Compositions, Random Sums and Continued Random Fractions of Poisson and Fractional Poisson Processes

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Abstract

In this paper we consider the relation between random sums and compositions of different processes. In particular, for independent Poisson processes $N_\alpha(t)$, $N_\beta(t)$, $t>0$, we show that $N_\alpha(N_\beta(t)) \overset{\text{d}}{=} \sum_{j=1}^{N_\beta(t)} X_j$, where the X_js are Poisson random variables. We present a series of similar cases, the most general of which is the one in which the outer process is Poisson and the inner one is a nonlinear fractional birth process. We highlight generalisations of these results where the external process is infinitely divisible. A section of the paper concerns compositions of the form $N_\alpha(\tau_k^\nu)$, $\nu \in (0,1]$, where τ_k^ν is the inverse of the fractional Poisson process, and we show how these compositions can be represented as random sums. Furthermore we study compositions of the form $\Theta(N(t))$, $t>0$, which can be represented as random products. The last section is devoted to studying continued fractions of Cauchy random variables with a Poisson number of levels. We evaluate the exact distribution and derive the scale parameter in terms of ratios of Fibonacci numbers.

Keywords: Fractional birth process, Bell polynomials, Mittag–Leffler functions, Fibonacci numbers, continued fractions, Poisson random fields, golden ratio, Linnik distribution, discrete Mittag–Leffler distribution, logarithmic random variables, negative binomial distribution, Mellin transforms.

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1 Introduction

Publications in the field of probability have devoted considerable attention to compositions of different processes, as e.g. Brownian motions, fractional Brownian motions and telegraph processes. Also, more general cases as stable processes, combined in different ways have been investigated and the p.d.e. connections analysed. In the present paper we focus our attention on the composition of point processes, e.g. Poisson processes, fractional Poisson processes, and others. For independent homogeneous Poisson processes $N_\alpha(t)$, $N_\beta(t)$, $t>0$, we are able to show that $N_\alpha(N_\beta(t))$ has a remarkable connection, with respect to distributional properties, with random sums, that is, we prove that

$$\hat{N}(t) = N_\alpha(N_\beta(t)) \overset{\mathrm{d}}{=} \sum_{j=1}^{N_\beta(t)} X_j. \tag{1.1}$$

The first part of the paper is devoted to the presentation of similar results involving more general processes as e.g. nonlinear fractional birth processes $\mathscr{Y}^\nu(t)$, $t > 0$, $\nu \in (0,1]$, so as to obtain

$$N_\alpha(\mathscr{Y}^\nu(t)) \overset{\mathrm{d}}{=} \sum_{j=1}^{\mathscr{Y}^\nu(t)} X_j, \tag{1.2}$$

where the X_js appearing in (1.1) and (1.2) are independent Poisson random variables of parameter λ_α. A more general result is obtained when the external process is replaced by a process Θ possessing infinitely divisible distribution. In this case, we are able to show that

$$\Theta(N_\beta(t)) \overset{\mathrm{d}}{=} \sum_{j=1}^{N_\beta(t)} \xi_j, \tag{1.3}$$

where the random variables ξ_js are the components of the infinitely divisible random variable $\Theta(1)$, and $N_\beta(t)$, $t > 0$, is a homogeneous Poisson process. The representation (1.1) is remarkable in that it results in the explicit law of a random sum which is rarely possible in general. The distribution of (1.1) can be given as

$$\begin{aligned} \Pr\{N_\alpha(N_\beta(t)) = k\} &= \frac{\lambda_\alpha^k}{k!} e^{-\lambda_\beta t} \sum_{r=0}^{\infty} \frac{e^{-\lambda_\alpha r} r^k (\lambda_\beta t)^r}{r!} \\ &= \frac{\lambda_\alpha^k}{k!} e^{-\lambda_\beta t(1-e^{-\lambda_\alpha})} \mathfrak{B}_k\left(\lambda_\beta t e^{-\lambda_\alpha}\right), \end{aligned} \tag{1.4}$$

where

$$\mathfrak{B}_k(x) = e^{-x} \sum_{r=0}^{\infty} \frac{r^k x^r}{r!}, \tag{1.5}$$

are the so-called Bell polynomials, and λ_α, λ_β, are the parameters of, respectively, $N_\alpha(t)$ and $N_\beta(t)$. The composition (1.1) produces a process with linearly increasing mean value and variance:

$$\begin{cases} \mathbb{E} N_\alpha(N_\beta(t)) = \lambda_\alpha \lambda_\beta t, \\ \mathbb{V}\mathrm{ar} N_\alpha(N_\beta(t)) = \lambda_\alpha(\lambda_\alpha + 1)\lambda_\beta t. \end{cases} \tag{1.6}$$

For the iterated Poisson process we find that the first-passage time

$$T_k = \inf(s : N_\alpha(N_\beta(s)) = k), \qquad k \geq 1, \tag{1.7}$$

has distribution

$$\Pr\{T_k \in \mathrm{d}s\} = \mathrm{d}s\, \lambda_\beta e^{-\lambda_\alpha} e^{-\lambda_\beta s} \frac{\lambda_\alpha^k}{k!} \sum_{j=0}^{\infty} e^{-\lambda_\alpha j} \left[(j+1)^k - j^k\right] \frac{(\lambda_\beta s)^j}{j!}, \qquad s < 0. \tag{1.8}$$

Furthermore, $\Pr\{T_k < \infty\} < 1$, for all $k \geq 1$, so that there is a positive probability of never hitting the level k because the iterated Poisson process can take jumps of arbitrary integer-valued size.

Finally, we note that the iterated Poisson process produces a Galton–Watson process (continuous in time).
In the case in which $N_\beta(t)$ is replaced by a non-homogeneous Poisson process $\mathfrak{N}(t)$, with rate $\lambda(t)$, $t>0$, we still have a representation of the composition $N_\alpha(\mathfrak{N}(t))$ as a random sum, i.e.

$$N_\alpha(\mathfrak{N}(t)) \overset{\mathrm{d}}{=} \sum_{j=1}^{\mathfrak{N}(t)} X_j, \tag{1.9}$$

where the X_js are independent Poisson random variables of parameter λ_α. Since the probability generating function of (1.9) reads

$$\mathbb{E}u^{N_\alpha(\mathfrak{N}(t))} = e^{\int_0^t \lambda(w)\mathrm{d}w\left[e^{\lambda_\alpha(u-1)}-1\right]} \tag{1.10}$$

we have also that

$$\begin{cases} \mathbb{E}N_\alpha((t)) = \lambda_\alpha \int_0^t \lambda(w)\mathrm{d}w, \\ \mathbb{V}\mathrm{ar} = \lambda_\alpha(\lambda_\alpha+1)\int_0^t \lambda(w)\mathrm{d}w. \end{cases} \tag{1.11}$$

The process (1.1) has non-decreasing sample paths with jumps of integer-valued size and thus differs and extends the classical homogeneous Poisson process. We note that, for $\lambda_\alpha \neq \lambda_\beta$, we have that

$$\Pr\left\{N_\alpha(N_\beta(t)) \neq N_\beta(N_\alpha(t))\right\} = 1, \tag{1.12}$$

as can be inferred from the structure of the probability generating function. The interchange of the non-homogeneous Poisson process $\mathfrak{N}(t)$, with the homogeneous one produces a composed process

$$\mathfrak{N}(N_\alpha(t)), \tag{1.13}$$

whose probability generating function can be written as

$$\mathbb{E}u^{\mathfrak{N}(N_\alpha(t))} = \sum_{r=0}^\infty e^{(u-1)\int_0^r \lambda(w)\mathrm{d}w} \frac{(\lambda_\alpha t)^r}{r!} e^{-\lambda_\alpha t}. \tag{1.14}$$

Even the mean value of (1.13) differs from (1.11) since

$$\mathbb{E}\mathfrak{N}(N_\alpha(t)) = e^{-\lambda_\alpha t} \sum_{r=1}^\infty \left[\int_0^r \lambda(w)\mathrm{d}w\right] \frac{(\lambda_\alpha t)^r}{r!} = \sum_{j=1}^\infty \left[\int_{j-1}^j \lambda(w)\mathrm{d}w\right] \Pr\{N_\alpha(t) \geq j\}. \tag{1.15}$$

The third section deals with the inverse of the fractional Poisson process $N^\nu(t)$, $t>0$. The process $N^\nu(t)$ can be viewed as a renewal process with Mittag–Leffler distributed intertimes and the following probability distribution (see Mainardi et al. [2004], Beghin and Orsingher [2009])

$$\begin{aligned} \Pr\{N^\nu(t)=m\} &= \sum_{j=m}^\infty (-1)^{j-m}\binom{j}{m} \frac{(\lambda_\beta t^\nu)^j}{\Gamma(\nu j+1)} \\ &= (\lambda_\beta t^\nu)^m \sum_{j=0}^\infty \binom{j+m}{j} \frac{(-\lambda_\beta t^\nu)^j}{\Gamma\left(\nu(m+j)+1\right)}, \qquad t>0,\ \nu\in(0,1]. \end{aligned} \tag{1.16}$$

The inverse of the fractional Poisson process is defined as

$$\tau_k^\nu = \inf(t : N^\nu(t) = k), \qquad k \geq 1,\ \nu \in (0,1], \tag{1.17}$$

with distribution

$$\Pr\{\tau_k^\nu \in \mathrm{d}s\}/\mathrm{d}s = \lambda_\beta^k \sum_{j=0}^{\infty} \binom{-k}{j} \lambda_\beta^j \frac{s^{\nu(k+j)-1}}{\Gamma(\nu(k+j))} \tag{1.18}$$

and moment generating function

$$\mathbb{E}e^{-\mu\tau_k^\nu} = \left(\frac{\mu^\nu}{\lambda_\beta} + 1\right)^{-k}. \tag{1.19}$$

It should be noted that for $\nu = 1$ this coincides with the Erlang process. The composed process $N_\alpha(\tau_k^\nu)$ has probability generating function

$$\mathbb{E}u^{N_\alpha(\tau_k^\nu)} = \left[1 + (1-u)^\nu \frac{\lambda_\alpha^\nu}{\lambda_\beta}\right]^{-k} \tag{1.20}$$

which suggests the following interesting representation:

$$N_\alpha(\tau_k^\nu) \overset{\mathrm{d}}{=} \sum_{j=1}^{k} \xi_j, \tag{1.21}$$

where the independent random variables ξ_j, $1 \le j \le k$, are discrete Mittag–Leffler (see Pillai and Jayakumar [1995]). From (1.20), we can infer that $N_\alpha(\tau_k^\nu)$ has Linnik distribution and, for $\nu = 1$, this coincides with the negative binomial distribution having parameters k and $\lambda_\alpha/(\lambda_\alpha + \lambda_\beta)$. For the special case $\nu = 1$, we also have the following representation of $N_\alpha(\tau_k^\nu)$:

$$N_\alpha(\tau_k^\nu) \overset{\mathrm{d}}{=} X_1 + \cdots + X_N, \tag{1.22}$$

where N is a Poisson random variable of parameter $\mu = \log((\lambda_\alpha + \lambda_\beta)/\lambda_\beta)^k$, and the X_js are independent and possess logarithmic distribution of parameter $\eta = \lambda_\alpha/(\lambda_\alpha + \lambda_\beta)$. Furthermore, the case $N_\alpha(\phi_k^\nu)$ where the inner process is the inverse ϕ_k^ν of a fractional linear pure birth process $Y_\beta^\nu(t)$, is examined.
In particular, we show that

$$\mathbb{E}u^{N_\alpha(Y_\beta^\nu(t))} = k! \frac{\Gamma\left(\frac{\lambda_\alpha^\nu(1-u)^\nu}{\lambda_\beta} + 1\right)}{\Gamma\left(\frac{\lambda_\alpha^\nu(1-u)^\nu}{\lambda_\beta} + 1 + k\right)}, \qquad |u| < 1. \tag{1.23}$$

The final section of this paper deals with random products of non negative i.i.d. random variables of the form

$$N_\pi(t) = \prod_{i=1}^{N(t)} X_j, \qquad t > 0, \tag{1.24}$$

where $N(t)$ is a homogeneous Poisson process. We show that the Mellin transform of $N_\pi(t)$ is

$$\mathbb{E}\left(N_\pi(t)\right)^{\eta-1} = e^{\lambda t\left[\mathbb{E}X^{\eta-1}-1\right]}. \tag{1.25}$$

Furthermore, we can evaluate the covariance function of the process $N_\pi(t)$ which reads

$$\mathbb{C}\mathrm{ov}\left[N_\pi(t), N_\pi(s)\right] = e^{\lambda t(\mathbb{E}X-1)}\left(e^{\lambda s \mathbb{E}X(X-1)} - e^{\lambda s(\mathbb{E}X-1)}\right). \tag{1.26}$$

Clearly, the random products can be viewed as compositions of the form $\Xi(N(t))$, where

$$\Xi(k) = \begin{cases} 1, & k = 0, \\ \prod_{j=1}^{k} X_j = e^{\sum_{j=1}^{k} \log X_j}, & k > 1. \end{cases} \tag{1.27}$$

Finally, we consider continuous fractions with a random number of levels:

$$[X_1; X_2, \ldots, X_{N(t)}] = X_1 + \cfrac{1}{X_2 + \cfrac{1}{\ddots + X_{N(t)-1} + \frac{1}{X_{N(t)}}}}, \tag{1.28}$$

where the X_js are independent standard Cauchy random variables and $N(t)$ is a homogeneous Poisson process. We show that the conditional distribution of (1.28) is Cauchy in which the scale parameter equals F_{n+1}/F_n, where F_n are Fibonacci numbers. This permits us to give a stochastic representation of (1.28) in the form

$$[X_1; X_2, \ldots, X_{N(t)}] \overset{\text{d}}{=} \sum_{j=1}^{F_{N(t)+1}} Y_{j,N(t)}, \tag{1.29}$$

where $Y_{j,N(t)}$ are Cauchy random variables with scale parameter equal to $1/F_{N(t)}$.

