STURMIAN MULTIPLE ZEROS FOR STOKES AND NAVIER–STOKES EQUATIONS IN $\mathbb{R}^3$ VIA SOLENOIDAL HERMITE POLYNOMIALS

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Abstract. The Cauchy problem for the 3D *Stokes* and *Navier–Stokes equations*,

$$
\begin{aligned}
&\mathbf{u}_t = -\nabla p + \Delta \mathbf{u}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad \mathbb{R}^3 \times (-1,0], \quad \text{and} \\
&\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \Delta \mathbf{u}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad \mathbb{R}^3 \times (-1,0],
\end{aligned} \tag{0.1}
$$

where $\mathbf{u} = [u,v,w]^T$ is the vector field and p is the pressure, is considered. A smooth bounded initial data $\mathbf{u}_0(x)$, with $\operatorname{div} \mathbf{u}_0 = 0$, are prescribed at $t=-1$.

The problem of formation of *multiple zeros* at a given point $(x,t) = (0,0^-)$ of the components of $\mathbf{u}(x,t)$ is considered. In recent years, such a classic problem, which, for the 1D heat equation, was solved by Sturm in 1836, was under scrutiny for a number of parabolic, hyperbolic, elliptic, and dispersion PDEs. As usual, such an analysis gives insight into a "microscopic blow-up scale properties" of the equations (0.1) under consideration. It is shown that formation of multiple zeros of solutions can follow "self-focusing" of nodal sets moving according to zero surfaces of the corresponding *solenoidal Hermite polynomials* as eigenfunctions of a rescaled adjoint Hermite operator. This is shown to be always the case for the first problem in (0.1), but not always for the second one. Using such blow-up asymptotics allows to state a *unique continuation theorem.*

A similar phenomenon is studied for the well-posed Stokes and Burnett equations with the bi-Laplacian $-\Delta^2 \mathbf{u}$ instead of the standard viscosity operator $\Delta \mathbf{u}$ in (0.1).

1. Introduction: towards micro-scale structure of smooth solutions

1.1. Stokes, Navier–Stokes, and Burnett equations.

We consider the Cauchy problem for the *three-dimensional linear Stokes* and *Navier–Stokes equations* (NSEs)

$$\mathbf{u}_t = -\nabla p + \Delta \mathbf{u}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad \mathbb{R}^3 \times (-1,0], \quad \text{and} \tag{1.1}$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \Delta \mathbf{u}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad \mathbb{R}^3 \times (-1,0]. \tag{1.2}$$

where $\mathbf{u} = [u,v,w]^T$ is the vector field and p is the corresponding pressure. A smooth bounded initial data $\mathbf{u}_0(x)$ are prescribed at $t=-1$, with $\operatorname{div} \mathbf{u}_0 = 0$.

The problem of formation of *multiple zeros* at the point $(x,t) = (0,0^-)$ of the vector field $\mathbf{u}(x,t)$ is considered. In recent years, such a classic problem, which, for the 1D heat equation, was solved by C. Sturm (1836), was under scrutiny for a number of parabolic,

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hyperbolic, elliptic, and dispersion PDEs; see short surveys for each type of linear and nonlinear PDEs in [17], and also [20], containing a most recent survey. As usual, such analysis gives insight into a "microscopic scale properties" of the equations (0.1) under consideration. Indeed, such a microscopic blow-up approach was and is key in classic regularity problems including fundamental questions of the regularity of characteristic boundary points and singular points in potential and other related PDE theory; we refer to recent surveys in [17, 19, 20], where matching blow-up techniques were principally used.

In the present paper, in a similar manner, we show that formation of multiple zeros of solutions can follow "self-focusing" of nodal sets moving according to zero surfaces of the corresponding *solenoidal Hermite polynomials* as eigenfunctions of a rescaled adjoint Hermite operator. It turns out that this is always the case for the Stokes problem. For the NSEs, the situation is shown to be more complicated, though similar zeros do exist. This allows us to state a non-standard *unique continuation theorem* for such problems.

For more clear expressing our main "blow-up" techniques and their applicability in general PDE theory, we develop in Appendix A (the B one contains the corresponding Hermitian spectral analysis) at the paper end, as a natural extension, a similar multiple zeros analysis of the Cauchy problem for the *fourth-order Stokes-like equations* and *well-posed Burnett equations*:

$$
(1.3)\quad \begin{aligned}
&\mathbf{u}_t = -\nabla p - \Delta^2 \mathbf{u}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad \mathbb{R}^N \times (-1, 0], \quad \mathbf{u}(x, -1) = \mathbf{u}_0(x),\\
&\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p - \Delta^2 \mathbf{u}, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in} \quad \mathbb{R}^N \times (-1, 0], \quad \mathbf{u}(x, -1) = \mathbf{u}_0(x),
\end{aligned}
$$

where initial data $\mathbf{u}_0$ are sufficiently smooth and satisfy $\operatorname{div} \mathbf{u}_0 = 0$. Here, we have the *bi-harmonic* diffusion operators on the right-hand side of the $\mathbf{u}$-equations. It turns out that our general scheme describing multiple zeros analysis can be applied, however, requiring another non-self-adjoint spectral theory for the corresponding rescaled operator, where *generalized* solenoidal Hermite polynomials naturally occur (Appendix B).

