free with generator σ. Then the restriction of σ to S_i is of the form

$$\sigma|_{S_i} = \frac{1}{x_i^{2m}y_i^{2m}(x_i^{2a}+y_i^{2a})^m}\bigl(dx_i \wedge dy_i\bigr)^{\otimes 2m} \cdot f_i(x_i, y_i, z_i)$$

for some f_i such that $f_i(0, 0, 0) \neq 0$. Furthermore,

$$\tau^*\Bigl(\sigma|_{S_2}\Bigr)|_{C_2} = \Bigl(\sigma|_{S_1}\Bigr)|_{C_1}.$$

Further restricting to the x-axis, this gives

$$\frac{1}{(\lambda x_1)^{2m+2am}}\bigl(\lambda dx_1\bigr)^{\otimes 2m} f_2(\lambda x_1, 0, 0) = \frac{1}{x_1^{2m+2am}}\bigl(dx_1\bigr)^{\otimes 2m} f_1(x_1, 0, 0).$$

which implies that

$$f_2(0, 0, 0) = \lambda^{2am} f_1(0, 0, 0).$$

Similarly, computing on the y-axis we obtain that

$$f_2(0, 0, 0) = \mu^{2am} f_1(0, 0, 0).$$

If $\lambda^{2am} \neq \mu^{2am}$, these imply that $f_1(0, 0, 0) = f_2(0, 0, 0) = 0$, hence $\omega^{[2m]}_{T(\lambda,\mu)}$ is not locally free. If $\lambda^{2am} = \mu^{2am}$ then $f_1(x_1, y_1, z_1) \equiv 1$ and $f_2(x_2, y_2, z_2) \equiv \lambda^{2am}$ give a global generator of $\omega^{[2m]}_{T(\lambda,\mu)}$.

For $a \geq 1$, we have our required examples. As λ, μ vary in $\mathbb{C}^* \times \mathbb{C}^*$, we get a flat family of seminormal surfaces $T(\lambda, \mu)$. The set of pairs (λ, μ) such that λ/μ is a root of unity is a Zariski dense subset of $\mathbb{C}^* \times \mathbb{C}^*$ whose complement is also Zariski dense.

Note, however, that for $a = 0$, $\omega^{[2]}_{T(\lambda,\mu)}$ is locally free for every λ, μ. In this case, $S := \bigl(z^2 = xy\bigr) \subset \mathbb{A}^3$ is a quadric cone and $T(\lambda, \mu)$ is slc. (In fact $T(\lambda, \mu)$ is isomorphic to the reducible quartic cone $(x^2 + y^2 + z^2 + t^2 = xy = 0) \subset \mathbb{A}^4$ for every λ, μ.)

Acknowledgments. This paper was written while I visited RIMS, Kyoto University. I thank S. Mori and S. Mukai for the invitation and their hospitality. I am grateful to O. Fujino, Ch. Hacon, S. Mori, Y. Odaka, V. Shokurov and Ch. Xu for many comments and corrections. Partial financial support was provided by the NSF under grant number DMS-0758275.

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