2 Composition of Poisson processes with different point processes

In this section we examine the following compositions:

1. $N_\alpha(N_\beta(t))$, where N_α and N_β are independent Poisson processes;
2. $N_\alpha(\mathfrak{N}(t))$, where $\mathfrak{N}(t)$ is a non-homogeneous Poisson process with rate $\lambda(t)$, $t > 0$;
3. $N_\alpha(\mathscr{Y}^\nu(t))$, where $\mathscr{Y}^\nu(t)$ is a nonlinear fractional birth process with birth rates $\lambda_1, \ldots, \lambda_k, \ldots$, and $0 < \nu \leq 1$;
4. $N_\alpha(Y^\nu(t))$, where $Y^\nu(t)$ is a linear fractional birth process. This is a special case of the preceding point.
5. $N_\alpha(\mathscr{F}(\mathscr{B}))$, where $\mathscr{F}(\mathscr{B})$ is a Poisson field.
6. $\Theta(N_\beta(t))$, where $\Theta(1)$ is an infinitely divisible random variable and $N(t)$ is an arbitrary point process.

We will establish some distributional relations between these composed processes and random sums.

2.1 Iterated Poisson process

We start our analysis by considering the iterated Poisson process $\hat{N}(t) = N_\alpha(N_\beta(t))$, $t > 0$. The sample paths of $\hat{N}(t)$ are non-decreasing with jumps of arbitrary integer-valued size. In the figures 1 and 2, we give the trajectories of $N_\alpha(t)$ and $N_\beta(t)$ separately and then the sample path of $\hat{N}(t)$ obtained by their composition.
We note that the iterated Poisson process $\hat{N}(t)$ jumps at the occurence of events of the inner process $N_\beta(t)$. Thus, if the rate of $N_\beta(t)$ is large, then $\hat{N}(t)$ has rapidly increasing trajectories for large λ_α. However high values of λ_α and low levels of λ_β can produce contradictory results and thus compensate each other.

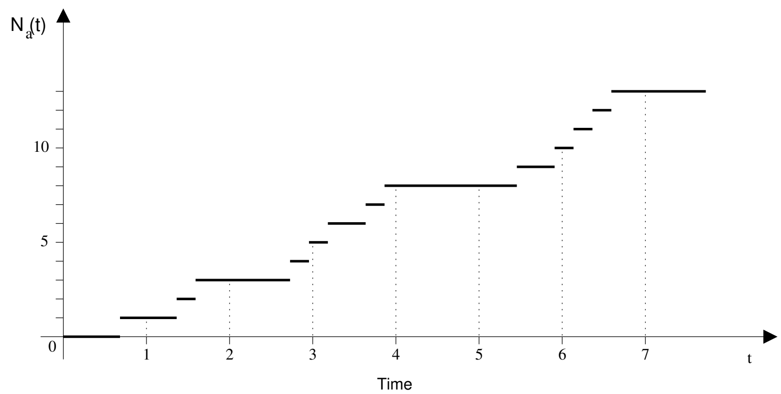

Figure 1: A realisation of the external Poisson process $N_\alpha(t)$, $t > 0$.

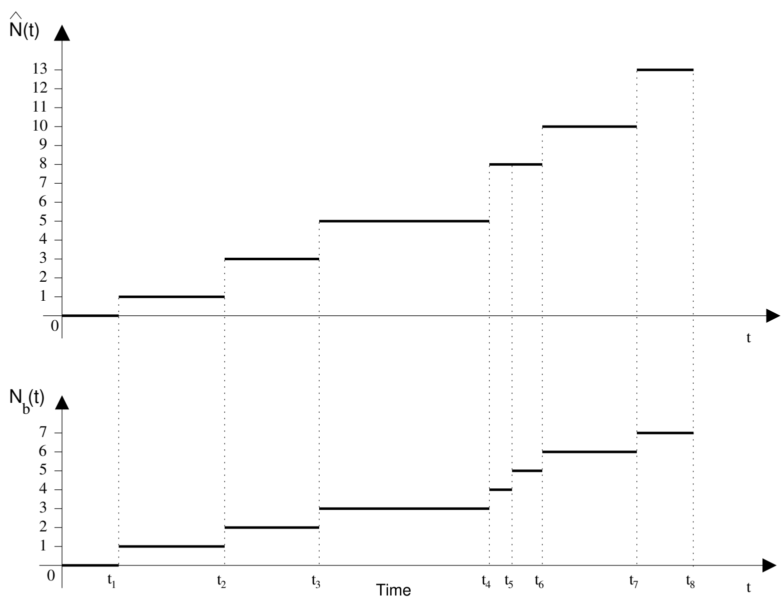

Figure 2: A path of the process $\hat{N}(t) = N_\alpha(N_\beta(t))$, $t > 0$, together with its relation for a specific path of the internal process $N_\beta(t)$, $t > 0$.

Theorem 2.1. *The distribution of* $\hat{N}(t) = N_\alpha(N_\beta(t))$ *reads*

$$
\begin{aligned}
Pr\{\hat{N}(t) = k\} &= \frac{\lambda_\alpha^k}{k!} e^{-\lambda_\beta t} \sum_{r=0}^{\infty} \frac{e^{-\lambda_\alpha r} r^k (\lambda_\beta t)^r}{r!} \\
&= \frac{\lambda_\alpha^k}{k!} e^{-\lambda_\beta t(1-e^{-\lambda_\alpha})} \mathfrak{B}_k \left(\lambda_\beta t e^{-\lambda_\alpha} \right), \qquad k \geq 0,\ t > 0,
\end{aligned}
\tag{2.1}
$$

where

$$
\mathfrak{B}_k(x) = e^{-x} \sum_{r=0}^{\infty} \frac{r^k x^r}{r!}, \tag{2.2}
$$

is the kth order Bell polynomial [Boyadzhiev, 2009]). The probability generating function of $\hat{N}(t)$ *has the form*

$$
\mathbb{E} u^{\hat{N}(t)} = e^{\lambda_\beta t \left(e^{\lambda_\alpha (u-1)} - 1 \right)}, \qquad |u| \leq 1. \tag{2.3}
$$

Proof.

$$
\begin{aligned}
\Pr\{\hat{N}(t) = k\} &= \sum_{r=0}^{\infty} \Pr\{N_\alpha(r) = k\} \Pr\{N_\beta(t) = r\} \\
&= \sum_{r=0}^{\infty} e^{-\lambda_\alpha r} \frac{(\lambda_\alpha r)^k}{k!} e^{-\lambda_\beta t} \frac{(\lambda_\beta t)^r}{r!}.
\end{aligned}
\tag{2.4}
$$

Furthermore

$$
\begin{aligned}
\mathbb{E} u^{\hat{N}(t)} &= \sum_{k=0}^{\infty} u^k \sum_{r=0}^{\infty} \frac{e^{-\lambda_\alpha r} (\lambda_\alpha r)^k}{k!} \cdot \frac{e^{-\lambda_\beta t} (\lambda_\beta t)^r}{r!} \\
&= \sum_{r=0}^{\infty} e^{-\lambda_\alpha r} e^{u \lambda_\alpha r} e^{\lambda_\beta t} \frac{(\lambda_\beta t)^r}{r!},
\end{aligned}
\tag{2.5}
$$

and thus (2.3) emerges. □

In Figure 3 we present the first four state probabilities as a function of time t for the iterated Poisson process in which $\lambda_\alpha = \lambda_\beta = 1$.

Theorem 2.2. *The following equality in distribution holds:*

$$
N_\alpha(N_\beta(t)) \stackrel{d}{=} \sum_{j=1}^{N_\beta(t)} X_j, \tag{2.6}
$$

where the X_j*s are i.i.d. Poisson random variables of parameter* λ_α.

Proof. The probability generating function of the random sum (2.6) is

$$
\mathbb{E} u^{\sum_{j=1}^{N_\beta(t)} X_j} = \sum_{k=0}^{\infty} \left(\mathbb{E} u^X \right)^k \Pr\{N_\beta(t) = k\} = e^{-\lambda_\beta t + \lambda_\beta t \mathbb{E} u^X} = e^{\lambda_\beta t \left(e^{\lambda_\alpha (u-1)} - 1 \right)}, \tag{2.7}
$$

and this coincides with (2.3). □

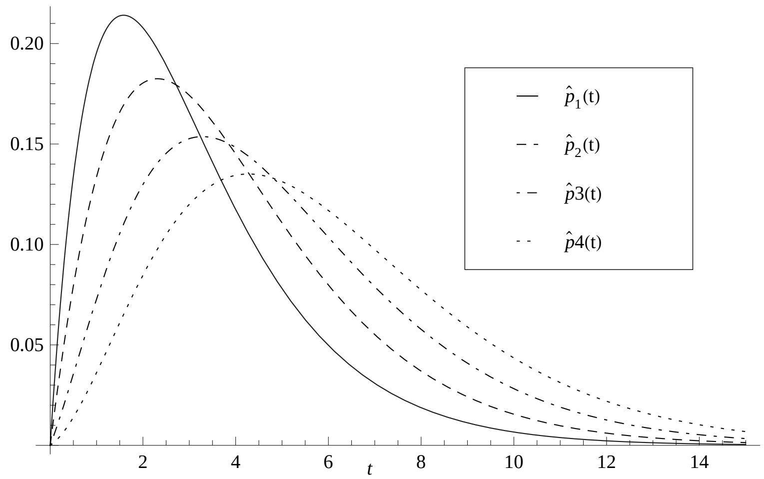

Figure 3: The first four state probabilities (Iterated Poisson). The parameters are $\lambda_\alpha = 1$, $\lambda_\beta = 1$.

Remark 2.1. *For the iterated Poisson process we can write*

$$\mathbb{E}\hat{N}(t) = \mathbb{E}N_\alpha(N_\beta(t)) = \lambda_\alpha \lambda_\beta t, \tag{2.8}$$

$$\mathbb{V}ar\hat{N}(t) = \lambda_\alpha(1+\lambda_\alpha)\lambda_\beta t. \tag{2.9}$$

We can obtain (2.8) *directly, by means of the probability generating function, and also by applying Wald's formula for random sums.*

$$\begin{aligned}\mathbb{E}N_\alpha(N_\beta(t)) &= \sum_{k=0}^{\infty} k \sum_{r=0}^{\infty} e^{-\lambda_\alpha r} \frac{(\lambda_\alpha r)^k}{k!} e^{-\lambda_\beta t} \frac{(\lambda_\beta t)^r}{r!} \\ &= e^{-\lambda_\beta t} \sum_{r=0}^{\infty} e^{-\lambda_\beta r} \frac{(\lambda_\beta t)^r}{r!} \sum_{k=1}^{\infty} \frac{(\lambda_\alpha r)^k}{(k-1)!} \\ &= e^{-\lambda_\beta t} \sum_{r=1}^{\infty} \lambda_\alpha^r \frac{(\lambda_\beta t)^r}{r!} = \lambda_\alpha \lambda_\beta t.\end{aligned} \tag{2.10}$$

Alternatively

$$\mathbb{E}N_\alpha(N_\beta(t)) = \frac{\mathrm{d}}{\mathrm{d}u} \mathbb{E}u^{\hat{N}(t)} \bigg|_{u=1} = \lambda_\alpha \lambda_\beta t \left(e^{\lambda_\alpha(u-1)}\right) e^{\lambda_\beta t \left(e^{\lambda_\alpha(u-1)-1}\right)} \bigg|_{u=1} = \lambda_\alpha \lambda_\beta t \tag{2.11}$$

and, by applying the Wald's formula

$$\mathbb{E}X\mathbb{E}N_\beta(t) = \mathbb{E}\hat{N}(t) = \lambda_\alpha \lambda_\beta t. \tag{2.12}$$

For the second moment, analogously we obtain

$$\mathbb{E}\hat{N}^2(t) = \sum_{k=0}^{\infty} k^2 \sum_{r=0}^{\infty} e^{-\lambda_\alpha r} \frac{(\lambda_\alpha r)^k}{k!} e^{-\lambda_\beta t} \frac{(\lambda_\beta t)^r}{r!} \tag{2.13}$$

$$
\begin{aligned}
&= e^{-\lambda_\beta t} \sum_{r=0}^{\infty} e^{-\lambda_\alpha r} \frac{(\lambda_\beta t)^r}{r!} \sum_{k=0}^{\infty} \frac{k^2}{k!} (\lambda_\alpha r)^k \\
&= e^{-\lambda_\beta t} \sum_{r=0}^{\infty} e^{-\lambda_\alpha r} \frac{(\lambda_\beta t)^r}{r!} \sum_{k=0}^{\infty} \frac{k+1}{k!} (\lambda_\alpha r)^{k+1} \\
&= e^{-\lambda_\beta t} \sum_{r=0}^{\infty} e^{-\lambda_\alpha r} \frac{(\lambda_\beta t)^r}{r!} \left[\sum_{k=0}^{\infty} \frac{(\lambda_\alpha r)^{k+2}}{k!} + \sum_{k=0}^{\infty} \frac{(\lambda_\alpha r)^{k+1}}{k!} \right] \\
&= e^{-\lambda_\beta t} \sum_{r=0}^{\infty} \frac{(\lambda_\beta t)^r}{r!} \left[(\lambda_\alpha r)^2 + (\lambda_\alpha r) \right] \\
&= e^{-\lambda_\beta t} \left[\sum_{r=0}^{\infty} \frac{(\lambda_\beta t)^{r+1}}{r!} \lambda_\alpha^2 (r+1) + \sum_{r=0}^{\infty} \frac{(\lambda_\beta t)^{r+1}}{r!} \lambda_\alpha \right] \\
&= \lambda_\alpha \lambda_\beta t + \lambda_\alpha^2 \lambda_\beta t + \lambda_\alpha^2 \lambda_\beta^2 t^2.
\end{aligned}
$$

Therefore

$$
\mathbb{V}ar\hat{N}(t) = \mathbb{E}\hat{N}^2(t) - \left[\mathbb{E}\hat{N}(t)\right]^2 = \lambda_\alpha \lambda_\beta (1+\lambda_\alpha) t. \tag{2.14}
$$

This result can be confirmed by applying the Wald's formula for the variance.

$$
\mathbb{V}ar\hat{N}(t) = \mathbb{V}arN_\beta(t) \left[\mathbb{E}X\right]^2 + \mathbb{V}arX \mathbb{E}N_\beta(t) = \lambda_\alpha \lambda_\beta (1+\lambda_\alpha) t, \tag{2.15}
$$

because $\mathbb{E}X = \mathbb{V}arX = \lambda_\alpha$*.*

Remark 2.2. *The state probabilities* $\hat{p}_k(t) = Pr\{\hat{N}(t) = k\}$ *satisfy the difference-differential equations*

$$
\frac{\mathrm{d}}{\mathrm{d}t} \hat{p}_k(t) = -\lambda_\beta \hat{p}_k(t) + \lambda_\beta e^{-\lambda_\alpha} \sum_{m=0}^{k} \frac{\lambda_\alpha^m}{m!} \hat{p}_{k-m}(t), \qquad k \geq 0. \tag{2.16}
$$

In order to prove (2.16) *we write*

$$
\hat{p}_k(t) = \sum_{r=0}^{\infty} Pr\{N_\alpha(r) = k\} Pr\{N_\beta(t) = r\}, \tag{2.17}
$$

so that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \hat{p}_k(t) &= \sum_{r=0}^{\infty} Pr\{N_\alpha(r) = k\} \frac{\mathrm{d}}{\mathrm{d}t} Pr\{N_\beta(t) = r\} \qquad (2.18) \\
&= -\lambda_\beta \sum_{r=0}^{\infty} Pr\{N_\alpha(r) = k\} Pr\{N_\beta(t) = r\} + \lambda_\beta \sum_{r=0}^{\infty} Pr\{N_\alpha(r) = k\} Pr\{N_\beta(t) = r-1\} \\
&= -\lambda_\beta \hat{p}_k(t) + \lambda_\beta \frac{\lambda_\alpha^k}{k!} e^{-\lambda_\beta t} \sum_{r=1}^{\infty} e^{-\lambda_\alpha r} \frac{r^k (\lambda_\beta t)^{r-1}}{(r-1)!} \\
&= -\lambda_\beta \hat{p}_k(t) + \lambda_\beta \frac{\lambda_\alpha^k}{k!} e^{-\lambda_\beta t} \sum_{r=0}^{\infty} e^{-\lambda_\alpha (r+1)} \frac{(r+1)^k (\lambda_\beta t)^r}{r!} \\
&= -\lambda_\beta \hat{p}_k(t) + \lambda_\beta e^{-\lambda_\beta t} \frac{\lambda_\alpha^k}{k!} e^{-\lambda_\alpha} \sum_{m=0}^{k} \frac{k!}{m!(k-m)!} \sum_{r=0}^{\infty} e^{-\lambda_\alpha r} r^m \frac{(\lambda_\beta t)^r}{r!}
\end{aligned}
$$

$$= -\lambda_\beta \hat{p}_k(t) + \lambda_\beta \lambda_\alpha^k e^{-\lambda_\alpha} \sum_{m=0}^{k} \frac{\lambda_\alpha^{-m}}{(k-m)!} \hat{p}_m(t)$$
$$= -\lambda_\beta \hat{p}_k(t) + \lambda_\beta e^{-\lambda_\alpha} \sum_{m=0}^{k} \frac{\lambda_\alpha^m}{m!} \hat{p}_{k-m}(t).$$

It is now easy to show that

$$\hat{G}(u,t) = \sum_{k=0}^{\infty} u^k \hat{p}_k(t) \tag{2.19}$$

satisfies the partial differential equation

$$\frac{\partial}{\partial t} \hat{G}(u,t) = -\lambda_\beta \hat{G}(u,t) + \lambda_\beta e^{\lambda_\alpha(u-1)} \hat{G}(u,t) = \lambda_\beta \hat{G}(u,t) \left(e^{\lambda_\alpha(u-1)} - 1 \right). \tag{2.20}$$

Remark 2.3. *For the composition $N_\alpha(\mathfrak{N}(t))$ we have that*

$$\sum_{k=0}^{\infty} u^k Pr\{N_\alpha(\mathfrak{N}(t)) = k\} = \sum_{k=0}^{\infty} u^k \sum_{r=0}^{\infty} \frac{(\lambda_\alpha r)^k}{k!} e^{-\int_0^k \lambda(w) \mathrm{d}w} \frac{\left[\int_0^t \lambda(w) \mathrm{d}w\right]^r}{r!} e^{-\lambda_\alpha r} \tag{2.21}$$
$$= \sum_{r=0}^{\infty} e^{-\lambda_\alpha r} \frac{\left[\int_0^t \lambda(w) \mathrm{d}w\right]^r}{r!} e^{\lambda_\alpha r u} e^{-\int_0^t \lambda(w) \mathrm{d}w}$$
$$= e^{\left[e^{\lambda_\alpha(u-1)} - 1\right] \int_0^t \lambda(w) \mathrm{d}w}.$$

We can also ascertain that

$$N_\alpha(\mathfrak{N}(t)) \stackrel{d}{=} \sum_{j=1}^{\mathfrak{N}(t)} X_j \tag{2.22}$$

by using result (2.21).

2.1.1 Hitting time for the iterated Poisson process

Here we study the distribution of the random variable

$$T_k = \inf(s : N_\alpha(N_\beta(s)) = k), \qquad k \geq 1, \tag{2.23}$$

which represents the first-passage time of the iterated Poisson process at level k. In the next theorem we state the main result.

Theorem 2.3.

$$Pr\{T_k \in \mathrm{d}s\} = \mathrm{d}s \lambda_\beta e^{-\lambda_\alpha} \frac{\lambda_\alpha^k}{k!} e^{-\lambda_\beta s} \sum_{j=0}^{\infty} e^{-\lambda_\alpha j} \left[(j+1)^k - j^k\right] \frac{(\lambda_\beta s)^j}{j!}, \qquad s > 0. \tag{2.24}$$

Proof. In order to arrive at result (2.24), we first write

$$\Pr\{T_k \in \mathrm{d}s\} = \sum_{h=1}^{k} \Pr\{N_\alpha(N_\beta(s)) = k-h, N_\alpha(N_\beta(s+\mathrm{d}s)) = k\} \tag{2.25}$$

$$= \sum_{h=1}^{k} \Pr\{N_\alpha(N_\beta(s)) = k-h, N_\alpha(N_\beta(s)+\mathrm{d}N_\beta(s)) = k\}.$$

Clearly, $\mathrm{d}N_\beta(s)$ either takes the value 0 with probability $1-\lambda_\beta \mathrm{d}s$ (and, in this case, all events appearing in (2.25) are mutually exclusive) or the value 1. Therefore

$$\Pr\{T_k \in \mathrm{d}s\} = \lambda_\beta \mathrm{d}s \sum_{h=1}^{k} \Pr\{N_\alpha(N_\beta(s)) = k-h, N_\alpha(N_\beta(s)+1) = k\}. \tag{2.26}$$

In the interval $(N_\beta(s), N_\beta(s)+1)$, the external process can take all possible values $0 \le k-h \le k-1$. The value k must be excluded because if at time s, $N_\alpha(N_\beta(s)) = k$, the first attainment of value k cannot be recorded during the interval $(s, s+\mathrm{d}s]$. Furthermore

$$\begin{aligned}
&\Pr\{N_\alpha(N_\beta(s)) = k-h, N_\alpha(N_\beta(s)+1) = k | N_\beta(s) = j\} \qquad (2.27)\\
&= \Pr\{N_\alpha(j) = k-h, N_\alpha(j+1) = k\}\\
&= \Pr\{N_\alpha(j) = k-h\}\Pr\{N_\alpha(j+1) - N_\alpha(j) = h\}\\
&= e^{-\lambda_\alpha j}\frac{(\lambda_\alpha j)^{k-h}}{(k-h)!} e^{-\lambda_\alpha}\frac{\lambda_\alpha^h}{h!}.
\end{aligned}$$

By inserting (2.27) into (2.25), we arrive at

$$\begin{aligned}
\Pr\{T_k \in \mathrm{d}s\} &= \mathrm{d}s\, \lambda_\beta e^{-\lambda_\beta s} \sum_{j=0}^{\infty}\sum_{h=1}^{k} e^{-\lambda_\alpha j}\frac{(\lambda_\alpha j)^{k-h}}{(k-h)!} e^{-\lambda_\alpha}\frac{\lambda_\alpha^h}{h!}\frac{(\lambda_\beta s)^j}{j!} \qquad (2.28)\\
&= \mathrm{d}s\, \lambda_\beta e^{-\lambda_\alpha} e^{-\lambda_\beta s}\frac{\lambda_\alpha^k}{k!}\sum_{j=0}^{\infty} e^{-\lambda_\alpha j}\left[(j+1)^k - j^k\right]\frac{(\lambda_\beta s)^j}{j!}.
\end{aligned}$$

We note that $\Pr\{T_k < \infty\} < 1$, for all $k \ge 1$. From (2.24), we have that

$$\begin{aligned}
\Pr\{T_k < \infty\} &= e^{-\lambda_\alpha}\frac{\lambda_\alpha^k}{k!}\sum_{j=0}^{\infty} e^{-\lambda_\alpha j}\left[(j+1)^k - j^k\right] \qquad (2.29)\\
&= e^{-\lambda_\alpha}\frac{\lambda_\alpha^k}{(k-1)!}\sum_{j=0}^{\infty} e^{-\lambda_\alpha j}\int_j^{j+1} x^{k-1}\mathrm{d}x\\
&= \frac{\lambda_\alpha^k}{(k-1)!}\sum_{j=0}^{\infty}\int_j^{j+1} e^{-\lambda_\alpha(j+1)} x^{k-1}\mathrm{d}x\\
&< \frac{\lambda_\alpha^k}{(k-1)!}\int_0^{\infty} e^{-\lambda_\alpha x} x^{k-1}\mathrm{d}x = 1.
\end{aligned}$$

□

Remark 2.4. *The previous result shows that there is a positive probability of never reaching level k for the iterated Poisson process. For some cases this probability can be evaluated explicitly.*

$$Pr\{T_1 < \infty\} = \lambda_\alpha e^{-\lambda_\alpha}\sum_{j=0}^{\infty} e^{-\lambda_\alpha j} = \frac{\lambda_\alpha e^{-\lambda_\alpha}}{1-e^{-\lambda_\alpha}} < 1. \tag{2.30}$$

This is because

$$0 < 1 - e^{-\lambda_\alpha} - \lambda_\alpha e^{-\lambda_\alpha} = 1 - Pr\{N_\alpha(1) = 0\} - Pr\{N_\alpha(1) = 1\}. \tag{2.31}$$

From (2.25) *we can evaluate the distribution of T_1 as follows.*

$$Pr\{T_1 \in \mathrm{d}s\} = \mathrm{d}s\, \lambda_\alpha e^{-\lambda_\alpha} \lambda_\beta e^{-\lambda_\beta s(1-e^{-\lambda_\alpha})}, \qquad s > 0. \tag{2.32}$$

By similar calculations we have also that

$$Pr\{T_2 \in \mathrm{d}s\} = \mathrm{d}s\, \lambda_\beta \frac{\lambda_\alpha^2 e^{-\lambda_\alpha}}{2} e^{-\lambda_\beta s(1-e^{-\lambda_\alpha})} \left[1 + 2(\lambda_\beta s) e^{-\lambda_\alpha}\right]. \tag{2.33}$$

Furthermore

$$Pr\{T_2 < \infty\} = \left[Pr\{T_1 < \infty\}\right]^2 + \frac{\lambda_\alpha}{2} Pr\{T_1 < \infty\}. \tag{2.34}$$

Finally, an alternative form of (2.25) reads

$$\Pr\{T_k \in \mathrm{d}s\} = \lambda_\beta \mathrm{d}s \sum_{j=0}^{\infty} \int_j^{j+1} e^{-\lambda_\alpha(j+1-x)} \Pr\{T_k^\alpha \in \mathrm{d}s\}, \tag{2.35}$$

where $T_k^\alpha = \inf\{s \colon N_\alpha(s) = k\}$ and $\Pr\{T_k^\alpha \in \mathrm{d}s\}$ is the Erlang distribution for the external Poisson process.

2.2 Subordination of a homogeneous Poisson process to a fractional pure birth process

Let $\mathscr{Y}^\nu(t)$, $t > 0$, $0 < \nu \le 1$, be a nonlinear fractional pure birth process with rates $\lambda_j > 0$, $j \in \mathbb{N}$ (representing an extension of the classical nonlinear pure birth process). It has been shown in Orsingher and Polito [2010] that

$$\Pr\{\mathscr{Y}^\nu(t) = k | \mathscr{Y}^\nu(0) = 1\} = \begin{cases} \prod_{j=1}^{k-1} \lambda_j \sum_{m=1}^{k} \frac{E_{\nu,1}(-\lambda_m t^\nu)}{\prod_{l=1, l\neq m}^{k} (\lambda_l - \lambda_m)}, & k > 1, \\ E_{\nu,1}(-\lambda_1 t^\nu), & k = 1, \end{cases} \tag{2.36}$$

where $E_{\nu,1}(x)$ is the Mittag–Leffler function. For the process $N_\alpha(\mathscr{Y}^\nu(t))$ we have the following result.

Theorem 2.4. *For the composition $N_\alpha(\mathscr{Y}^\nu(t))$ we have that*

$$N_\alpha(\mathscr{Y}^\nu(t)) \overset{d}{=} \sum_{j=1}^{\mathscr{Y}^\nu(t)} X_j, \tag{2.37}$$

where the X_j, $j \ge 1$, are i.i.d. Poisson random variables with rate λ_α.

Proof. In order to prove (2.37), we evaluate the probability generating function of both members.

$$\begin{aligned} \mathbb{E}u^{N_\alpha(\mathscr{Y}^\nu(t))} &= \sum_{k=0}^{\infty} u^k \left[\frac{e^{-\lambda_\alpha} \lambda_\alpha^k}{k!} E_{\nu,1}(-\lambda_1 t^\nu) + \sum_{r=2}^{\infty} e^{-\lambda_\alpha r} \frac{(\lambda_\alpha r)^k}{k!} \prod_{j=1}^{r-1} \lambda_j \sum_{m=1}^{r} \frac{1}{\prod_{l=1, l\neq m}^{r} (\lambda_l - \lambda_m)} E_{\nu,1}(-\lambda_m t^\nu) \right] \quad (2.38) \\ &= e^{\lambda_\alpha(u-1)} E_{\nu,1}(-\lambda_1 t^\nu) + \sum_{r=2}^{\infty} \left[e^{\lambda_\alpha(u-1)}\right]^r \prod_{j=1}^{r-1} \lambda_j \sum_{m=1}^{r} \frac{1}{\prod_{l=1, l\neq m}^{r} (\lambda_l - \lambda_m)} E_{\nu,1}(-\lambda_m t^\nu). \end{aligned}$$

The probability generating function of the right-hand side of (2.37) reads

$$\mathbb{E}u^{\sum_{j=1}^{\mathscr{Y}^\nu(t)} X_j} = e^{\lambda_\alpha(u-1)}\Pr\{\mathscr{Y}^\nu(t)=1\} + \sum_{r=2}^{\infty} e^{\lambda_\alpha(u-1)r}\Pr\{\mathscr{Y}^\nu(t)=r\}, \tag{2.39}$$

as $\mathbb{E}u^X = e^{\lambda_\alpha(u-1)}$, because X is a Poisson random variable of parameter λ_α. By comparing (2.38) with (2.39), in view of (2.36), we have the claimed result. □

Remark 2.5. *For the fractional linear birth process $Y^\nu(t)$, $t>0$, the distribution* (2.36) *specialises and takes the form*

$$Pr\{Y^\nu(t)=k|Y^\nu(0)=1\} = \sum_{j=1}^{k}\binom{k-1}{j-1}(-1)^{j-1}E_{\nu,1}(-\lambda_\beta j t^\nu). \tag{2.40}$$

For $\nu=1$, the distribution (2.40) *reduces to the geometric distribution of the classical Yule–Furry process. The probability generating function of the composed process $N_\alpha(Y^\nu(t))$ becomes*

$$\begin{aligned}
\mathbb{E}\left[u^{N_\alpha(Y^\nu(t))}\right] &= \sum_{k=0}^{\infty}u^k\sum_{r=1}^{\infty}e^{-\lambda_\alpha r}\frac{(\lambda_\alpha r)^k}{k!}\sum_{j=1}^{r}\binom{r-1}{j-1}(-1)^{j-1}E_{\nu,1}(-\lambda_\beta j t^\nu) \\
&= \sum_{r=1}^{\infty}e^{-\lambda_\alpha r}e^{\lambda_\alpha r u}\sum_{j=1}^{r}\binom{r-1}{j-1}(-1)^{j-1}E_{\nu,1}(-\lambda_\beta j t^\nu) \\
&= \sum_{j=1}^{\infty}(-1)^{j-1}E_{\nu,1}(-\lambda_\beta j t^\nu)\sum_{r=j}^{\infty}\binom{r-1}{j-1}e^{-\lambda_\alpha(1-u)r} \\
&= \sum_{j=1}^{\infty}(-1)^{j-1}E_{\nu,1}(-\lambda_\beta j t^\nu)e^{-\lambda_\alpha(i-u)j}\sum_{r=0}^{\infty}\binom{j+r-1}{r}e^{-\lambda_\alpha(1-u)r} \\
&= \sum_{j=1}^{\infty}(-1)^{j-1}E_{\nu,1}(-\lambda_\beta j t^\nu)e^{-\lambda_\alpha(1-u)j}\sum_{r=0}^{\infty}\binom{-j}{r}(-1)^r e^{-\lambda_\alpha(1-u)r} \\
&= \sum_{j=1}^{\infty}(-1)^{j-1}e^{-\lambda_\alpha(1-u)j}\left(1-e^{-\lambda_\alpha(1-u)}\right)^{-j}E_{\nu,1}(-\lambda_\beta j t^\nu) \\
&= \sum_{j=1}^{\infty}(-1)^{j-1}\left[\frac{e^{\lambda_\alpha(u-1)}}{1-e^{\lambda_\alpha(u-1)}}\right]^j E_{\nu,1}(-\lambda_\beta j t^\nu).
\end{aligned} \tag{2.41}$$

2.3 Subordination with a Poisson field

Let B a Borel set and let us indicate with $\Lambda(B)$ its Lebesgue measure. The aim of this section is to study the process

$$N_\alpha(\mathscr{F}(B)), \tag{2.42}$$

where $N_\alpha(t)$, $t>0$, is a homogeneous Poisson process with rate $\lambda_\alpha>0$, and $\mathscr{F}(B)$, is a homogeneous Poisson field with rate $\lambda>0$. Analogously to the iterated Poisson process, the representation

$$N_\alpha(\mathscr{F}(B)) \overset{\text{d}}{=} X_1+\cdots+X_{\mathscr{F}(B)}, \tag{2.43}$$

where the random variables X_j are i.i.d. Poisson distributed of parameter $\lambda_\alpha > 0$, is still valid. Therefore, the state probabilities can be written as

$$\Pr\{N_\alpha(\mathscr{F}(B)) = k\} = \frac{\lambda_\alpha^k}{k!} e^{-\lambda\Lambda(B)(1-e^{-\lambda_\alpha})} \mathfrak{B}_k \left(\lambda\Lambda(Be^{-\lambda_\alpha})\right), \quad k \geq 0, \tag{2.44}$$

and the probability generating function reads

$$G(u, B) = e^{\lambda\Lambda(B)\left(e^{\lambda_\alpha(u-1)}-1\right)}, \qquad |u| \leq 1. \tag{2.45}$$

The emptiness probability is given by:

$$\Pr\{N_\alpha(\mathscr{F}(B)) = 0\} = e^{-\lambda\Lambda(B)\left(1-e^{-\lambda_\alpha}\right)}. \tag{2.46}$$

Let now B_l be the disc with radius l, centred in the origin. The first-contact distribution $H(l)$, $l \in \mathbb{R}^+$, (with the first point) is

$$H(l) = 1 - \Pr\{N_\alpha(\mathscr{F}(B_l)) = 1\} = 1 - e^{-\lambda\pi l^2(1-e^{-\lambda_\alpha})}, \qquad l \in \mathbb{R}^+. \tag{2.47}$$

The probability density is in turn

$$h(l) = 2\lambda\pi l(1 - e^{-\lambda_\alpha})e^{-\lambda\pi l^2(1-e^{-\lambda_\alpha})}, \qquad l \in \mathbb{R}^+, \tag{2.48}$$

that is, a Rayleigh distribution (see, for the classical non subordinated case, Stoyan and Stoyan [1994], page 213).

3 Compositions of Poisson processes with first-passage time of different point processes

In this section, we consider a fractional Poisson process $N^\nu(t)$ (see Beghin and Orsingher [2009] for information on this process) whose first-passage time

$$\tau_k^\nu = \inf(t : N^\nu(t) = k), \qquad k \geq 1, \tag{3.1}$$

is composed either with a homogeneous Poisson process $N_\alpha(t)$ ($\lambda_\alpha > 0$ is its rate) or with a Yule–Furry process $Y_\alpha(t)$.
The distribution of (3.1) has the following density

$$\Pr(\tau_k^\nu \in \mathrm{d}t)/\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}t}\Pr\{N^\nu(t) \geq k\} = \lambda_\beta^k t^{\nu k-1} E_{\nu,\nu k}^k(-\lambda_\beta t^\nu), \qquad t > 0,\ \lambda_\beta > 0, \tag{3.2}$$

[Beghin and Orsingher, 2010, formula (1.6)], where

$$E_{\xi,\gamma}^\delta(z) = \sum_{r=0}^\infty \frac{(\delta)_r z^r}{\Gamma(\xi r + \gamma) r!}, \qquad \xi, \gamma, \delta \in \mathbb{C},\ \Re(\xi) > 0. \tag{3.3}$$

The function $E_{\xi,\gamma}^\delta(z)$ is called generalised Mittag–Leffler function (see Mathai and Haubold [2008, page 91])

Theorem 3.1. *The composed process $\tilde{N}^\nu(k) = N_\alpha(\tau_k^\nu)$ has the distribution*

$$Pr\{\tilde{N}^\nu(k) = r\} = \frac{1}{r!} \sum_{j=0}^\infty \binom{-k}{j} \left(\frac{\lambda_\beta}{\lambda_\alpha^\nu}\right)^{k+j} \frac{\Gamma(\nu(k+j)+r)}{\Gamma(\nu(k+j))}, \qquad r \geq 0, \tag{3.4}$$

and possesses probability generating function

$$\mathbb{E}u^{\tilde{N}^\nu(k)} = \left[1+(1-u)^\nu \frac{\lambda_\alpha^\nu}{\lambda_\beta}\right]^{-k}, \qquad |u| \leq 1. \tag{3.5}$$

Proof. We start our proof with the following relation:

$$\Pr\left\{\tilde{N}^\nu(k)=r\right\} = \int_0^\infty \Pr\left\{N_\alpha(s)=r\right\}\Pr\left\{\tau_k^\nu \in \mathrm{d}s\right\}, \qquad k\geq 1,\ r\geq 0. \tag{3.6}$$

Instead of writing the distribution of τ_k^ν in terms of generalised Mittag–Leffler functions, it is more convenient to work with the following expression which we derive from scratch.
From Beghin and Orsingher [2009], we know that the distribution of the fractional Poisson process is

$$\begin{aligned}\Pr\{N^\nu(t)=m\} &= \sum_{j=m}^\infty (-1)^{j-m}\binom{j}{m}\frac{(\lambda_\beta t^\nu)^j}{\Gamma(\nu j+1)} \qquad (3.7)\\ &= (\lambda_\beta t^\nu)^m \sum_{j=0}^\infty \binom{j+m}{j}\frac{(-\lambda_\beta t^\nu)^j}{\Gamma\left(\nu(m+j)+1\right)}, \qquad t>0,\ \nu\in(0,1].\end{aligned}$$

By considering that $\Pr\{\tau_k^\nu < s\} = \Pr\{N^\nu(s)\geq k\}$, we have that

$$\begin{aligned}\Pr\{\tau_k^\nu \in \mathrm{d}s\}/\mathrm{d}s &= \frac{\mathrm{d}}{\mathrm{d}s}\sum_{m=k}^\infty \Pr\{N^\nu(s)=m\} \qquad (3.8)\\
&= \frac{\mathrm{d}}{\mathrm{d}s}\sum_{m=k}^\infty (\lambda_\beta s^\nu)^m \sum_{j=0}^\infty \binom{j+m}{j}\frac{(-\lambda_\beta t^\nu)^j}{\Gamma\left(\nu(m+j)+1\right)}\\
&= \frac{\mathrm{d}}{\mathrm{d}s}\sum_{m=k}^\infty \lambda_\beta^m \sum_{j=0}^\infty \binom{j+m}{j}(-\lambda_\beta)^j\frac{(s^\nu)^{m+j}}{\Gamma(\nu(m+j)+1)}\\
&\overset{h=j+m}{=} \frac{\mathrm{d}}{\mathrm{d}s}\sum_{m=k}^\infty \lambda_\beta^m \sum_{h=m}^\infty \binom{h}{h-m}(-\lambda_\beta)^{h-m}\frac{s^{\nu h}}{\Gamma(\nu h+1)}\\
&= \frac{\mathrm{d}}{\mathrm{d}s}\sum_{h=k}^\infty (-\lambda_\beta)^h \frac{s^{\nu h}}{\Gamma(\nu h+1)}\sum_{m=k}^h \binom{h}{m}(-1)^m\\
&= \frac{\mathrm{d}}{\mathrm{d}s}\sum_{h=k}^\infty (-\lambda_\beta)^h \frac{s^{\nu h}}{\Gamma(\nu h+1)}(-1)^k\binom{h-1}{k-1}\\
&= \sum_{h=k}^\infty (-\lambda_\beta)^h(-1)^k\binom{h-1}{k-1}\frac{\nu h s^{\nu h-1}}{\Gamma(\nu h+1)}\\
&= \sum_{h=k}^\infty (-\lambda_\beta)^h(-1)^k\binom{h-1}{k-1}\frac{s^{\nu h-1}}{\Gamma(\nu h)}\\
&\overset{j=h-k}{=} \sum_{j=0}^\infty (-\lambda_\beta)^{j+k}(-1)^k\binom{j+k-1}{k-1}\frac{s^{\nu(k+j)-1}}{\Gamma(\nu(k+j))}\\
&= \lambda_\beta^k \sum_{j=0}^\infty \binom{-k}{j}\lambda_\beta^j \frac{s^{\nu(k+j)-1}}{\Gamma(\nu(k+j))}.\end{aligned}$$

In the fifth step of (3.8), we used the following formula which is interesting in itself

$$\sum_{m=k}^{h} \binom{h}{m} (-1)^m = (-1)^k \binom{h-1}{k-1}. \tag{3.9}$$

We provide here a proof of (3.9) in the following way:

$$\begin{aligned}
\sum_{m=k}^{h} \binom{h}{m} (-1)^m &= \sum_{m=0}^{h} \binom{h}{m} (-1)^m - \sum_{m=0}^{k-1} \binom{h}{m} (-1)^m = \sum_{m=0}^{k-1} \binom{h}{m} (-1)^{m+1} \qquad (3.10)\\
&= -1 + h - \frac{h(h-1)}{2} + \frac{h(h-1)(h-2)}{2\cdot 3} - \frac{h(h-1)(h-2)(h-3)}{2\cdot 3\cdot 4} \\
&\quad + \dots + (-1)^k \frac{h(h-1)(h-2)\dots(h-k+2)}{2\cdot 3\cdots(k-1)} \\
&= (h-1)\left[1 - \frac{h}{2} + \frac{h(h-2)}{2\cdot 3} - \frac{h(h-2)(h-3)}{2\cdot 3\cdot 4} \right.\\
&\quad \left. + \dots + (-1)^k \frac{h(h-2)\dots(h-k+2)}{2\cdot 3\cdots(k-1)}\right] \\
&= \frac{(h-1)(h-2)}{2} \left[-1 + \frac{h}{3} - \frac{h(h-3)}{3\cdot 4} + \dots + (-1)^k \frac{h(h-3)\dots(h-k+2)}{3\cdot 4\cdots(k-1)}\right] \\
&= \frac{(h-1)(h-2)(h-3)}{2\cdot 3} \left[1 - \frac{h}{4} + \dots + (-1)^k \frac{h(h-4)\dots(h-k+2)}{4\cdot 5\cdots(k-1)}\right] = \dots = \\
&= \frac{(h-1)(h-2)(h-3)\dots(h-k+2)}{(k-2)!} \left[(-1)^{k-1} + (-1)^k \frac{h}{k-1}\right] \\
&= \frac{(h-1)(h-2)(h-3)\dots(h-k+2)}{(k-2)!} (-1)^k \left[-1 + \frac{h}{k-1}\right] \\
&= (-1)^k \frac{(h-1)!}{(k-1)!(h-k)!} = (-1)^k \binom{h-1}{k-1}.
\end{aligned}$$

By inserting (3.8) into (3.6) we obtain that

$$\begin{aligned}
\Pr\{\tilde{N}^\nu(k) = r\} &= \int_0^\infty e^{-\lambda_\alpha s} \frac{(\lambda_\alpha s)^r}{r!} \lambda_\beta^k \sum_{j=0}^\infty \binom{-k}{j} \lambda_\beta^j \frac{s^{\nu(k+j)-1}}{\Gamma(\nu(k+j))} \mathrm{d}s \qquad (3.11)\\
&= \frac{\lambda_\alpha^r}{r!} \lambda_\beta^k \sum_{j=0}^\infty \binom{-k}{j} \lambda_\beta^j \frac{\Gamma(\nu(k+j)+r)}{\lambda_\alpha^{r+\nu(k+j)} \Gamma(\nu(k+j))} \\
&= \frac{1}{r!} \sum_{j=0}^\infty \binom{-k}{j} \left(\frac{\lambda_\beta}{\lambda_\alpha^\nu}\right)^{k+j} \frac{\Gamma(\nu(k+j)+r)}{\Gamma(\nu(k+j))}.
\end{aligned}$$

This proves formula (3.4). The probability generating function

$$\begin{aligned}
\mathbb{E} u^{\tilde{N}^\nu(k)} &= \sum_{r=0}^\infty u^r \int_0^\infty \frac{e^{-\lambda_\alpha s} (\lambda_\alpha s)^r}{r!} \lambda_\beta^k \sum_{j=0}^\infty \binom{-k}{j} \lambda_\beta^j \frac{s^{\nu(k+j)-1}}{\Gamma(\nu(k+j))} \mathrm{d}s \qquad (3.12)\\
&= \int_0^\infty e^{-\lambda_\alpha s} e^{\lambda_\alpha s u} \lambda_\beta^k \sum_{j=0}^\infty \binom{-k}{j} \lambda_\beta^j \frac{s^{\nu(k+j)-1}}{\Gamma(\nu(k+j))} \mathrm{d}s
\end{aligned}$$

$$= \left(\frac{\lambda_\beta}{\lambda_\alpha^\nu(1-u)^\nu}\right)^k \sum_{j=0}^\infty \binom{-k}{j} \left(\frac{\lambda_\beta}{\lambda_\alpha^\nu(1-u)^\nu}\right)^j$$
$$= \left(\frac{\lambda_\beta}{\lambda_\alpha^\nu(1-u)^\nu}\right)^k \left(1+\frac{\lambda_\beta}{\lambda_\alpha^\nu(1-u)^\nu}\right)^{-k}$$
$$= \left[1+(1-u)^\nu \frac{\lambda_\alpha^\nu}{\lambda_\beta}\right]^{-k}.$$

□

Remark 3.1. *The waiting time* τ_k^ν, $k \geq 1$, *for the kth event of the fractional Poisson process can be viewed as the sum of independent waiting times* $\tau_{1,j}^\nu$ *separating the events of the Poisson flow, i.e.*

$$\tau_k^\nu = \sum_{j=1}^k \tau_{1,j}^\nu. \tag{3.13}$$

It is well-known (see e.g. Mainardi et al. [2004] or Beghin and Orsingher [2009])

$$\begin{aligned} Pr\{\tau_{1,j}^\nu \in \mathrm{d}s\} &= Pr\{\tau_1^\nu \in \mathrm{d}s\} \\ &= \lambda_\beta s^{\nu-1} \sum_{j=0}^\infty \frac{(-\lambda_\beta s^\nu)^j}{\Gamma(\nu(j+1))} \\ &= \lambda_\beta s^{\nu-1} E_{\nu,\nu}(-\lambda_\beta s^\nu) \\ &= -\frac{\mathrm{d}}{\mathrm{d}s} E_{\nu,1}(-\lambda_\beta s^\nu). \end{aligned} \tag{3.14}$$

From (3.14) *it follows that*

$$\int_0^\infty Pr\{\tau_{1,j}^\nu \in \mathrm{d}s\} = 1, \tag{3.15}$$

and, by writing the distribution of τ_k^ν *as convolution of the terms pertaining to* $\tau_{1,j}^\nu$, *we have also that*

$$\int_0^\infty Pr\{\tau_k^\nu \in \mathrm{d}s\} = 1. \tag{3.16}$$

The Laplace transform of (3.8) *is easily calculated and reads*

$$\begin{aligned} \mathbb{E}e^{-\mu\tau_k^\nu} &= \int_0^\infty e^{-\mu s} \lambda_\beta^k \sum_{j=0}^\infty \binom{-k}{j} \lambda_\beta^j \frac{s^{\nu(k+j)-1}}{\Gamma(\nu(k+j))} \mathrm{d}s \\ &= \frac{\lambda_\beta^k}{\mu^{\nu k}} \sum_{j=0}^\infty \binom{-k}{j} \left(\frac{\lambda_\beta}{\mu^\nu}\right)^j = \left[\frac{\mu^\nu}{\lambda_\beta}+1\right]^{-k}, \end{aligned} \tag{3.17}$$

and this clearly confirms the additive structure (3.13).

Remark 3.2. *Result* (3.17) *suggests the following representation for the composed process* $\tilde{N}^\nu(k)$:

$$\tilde{N}^\nu(k) \overset{d}{=} \xi_1^\nu + \cdots + \xi_k^\nu, \tag{3.18}$$

where the ξ_j^ν are independent random variables which are called discrete Mittag–Leffler random variables (see Pillai and Jayakumar [1995]) having parameters ν and $\lambda_\alpha^\nu/\lambda_\beta$. The discrete Mittag–Leffler reduces to the geometric random variable of parameter $q = \lambda_\alpha/(\lambda_\alpha+\lambda_\beta)$ when $\nu = 1$. We now give the explicit distribution of the generalised geometric random variables ξ_j^ν.

$$\begin{aligned} Pr\{\xi^\nu = r\} &= \frac{1}{r!}\sum_{j=0}^{\infty}\left(\frac{\lambda_\beta}{\lambda_\alpha^\nu}\right)^{j+1}\frac{\Gamma(\nu(j+1)+r)}{\Gamma(\nu(j+1))} \\ &= \frac{\lambda_\beta}{\lambda_\alpha^\nu r!}\int_0^\infty e^{-w}\sum_{j=0}^{\infty}\frac{w^{\nu(j+1)+r-1}}{\Gamma(\nu(j+1))}\left(-\frac{\lambda_\beta}{\lambda_\alpha^\nu}\right)^j \mathrm{d}w \\ &= \frac{\lambda_\beta}{\lambda_\alpha^\nu}\int_0^\infty e^{-w}\frac{w^{\nu+r-1}}{r!}E_{\nu,\nu}\left(-\frac{\lambda_\beta}{\lambda_\alpha^\nu}w^\nu\right)\mathrm{d}w. \end{aligned} \tag{3.19}$$

For $\nu = 1$, we extract from (3.19) *the geometric distribution.*

$$Pr\{\xi^1 = r\} = \frac{\lambda_\beta}{\lambda_\alpha}\int_0^\infty e^{-w}\frac{w^r}{r!}e^{-\frac{\lambda_\beta w}{\lambda_\alpha}w}\mathrm{d}w = \frac{\lambda_\beta/\lambda_\alpha}{\left(1+\frac{\lambda_\beta}{\lambda_\alpha}\right)^r} = pq^{r-1}, \tag{3.20}$$

where $p = \lambda_\beta/(\lambda_\alpha+\lambda_\beta)$. In order to check that the generalised geometric law (3.19) *sums up to unity, we write*

$$\begin{aligned} \sum_{r=0}^{\infty}Pr\{\xi^\nu = r\} &= \frac{\lambda_\beta}{\lambda_\alpha^\nu}\int_0^\infty e^{-w}w^{\nu-1}E_{\nu,\nu}\left(-\frac{\lambda_\beta}{\lambda_\alpha^\nu}\right)\mathrm{d}w \\ &= \int_0^\infty -\frac{\mathrm{d}}{\mathrm{d}w}E_{\nu,1}\left(-\frac{\lambda_\beta}{\lambda_\alpha^\nu}\right)\mathrm{d}w = \left|-E_{\nu,1}\left(-\frac{\lambda_\beta}{\lambda_\alpha^\nu}\right)\right|_{w=0}^{w=\infty} = 1. \end{aligned} \tag{3.21}$$

Furthermore, formula (3.5) *shows that $\tilde{N}^\nu(t)$ possesses Linnik distribution and its form is explicitly given by* (3.4)*. Finally, the distribution* (3.4)*, for $\nu = 1$, becomes the negative binomial distribution having parameters k and $\lambda_\alpha/(\lambda_\alpha+\lambda_\beta)$. Indeed, from* (3.4)*, we have that*

$$\begin{aligned} Pr\left(\tilde{N}^1(k) = r\right) &= \sum_{m=0}^{\infty}(-1)^m\left(\frac{\lambda_\beta}{\lambda_\alpha}\right)^{k+m}\frac{(k+m+r-1)!}{r!m!(k-1)!} \\ &= \binom{k+r-1}{r}\sum_{m=0}^{\infty}\left(\frac{\lambda_\beta}{\lambda_\alpha}\right)^{k+m}(-1)^m\binom{k+m+r-1}{m} \\ &= \binom{k+r-1}{r}\left(\frac{\lambda_\beta}{\lambda_\alpha}\right)^k\sum_{m=0}^{\infty}\left(\frac{\lambda_\beta}{\lambda_\alpha}\right)^m\binom{-(k+r)}{m} \\ &= \binom{k+r-1}{r}\left(\frac{\lambda_\beta}{\lambda_\alpha}\right)^k\left(1+\frac{\lambda_\beta}{\lambda_\alpha}\right)^{-(k+r)} \\ &= \binom{k+r-1}{r}\left(\frac{\lambda_\alpha}{\lambda_\alpha+\lambda_\beta}\right)^r\left(1-\frac{\lambda_\alpha}{\lambda_\alpha+\lambda_\beta}\right)^k. \end{aligned} \tag{3.22}$$

Remark 3.3. *We now find the distribution of a slightly modified first-passage time*

$$\hat{\tau}_k^\nu = \left(\frac{t}{k}\right)^{1/\nu}\tau_k^\nu, \tag{3.23}$$

where τ_k^ν *is defined in* (3.1) *and has distribution* (3.8).

$$Pr\{\hat{\tau}_k^\nu \in \mathrm{d}s\} = \mathrm{d}s \left(\frac{k}{t}\right)^{1/\nu} \lambda_\beta^k \sum_{j=0}^{\infty} \binom{-k}{j} \lambda_\beta^j \frac{\left[s\left(\frac{k}{t}\right)^{1/\nu}\right]^{\nu(k+j)-1}}{\Gamma(\nu(k+j))} \tag{3.24}$$
$$= \mathrm{d}s \sum_{j=0}^{\infty} \binom{-k}{j} \lambda_\beta^{k+j} \left(\frac{k}{t}\right)^{k+j} \frac{s^{\nu(k+j)-1}}{\Gamma(\nu(k+j))}.$$

The Laplace transform of (3.24) *becomes*

$$\int_0^\infty e^{-\mu s} Pr\{\hat{\tau}_k^\nu \in \mathrm{d}s\} = \left[1 + \lambda_\beta \frac{t\mu^\nu}{k}\right]^{-k}. \tag{3.25}$$

For $k \to \infty$ *we obtain the fine result*

$$\lim_{k\to\infty} \int_0^\infty e^{-\mu s} Pr\{\hat{\tau}_k^\nu \in \mathrm{d}s\} = e^{-\lambda_\beta t \mu^\nu}. \tag{3.26}$$

Result (3.26) *shows that the rescaled first-passage time* (3.23) *converges in distribution to a positively skewed stable law of order* $\nu \in (0,1)$.

We now consider the Yule–Furry process $Y_\alpha(t)$, with a single progenitor, subordinated to the first-passage time τ_k^ν. The distribution of $Y_\alpha(\tau_k^\nu)$ is given below and can be determined as follows. Bearing in mind the distribution (3.8), we have that

$$\Pr\{Y_\alpha(\tau_k^\nu) = r\} = \int_0^\infty e^{-\lambda_\alpha s}(1 - e^{-\lambda_\alpha s})^{r-1} \lambda_\beta^k \sum_{j=0}^{\infty} \binom{-k}{j} \lambda_\beta^j \frac{s^{\nu(k+j)-1}}{\Gamma(\nu(k+j))} \mathrm{d}s \tag{3.27}$$
$$= \int_0^\infty \sum_{h=1}^{r} \binom{r-1}{h-1} (-1)^{h-1} e^{-\lambda_\alpha h s} \lambda_\beta^k \sum_{j=0}^{\infty} \binom{-k}{j} \lambda_\beta^j \frac{s^{\nu(k+j)-1}}{\Gamma(\nu(k+j))} \mathrm{d}s$$
$$= \lambda_\beta^k \sum_{h=1}^{r} \binom{r-1}{h-1} (-1)^{h-1} \sum_{j=0}^{\infty} \binom{-k}{j} \lambda_\beta^j \frac{1}{(\lambda_\alpha h)^{\nu(k+j)}}$$
$$\left(\text{if } \lambda_\beta/\lambda_\alpha^\nu < 1\right)$$
$$= \sum_{h=1}^{r} \left[\frac{\lambda_\beta}{\lambda_\alpha^\nu h^\nu}\right]^k \binom{r-1}{h-1} (-1)^{h-1} \left[1 + \frac{\lambda_\beta}{\lambda_\alpha^\nu h^\nu}\right]^{-k}$$
$$= \sum_{h=1}^{r} \binom{r-1}{h-1} (-1)^{h-1} \left[1 + h^\nu \frac{\lambda_\alpha^\nu}{\lambda_\beta}\right]^{-k}, \qquad r \geq 0.$$

The probability generating function of the distribution (3.27) reads

$$\mathbb{E}u^{Y_\alpha(\tau_k^\nu)} = \sum_{r=1}^{\infty} u^r \sum_{h=1}^{r} \binom{r-1}{h-1} (-1)^{h-1} \left[1 + h^\nu \frac{\lambda_\alpha^\nu}{\lambda_\beta}\right]^{-k} \tag{3.28}$$
$$= \sum_{h=1}^{\infty} (-1)^{h-1} \left[1 + h^\nu \frac{\lambda_\alpha^\nu}{\lambda_\beta}\right]^{-k} \sum_{r=h}^{\infty} u^r \binom{r-1}{h-1}$$

$$= \sum_{h=1}^{\infty} (-1)^{h-1} \left(\frac{u}{1-u} \right)^h \left[1 + h^\nu \frac{\lambda_\alpha^\nu}{\lambda_\beta} \right]^{-k}, \qquad |u| < 1.$$

We remark that the inversion of sums in (3.28) is valid only for $|u| < 1$.

3.0.1 The classical case $\nu = 1$

For $\nu = 1$ we have special interesting results for $N_\alpha(\tau_k^1)$ and $Y_\alpha(\tau_k^1)$. For the first process we have the following result.

Theorem 3.2. *The composed process $N_\alpha(\tau_k^1)$ has the following representation:*

$$N_\alpha(\tau_k^1) \overset{d}{=} X_1 + \dots + X_N, \tag{3.29}$$

where N is a Poisson random variable of parameter

$$\mu = \log \left(\frac{\lambda_\alpha + \lambda_\beta}{\lambda_\beta} \right)^k, \tag{3.30}$$

and the X_js are i.i.d. random variables with logarithmic distribution of parameter $q = \lambda_\alpha / (\lambda_\alpha + \lambda_\beta)$.

Proof. The random variable $N_\alpha(\tau_k^1)$ is a negative binomial W (see (3.22)) with distribution

$$\Pr\{W = r\} = \binom{k+r-1}{r} p^k q^r. \tag{3.31}$$

In our case $p = \lambda_\beta / (\lambda_\alpha + \lambda_\beta)$ and $q = \lambda_\alpha / (\lambda_\alpha + \lambda_\beta)$. It is well-known that it can be expanded as a random sum of the form

$$N_\alpha(\tau_k^1) \overset{d}{=} X_1 + \dots + X_N, \tag{3.32}$$

where N is a Poisson random variable of parameter $\mu = -k \log p$ and X is a logarithmic distribution of parameter q. □

Remark 3.4. *From* (3.22) *we can infer that*

$$\mathbb{E}\tilde{N}(k) = \frac{\lambda_\alpha + \lambda_\beta}{\lambda_\beta} k, \tag{3.33}$$

and

$$\mathbb{V}ar\tilde{N}(k) = \frac{\lambda_\alpha(\lambda_\alpha + \lambda_\beta)}{\lambda_\beta^2} k. \tag{3.34}$$

These results can be confirmed by applying Wald's formula to the random sum (3.29).

Remark 3.5. *For $\nu = 1$, the distribution* (3.27) *becomes*

$$Pr\{Y_\alpha(\tau_k^1) = r\} = \sum_{h=1}^{r} \binom{r-1}{h-1} (-1)^{h-1} \left[1 + h \frac{\lambda_\alpha}{\lambda_\beta} \right]^{-k}, \qquad r \geq 1,\ k \geq 1. \tag{3.35}$$

We are able to give a fine expression of (3.35) *for* $k = 1$. *We have*

$$Pr\{Y_\alpha(\tau_1^1) = r\} = \sum_{h=1}^{r} \binom{r-1}{h-1} (-1)^{h-1} \frac{\lambda_\beta}{\lambda_\beta + h\lambda_\alpha} \tag{3.36}$$
$$= \sum_{h=0}^{r-1} \binom{r-1}{h} (-1)^h \frac{\lambda_\beta}{\lambda_\alpha} \frac{1}{\left(1 + \frac{\lambda_\beta}{\lambda_\alpha} + h\right)}.$$

In light of the formula (see e.g. Kirschenhofer [1996])

$$\sum_{k=0}^{N} \binom{N}{k} (-1)^k \frac{1}{x+k} = \frac{N!}{x(x+1)\cdots(x+N)}, \tag{3.37}$$

the probability (3.36) *becomes*

$$Pr\{Y_a(\tau_1^1) = r\} = \frac{\lambda_\beta}{\lambda_\alpha} \frac{(r-1)!\Gamma\left(\frac{\lambda_\beta}{\lambda_\alpha} + 1\right)}{\Gamma\left(\frac{\lambda_\beta}{\lambda_\alpha} + 1 + r\right)} = \frac{\lambda_\beta}{\lambda_\alpha} Beta\left(r, \frac{\lambda_\beta}{\lambda_\alpha} + 1\right), \qquad r \geq 1. \tag{3.38}$$

We can easily check that (3.38) *sums up to unity because*

$$\sum_{r=1}^{\infty} \frac{\lambda_\beta}{\lambda_\alpha} Beta\left(r, \frac{\lambda_\beta}{\lambda_\alpha} + 1\right) = \sum_{r=1}^{\infty} \frac{\lambda_\beta}{\lambda_\alpha} \int_0^1 x^{r-1} (1-x)^{\lambda_\beta/\lambda_\alpha} \mathrm{d}x \tag{3.39}$$
$$= \frac{\lambda_\beta}{\lambda_\alpha} \int_0^1 (1-x)^{\lambda_\beta/\lambda_\alpha - 1} = 1.$$

Remark 3.6. *The mean value and the variance of* $Y_\alpha(\tau_k^1)$ *can be obtained by means of the following calculations.*

$$\mathbb{E}Y_\alpha(\tau_k^1) = \frac{\lambda_\beta^k}{(k-1)!} \int_0^\infty e^{\lambda_\alpha s} s^{k-1} e^{-\lambda_\beta s} \mathrm{d}s = \frac{\lambda_\beta^k}{(k-1)!} \int_0^\infty e^{-(\lambda_\beta - \lambda_\alpha)s} s^{k-1} \mathrm{d}s = \left(\frac{\lambda_\beta}{\lambda_\beta - \lambda_\alpha}\right)^k, \tag{3.40}$$

if $\lambda_\beta > \lambda_\alpha$. *Analogously we have that*

$$\mathbb{V}arY_a(\tau_k^1) = \frac{\lambda_\beta^k}{(k-1)!} \int_0^\infty e^{\lambda_\alpha s} (1 + e^{\lambda_\alpha s}) s^{k-1} e^{-\lambda_\beta s} \mathrm{d}s = \left(\frac{\lambda_\beta}{\lambda\beta - \lambda_\alpha}\right)^k + \left(\frac{\lambda_\beta}{\lambda_\beta - 2\lambda_\alpha}\right)^k, \qquad \lambda_\beta > 2\lambda_\alpha. \tag{3.41}$$

3.1 Composition of Poisson processes with the inverse of an independent fractional linear birth process

Let $Y_\beta^\nu(t)$, $t > 0$, be a fractional linear pure birth process with rate $\lambda_\beta > 0$, studied in Orsingher and Polito [2010]. From Cahoy and Polito [2010], the distribution of

$$\phi_k^\nu = \inf(t : Y_\beta^\nu(t) = k), \tag{3.42}$$

is obtained and has the following probability density:

$$\Pr\{\phi_k^\nu \in \mathrm{d}t\}/\mathrm{d}t = \sum_{m=1}^{k} \sum_{l=1}^{m} \binom{m-1}{l-1} (-1)^{l-1} \lambda_\beta l t^{\nu-1} E_{\nu,\nu}(-\lambda_\beta l t^\nu) \tag{3.43}$$

$$= \sum_{l=1}^{k} (-1)^{l-1} \lambda_\beta l t^{\nu-1} E_{\nu,\nu}(-\lambda_\beta l t^\nu) \sum_{m=l}^{k} \binom{m-1}{l-1}$$
$$= \sum_{l=1}^{k} \binom{k}{l} (-1)^{l-1} \lambda_\beta l t^{\nu-1} E_{\nu,\nu}(-\lambda_\beta l t^\nu)$$
$$= \sum_{l=1}^{k} \binom{k}{l} (-1)^{l} \frac{\mathrm{d}}{\mathrm{d}t} E_{\nu,1}(-\lambda_\beta l t^\nu), \qquad t > 0,\ \nu \in (0,1].$$

The relation $\sum_{m=l}^{k} \binom{m-1}{l-1} = \binom{k}{l}$, used in the second step of (3.43), can be proved as follows:

$$\sum_{m=l}^{k} \binom{m-1}{l-1} = 1 + l + \frac{l(l+1)}{2} + \frac{l(l+1)(l+2)}{2\cdot 3} + \cdots + \frac{l(l+1)\ldots(k-1)}{2\cdot 3 \cdots (k-l)} \tag{3.44}$$
$$= (l+1)\left[1 + \frac{l}{2} + \frac{l(l+2)}{2\cdot 3} + \cdots + \frac{l(l+2)\ldots(k-1)}{2\cdot 3\cdots(k-l)}\right]$$
$$= \frac{(l+1)(l+2)}{2}\left[1 + \frac{l}{3} + \cdots + \frac{l(l+3)\ldots(k-1)}{3\cdot 4\cdots(k-l)}\right]$$
$$= \frac{(l+1)(l+2)(l+3)\ldots(k-1)}{(k-l-1)!}\left[1 + \frac{l}{k-l}\right] = \binom{k}{l}.$$

The distribution of $N_\alpha(\phi_k^\nu)$ therefore becomes

$$\Pr(N_\alpha(\phi_k^\nu) = r) = \int_0^\infty \frac{e^{-\lambda_\alpha s}(\lambda_\alpha s)^r}{r!} \sum_{l=1}^{k} \binom{k}{l} (-1)^{l-1} \lambda_\beta l s^{\nu-1} E_{\nu,\nu}(-\lambda_\beta l s^\nu) \mathrm{d}s \tag{3.45}$$
$$= \frac{1}{r!} \sum_{l=1}^{k} \binom{k}{l} (-1)^l \sum_{n=0}^{\infty} \left(-\frac{\lambda_\beta l}{\lambda_\alpha^\nu}\right)^{n+1} \frac{\Gamma(\nu(n+1)+r)}{\Gamma(\nu(n+1))}.$$

The probability generating function of $N_\alpha(\phi_k^\nu)$ can be written in a neat form as

$$\mathbb{E}u^{N_\alpha(\phi_k^\nu)} = \sum_{r=0}^{\infty} u^r \int_0^\infty e^{-\lambda_\alpha s} \frac{(\lambda_\alpha s)^r}{r!} \sum_{l=1}^{k} \binom{k}{l} (-1)^{l-1} \lambda_\beta l s^{\nu-1} E_{\nu,\nu}(-\lambda_\beta l s^\nu) \mathrm{d}s \tag{3.46}$$
$$= \int_0^\infty e^{-\lambda_\alpha s(1-u)} \sum_{l=1}^{k} \binom{k}{l} (-1)^{l-1} \lambda_\beta l s^{\nu-1} E_{\nu,\nu}(-\lambda_\beta l s^\nu) \mathrm{d}s$$
$$= \sum_{l=1}^{k} \binom{k}{l} (-1)^{l-1} \frac{\lambda_\beta l}{[\lambda_\alpha(1-u)]^\nu + \lambda_\beta l} = k \sum_{l=0}^{k-1} \binom{k-1}{l} (-1)^l \frac{1}{\frac{\lambda_\alpha^\nu (1-u)^\nu}{\lambda_\beta} + 1 + l}$$
$$= \frac{k!}{\left(\frac{\lambda_\alpha^\nu(1-u)^\nu}{\lambda_\beta} + 1\right)\left(\frac{\lambda_\alpha^\nu(1-u)^\nu}{\lambda_\beta} + 2\right) \cdots \left(\frac{\lambda_\alpha^\nu(1-u)^\nu}{\lambda_\beta} + k\right)}$$
$$= \frac{k!\,\Gamma\left(\frac{\lambda_\alpha^\nu(1-u)^\nu}{\lambda_\beta} + 1\right)}{\Gamma\left(\frac{\lambda_\alpha^\nu(1-u)^\nu}{\lambda_\beta} + 1 + k\right)} = k \cdot \mathrm{Beta}\left(k, \frac{\lambda_\alpha^\nu(1-u)^\nu}{\lambda_\beta} + 1\right), \qquad |u| < 1.$$

4 Poisson random products and Poisson random continued fractions

4.1 Multiplicative compound Poisson process

In this section we consider a multiplicative compound Poisson process (denoted here π-compound Poisson process), defined as

$$N_\pi(t) = \prod_{j=1}^{N(t)} X_j, \qquad t > 0, \tag{4.1}$$

where the X_js are i.i.d. random variables and $N(t)$, $t > 0$, is a homogeneous Poisson process with rate $\lambda > 0$. We start by calculating the Mellin transform of $N_\pi(t)$.

$$\mathbb{E}\left[N_\pi(t)\right]^{\eta-1} = \sum_{k=0}^{\infty} \left[\mathbb{E}X^{\eta-1}\right]^k \frac{(\lambda t)^k}{k!} e^{-\lambda t} = e^{\lambda t(\mathbb{E}X^{\eta-1}-1)}. \tag{4.2}$$

The relation (4.2) can be rewritten as

$$\mathbb{E}e^{(\log N_\pi(t))(\eta-1)} = \mathbb{E}e^{(\eta-1)\sum_{j=1}^{N(t)}\log X_j} = e^{i\beta\sum_{j=1}^{N(t)}\log X_j} = e^{\lambda t\left(\mathbb{E}X^{i\beta}\right)}. \tag{4.3}$$

For the non-negative random variables X_js, the random sum $\sum_j = 1^{N(t)} \log X_j$ can be reduced to a Poisson random product for the random variables X_js possessing Mellin transform at point $\eta = i\beta + 1$,

$$\mathbb{E}X^{\eta-1} = \mathbb{E}X^{i\beta}. \tag{4.4}$$

We give the explicit form of the covariance function in the next theorem.

Theorem 4.1. *For $0 < s < t$, the covariance of the random product $N_\pi(t)$, $t > 0$, reads*

$$\mathbb{C}ov(N_\pi(t), N_\pi(s)) = e^{\lambda t(\mathbb{E}X-1)}\left[e^{\lambda s\mathbb{E}[X(X-1)]} - e^{\lambda s(\mathbb{E}X-1)}\right] = e^{\lambda t(\mathbb{E}X-1)}\int_{s\mathbb{E}(X-1)}^{s\mathbb{E}X(X-1)} \lambda e^{\lambda w}\mathrm{d}w. \tag{4.5}$$

Proof.

$$\begin{aligned}
\mathbb{E}\left[\prod_{j=1}^{N(t)} X_j \cdot \prod_{r=1}^{N(s)} X_r\right] &= \mathbb{E}\left[\prod_{j=1}^{N(s)} X_j \cdot \prod_{l=N(s)+1}^{N(t)} X_l \cdot \prod_{r=1}^{N(s)} X_r\right] \\
&= \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\left[\mathbb{E}X^2\right]^m [\mathbb{E}X]^{n-m} \Pr\{N(s) = m, N(t) = n\} \\
&= \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\left[\mathbb{E}X^2\right]^m [\mathbb{E}X]^{n-m} \Pr\{N(s) = m\}\Pr\{N(t-s) = n-m\} \\
&= \sum_{m=0}^{\infty}\left[\mathbb{E}X^2\right]^m \frac{e^{-\lambda s}(\lambda s)^m}{m!}\sum_{r=0}^{\infty}[\mathbb{E}X]^r \frac{e^{-\lambda(t-s)}(\lambda(t-s))^r}{r!} \\
&= e^{\lambda s(\mathbb{E}X^2-1)}e^{\lambda(t-s)(\mathbb{E}X-1)}.
\end{aligned} \tag{4.6}$$

Therefore

$$\mathbb{C}\text{ov}\left[\prod_{j=1}^{N(t)} X_j, \prod_{r=1}^{N(s)} X_r\right] = e^{\lambda s(\mathbb{E}X^2-1)}e^{\lambda(t-s)(\mathbb{E}X-1)} - e^{\lambda t(\mathbb{E}X-1)}e^{\lambda s(\mathbb{E}X-1)} \tag{4.7}$$

$$= e^{\lambda t(\mathbb{E}X-1)}\left[e^{\lambda s\mathbb{E}[X(X-1)]} - e^{\lambda s(\mathbb{E}X-1)}\right].$$

□

Remark 4.1. *Formula* (4.5) *shows that the process $N_\pi(t)$ is positively correlated.*
As a consequence of the previous calculations we have that

$$\mathbb{E}N_\pi(t) = e^{\lambda t(\mathbb{E}X-1)}, \qquad \mathbb{E}\left[N_\pi(t)\right]^2 = e^{\lambda t(\mathbb{E}X^2-1)}, \tag{4.8}$$

and

$$\mathbb{V}arN_\pi(t) = e^{\lambda t(\mathbb{E}X^2-1)} - e^{2\lambda t(\mathbb{E}X-1)} = e^{-\lambda t(1-\mathbb{E}X^2)}\left[1 - e^{-\lambda t\mathbb{E}(X-1)^2}\right]. \tag{4.9}$$

For $X \sim N(0,1)$, the covariance function of $N_\pi(t)$ takes the form

$$\mathbb{C}ovN_\pi(t) = 2\sinh\left[\lambda\min(s,t)\right]e^{-\lambda\min(s,t)} = 2\sinh\left[\lambda\mathbb{C}ov(N(t),N(s))\right]e^{-\lambda\mathbb{C}ov(N(t),N(s))}. \tag{4.10}$$

Remark 4.2. *If the random variables X_j, $j \geq 1$, are positively skewed stable with index $\nu \in (0,1)$, we are able to give an explicit form of the Mellin transform* (4.2).
Since

$$\mathbb{E}e^{-\mu X} = e^{-\mu^\nu}, \qquad \mu > 0,\ 0 < \nu < 1, \tag{4.11}$$

we have that the characteristic function of X reads

$$\mathbb{E}e^{i\beta X} = e^{-(i\beta)^\nu} = e^{-|\beta|^\nu e^{-\frac{i\pi\nu}{2}\operatorname{sgn}\beta}} = e^{|\beta|^\nu}\left[\cos\frac{\pi\nu}{2}\left(1 - \operatorname{sgn}\beta\tan\frac{\pi\nu}{2}\right)\right]. \tag{4.12}$$

Some manipulations as shown in D'Ovidio and Orsingher [2011] prove that

$$\mathbb{E}X^{\eta-1} = \frac{1}{\nu}\Gamma\left(\frac{1-\eta}{\nu}\right)\frac{1}{\Gamma(1-\eta)}. \tag{4.13}$$

This permits us to conclude that

$$\mathbb{E}\left[N_\pi(t)\right]^{\eta-1} = e^{\lambda t\left(\frac{1}{\nu}\Gamma\left(\frac{1-\eta}{\nu}\right)\frac{1}{\Gamma(1-\eta)}-1\right)}. \tag{4.14}$$

Remark 4.3. *When X_j are i.i.d Bernoulli random variables of parameter p, we have that the fractional moments of the compound process can be written as*

$$\mathbb{E}\left[N_\pi(t)\right]^\eta = e^{-\lambda t(1-p)}, \tag{4.15}$$

which do not depend on η. It follows that the mean value and the variance are

$$\mathbb{E}N_\pi(t) = e^{-\lambda t(1-p)}, \qquad \mathbb{V}N_\pi(t) = e^{-\lambda t(1-p)}\left(1 - e^{-\lambda t(1-p)}\right). \tag{4.16}$$

Note how the mean value and the variance formally coincide with those of a linear pure death process with a single progenitor.

Remark 4.4 (General case)**.** *Consider an infinitely divisible random variable Y in the sense of Mellin (or log-infinitely divisible), thus decomposable in product of i.i.d. random variables ζ_j. For Y we have that*

$$\mathbb{E}Y^{\eta-1} = \left[\mathbb{E}\zeta^{\eta-1}\right]^k. \tag{4.17}$$

The Mellin transform of the random product $\prod_{j=1}^{N(t)} \zeta_j$ is therefore

$$\mathbb{E}\left[\prod_{j=1}^{N(t)} \zeta_j\right]^{\eta-1} = \sum_k \left[\mathbb{E}\zeta^{\eta-1}\right]^k \frac{e^{\lambda_\beta t}(\lambda_\beta t)^k}{k!} = e^{-\lambda_\beta t} e^{\lambda_\beta t \mathbb{E}\zeta^{\eta-1}} = e^{-\lambda_\beta t\left[1-\mathbb{E}\zeta^{\eta-1}\right]}. \tag{4.18}$$

Let now $\Theta\colon \mathbb{N} \to \mathbb{N}$ such that for each $r \in \mathbb{N}$, $\Theta(r) = \prod_{j=1}^{r} \xi_j$. If the random variables ξ_js take integer values, then

$$\begin{aligned}\mathbb{E}\left[\Theta(N(t))\right]^{\eta-1} &= \sum_{m=0}^{\infty}\sum_{r=0}^{\infty} m^{\eta-1} Pr\{\Theta(r) = m\} Pr\{N(t) = r\} \\ &= \sum_{r=0}^{\infty}\left(\mathbb{E}\xi^{\eta-1}\right)^r Pr\{N(t) = r\} = e^{-\lambda_\beta t + \lambda_\beta \mathbb{E}\xi^{\eta-1}}.\end{aligned} \tag{4.19}$$

If $\Theta(r)$ is absolutely continuous the calculations follow in the same way and arrive at the Mellin transform (4.19)*:*

$$\begin{aligned}\mathbb{E}\left[\Theta(N(t))\right]^{\eta-1} &= \sum_{r=0}^{\infty}\int_0^{\infty} x^{\eta-1} Pr\{\Theta(r) \in \mathrm{d}x\} Pr\{N(t) = r\} \\ &= \sum_{r=0}^{\infty}\left(\mathbb{E}\xi^{\eta-1}\right)^r Pr\{N(t) = r\} = e^{-\lambda_\beta t + \lambda_\beta \mathbb{E}\xi^{\eta-1}}.\end{aligned} \tag{4.20}$$

In conclusion we have that

$$\Theta(N(t)) \stackrel{d}{=} \prod_{j=1}^{N(t)} \xi_j. \tag{4.21}$$

4.2 Continued fractions of Cauchy random variables with Poisson distributed levels

We consider in this section the random variables defined as

$$[X_1; X_2, \dots, X_{N(t)}] = X_1 + \cfrac{1}{X_2 + \cfrac{1}{\ddots + X_{N(t)-1} + \frac{1}{X_{N(t)}}}}, \tag{4.22}$$

where X_j, $j \geq 1$, are independent Cauchy random variables with scale parameter equal to unity and location parameter equal to zero. We will write $X \sim C(0,1)$. Furthermore $N(t)$, $t > 0$, is a homogeneous Poisson process independent of the Cauchy random variables X_j. For the convenience of the reader we note that

$$[X_1] = X_1, \tag{4.23}$$

$$[X_1; X_2] = X_1 + \frac{1}{X_2}, \tag{4.24}$$

$$[X_1; X_2, X_3] = X_1 + \frac{1}{X_2 + \frac{1}{X_3}}. \tag{4.25}$$

The standard Cauchy random variable has the remarkable property that $X \sim 1/X$, and this is the reason for which continued fractions can be treated when Cauchy random variables are involved (Cammarota and Orsingher [2010]).

For our analysis, we need the following result.

Lemma 4.1. *For a Cauchy random variable $C(a,b)$, $a \in \mathbb{R}$, $b \in \mathbb{R}^+$, the random variable $1/C(a,b) \sim C(a/(a^2+b^2), b/(a^2+b^2))$. In our case, $a=0$ and therefore $1/C(0,b) \sim C(a, 1/b)$.*

Our first result is stated in the next theorem.

Theorem 4.2. *The nth level fraction*

$$[X_1; X_2, \dots, X_n] = X_1 + \cfrac{1}{X_2 + \cfrac{1}{\ddots + X_{n-1} + \frac{1}{X_n}}}, \tag{4.26}$$

has Cauchy distribution with scale parameter $b_n = F_{n+1}/F_n$, where F_n are the Fibonacci numbers.

Proof. We proceed by induction.

$$[X_1; X_2] = X_1 + \frac{1}{X_2} \sim C(0,2). \tag{4.27}$$

In view of Lemma 4.1, we have that

$$[X_1; X_2, X_3] = X_1 + \frac{1}{X_2 + \frac{1}{X_3}} \sim C(0, 3/2). \tag{4.28}$$

Furthermore,

$$[X_1; X_2, X_3, X_4] = X_1 + \frac{1}{[X_2; X_3, X_4]} \sim C(0, 5/3). \tag{4.29}$$

In general we have that

$$\begin{aligned} [X_1; X_2, \dots, X_n] &= X_1 + \frac{1}{[X_2, X_3, \dots, X_n]} = X_1 + \frac{1}{C(0, F_{n-1}/F_n)} \\ &= X_1 + C(0, F_n/F_{n-1}) = C(0, 1 + F_n/F_{n-1}) = C(0, F_{n+1}/F_n), \end{aligned} \tag{4.30}$$

and in the last step we took into account the definition of Fibonacci numbers. □

Remark 4.5. *The Fibonacci numbers can be written in terms of the golden ration $\phi = (1+\sqrt{5})/2$ as*

$$F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}. \tag{4.31}$$

Therefore

$$\frac{F_{n+1}}{F_n} = \frac{\phi^{n+1} - (1-\phi)^{n+1}}{\phi^n + (1-\phi)^n} = \phi \frac{1 - \left(\frac{1-\phi}{\phi}\right)^{n+1}}{1 - \left(\frac{1-\phi}{\phi}\right)^n} \to_{n\to\infty} \phi. \tag{4.32}$$

This means that $[X_1; X_2, \dots, X_n] \xrightarrow{d} C(0, \phi)$.

Remark 4.6. *From the analysis above, we infer that $[X_1; X_2, \dots, X_{N(t)}]$, $t > 0$, is a process and, for each t, possesses distribution equal to*

$$Pr\{[X_1; X_2, \dots, X_{N(t)}]) \in \mathrm{d}x\}/\mathrm{d}x = \sum_n \frac{1}{\pi} \frac{F_{n+1}/F_n}{x^2 + (F_{n+1}/F_n)^2} e^{-\lambda t} \frac{(\lambda t)^n}{n!}. \tag{4.33}$$

We give now an alternative representation to the process $[X_1; X_2, \dots, X_{N(t)}]$, $t > 0$.

Theorem 4.3. *The characteristic function of the random continued fraction* $[X_1; X_2, \dots, X_{N(t)}]$ *reads*

$$\mathbb{E}e^{i\beta[X_1;X_2,\dots,X_{N(t)}]} = e^{-|\beta|\phi} \sum_{n=0}^{\infty} \prod_{j=1}^{\infty} e^{-|\beta|\sqrt{5}\left(\frac{1-\phi}{\phi}\right)^{nj}} Pr\{N(t) = n\}. \tag{4.34}$$

Proof. In view of Theorem 4.2, we have that

$$\begin{aligned}\mathbb{E}e^{i\beta[X_1;X_2,\dots,X_{N(t)}]} &= \sum_{n=0}^{\infty} \mathbb{E}e^{i\beta[X_1;X_2,\dots,X_n]} \Pr\{N(t) = n\} \\ &= \sum_{n=0}^{\infty} e^{-|\beta|\frac{F_{n+1}}{F_n}} \Pr\{N(t) = n\}.\end{aligned} \tag{4.35}$$

Since

$$\begin{aligned}\frac{F_{n+1}}{F_n} &= \phi \frac{1-\left(\frac{1-\phi}{\phi}\right)^{n+1}}{1-\left(\frac{1-\phi}{\phi}\right)^{n}} \\ &= \phi\left\{1-\left(\frac{1-\phi}{\phi}\right)^{n+1}\right\}\sum_{j=0}^{\infty}\left(\frac{1-\phi}{\phi}\right)^{nj} \\ &= \phi\left\{\sum_{j=0}^{\infty}\left(\frac{1-\phi}{\phi}\right)^{nj} - \frac{1-\phi}{\phi}\sum_{j=0}^{\infty}\left(\frac{1-\phi}{\phi}\right)^{n(j+1)}\right\} \\ &= \phi\left\{1+\left(1-\frac{1-\phi}{\phi}\right)\sum_{j=1}^{\infty}\left(\frac{1-\phi}{\phi}\right)^{nj}\right\} \\ &= \phi + \sqrt{5}\sum_{j=0}^{\infty}\left(\frac{1-\phi}{\phi}\right)^{nj},\end{aligned} \tag{4.36}$$

we have that

$$\begin{aligned}\mathbb{E}e^{i\beta[X_1;X_2,\dots,X_n]} &= \sum_{n=0}^{\infty} e^{-|\beta|\left[\phi+\sqrt{5}\sum_{j=1}^{\infty}\left(\frac{1-\phi}{\phi}\right)^{nj}\right]} \Pr\{N(t) = n\} \\ &= e^{-|\beta|\phi}\sum_{n=0}^{\infty}\prod_{j=1}^{\infty} e^{-|\beta|\sqrt{5}\left(\frac{1-\phi}{\phi}\right)^{nj}} \Pr\{N(t) = n\}.\end{aligned} \tag{4.37}$$

$\square$

Remark 4.7. *From* (4.34) *we can extract the following equality in distribution:*

$$[X_1; X_2, \dots, X_{N(t)}] \overset{d}{=} C(0,\phi) + \sqrt{5}\sum_{j=1}^{\infty} C_j\left(0, \left(\frac{1-\phi}{\phi}\right)^{N(t)}\right). \tag{4.38}$$

The second term in (4.38) *represents the effect of randomisation of the continued fraction.*

Remark 4.8. *The above analysis suggests an alternative representation of the random continued fraction as*

$$[X_1; X_2, \dots, X_{N(t)}] \stackrel{d}{=} \sum_{j=1}^{F_{N(t)+1}} Y_{j,N(t)}, \tag{4.39}$$

where the $Y_{j,N(t)}$ *are independent Cauchy random variables with scale parameter equal to* $1/F_{N(t)}$. *Clearly,* F_n *are the Fibonacci numbers.*
The equality (4.39) *can be ascertained by writing the characteristic function as follows:*

$$\begin{aligned}\mathbb{E}e^{i\beta[X_1;X_2,\dots,X_{N(t)}]} &= \mathbb{E}e^{i\beta\sum_{j=1}^{F_{N(t)+1}} Y_{j,N(t)}} = \mathbb{E}\left[\mathbb{E}e^{i\beta\sum_{j=1}^{F_{N(t)+1}} Y_{j,N(t)}}\middle| N(t)\right] \\ &= \sum_{n=0}^{\infty}\prod_{j=1}^{F_{n+1}} e^{-|\beta|\frac{1}{F_n}} Pr\{N(t)=n\} = \sum_{n=0}^{\infty} e^{-|\beta|\frac{F_{n+1}}{F_n}} Pr\{N(t)=n\},\end{aligned} \tag{4.40}$$

which coincides with (4.35).

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