1.2. **Leray blow-up rescaled variables: why Hermite polynomials occur on micro-scales.** For semilinear NSEs (1.2), we perform Leray's type [26][1] *nonstationary blow-up* scaling with the blow-up time $T = 0$:

$$
(1.4)\quad \mathbf{u}(x, t) = \tfrac{1}{\sqrt{-t}}\, \hat{\mathbf{u}}(y, \tau), \quad p(x, t) = \tfrac{1}{(-t)}\, P(y, \tau), \quad y = \tfrac{x}{\sqrt{-t}}, \quad \tau = -\ln(-t) \to +\infty
$$

as $t \to 0^-$. This yields the rescaled equations for $\hat{\mathbf{u}} = (\hat{u}^1, \hat{u}^2, \hat{u}^3)^T$ and P,

$$
(1.5)\quad \hat{\mathbf{u}}_\tau = \Delta \hat{\mathbf{u}} - \tfrac{1}{2}\,(y \cdot \nabla)\hat{\mathbf{u}} - \tfrac{1}{2}\,\hat{\mathbf{u}} - (\hat{\mathbf{u}} \cdot \nabla)\hat{\mathbf{u}} - \nabla P, \quad \operatorname{div} \hat{\mathbf{u}} = 0 \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}_+.
$$

As a standard next step, we exclude the pressure from the equations (1.5),

$$
(1.6)\quad \begin{aligned}
&\hat{\mathbf{u}}_\tau = \mathbf{H}(\hat{\mathbf{u}}) \equiv \big(\mathbf{B}^* - \tfrac{1}{2}\, I\big)\hat{\mathbf{u}} - \mathbb{P}\,(\hat{\mathbf{u}} \cdot \nabla)\hat{\mathbf{u}} \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}_+,\\
&\text{where} \quad \mathbb{P}\mathbf{v} = \mathbf{v} - \nabla \Delta^{-1}(\nabla \cdot \mathbf{v}) \quad (\|\mathbb{P}\| = 1)
\end{aligned}
$$

[1] In particular, Leray proposed not only to look at a self-similar blow-up as $t \to T^-$ but also at a further similarity extension for $t > T$, i.e., in the complementary limit $t \to T^+$, so that the blow-up factor $\sqrt{T-t}$ is replaced by $\sqrt{t-T}$. See some historical and further comments on Leray's blow-up scenario of 1934 for the 3D NSEs can be found in [18, § 2.2].

is the Leray–Hopf projector of $(L^2(\mathbb{R}^3))^3$ onto the subspace $\{\mathbf{w} \in (L^2)^3 :\ \operatorname{div} \mathbf{w} = 0\}$ of solenoidal vector fields[2]. Another representation is $\mathbb{P}\mathbf{v} = (v_1 - R_1\sigma, v_2 - R_2\sigma, v_3 - R_3\sigma)^T$, where R_j are the Riesz transforms, with symbols $\frac{\xi_j}{|\xi|}$, and $\sigma = R_1 v_1 + R_2 v_2 + R_3 v_3$. We then first apply $\mathbb{P}$ to the original velocity equation in (1.2) and next use the blow-up rescaling (1.4). Using the fundamental solution of Δ in $\mathbb{R}^3$

$$b_3(y) = -\tfrac{1}{4\pi}\,\tfrac{1}{|y|}, \tag{1.7}$$

the operator in (1.6) is written in the form of Leray's formulation [28, p. 32]

$$\begin{gathered} \mathbf{H}(\hat{\mathbf{u}}) \equiv \big(\mathbf{B}^* - \tfrac{1}{2}\, I\big)\hat{\mathbf{u}} - (\hat{\mathbf{u}} \cdot \nabla)\hat{\mathbf{u}} + C_3 \int\limits_{\mathbb{R}^3} \tfrac{y-z}{|y-z|^3}\, \mathrm{tr}(\nabla \hat{\mathbf{u}}(z,\tau))^2\, \mathrm{d}z, \\ \text{where} \quad \mathrm{tr}(\nabla \hat{\mathbf{u}}(z,\tau))^2 = \textstyle\sum_{(i,j)} \hat{u}^i_{z_j} \hat{u}^j_{z_i} \quad \text{and} \quad C_3 = \tfrac{1}{4\pi}. \end{gathered} \tag{1.8}$$

It follows from (1.8) that, in order to describe asymptotic behaviour of small solutions near multiple zeros, as a first step, one needs a spectral theory for the linearized *Hermite operator* $\mathbf{B}^*$ in a proper solenoidal functional space. Of course, this belongs to classic theory of self-adjoint operators; see Birman–Solomjak [4, p. 48]. Moreover, in a full capacity and specially for the NSEs in $\mathbb{R}^3$ and $\mathbb{R}^2$, this theory was already developed (though has been used for large-time behaviour of solutions in the opposite limit $t \to +\infty$) with eigenfunctions of the adjoint (in L^2-metric) operator $\mathbf{B}$. However, since this case is self-adjoint, we can directly use such a theory in the complementary blow-up limit $t \to 0^-$.

2. Hermitian spectral theory of the linear rescaled operator $\mathbf{B}^*$: point spectrum and solenoidal Hermite polynomials

Thus, approaching the point $(0,0)$ in the blow-up manner (1.4), one observes Hermite's operator $\mathbf{B}^*$ as the principal linear part of the rescaled equation (1.5). Writing it in the corresponding divergent form,

$$\mathbf{B}^*\mathbf{v} \equiv \tfrac{1}{\rho^*}\, \nabla \cdot (\rho^* \nabla \mathbf{v}), \tag{2.1}$$

where the weight is $\rho^*(y) = \mathrm{e}^{-\frac{|y|^2}{4}} > 0$, we observe that the actual rescaled evolution is now restricted to the weighted L^2-space $L^2_{\rho^*}(\mathbb{R}^3)$, with the exponentially decaying weight $\rho^*(y)$. Here, $\mathbf{B}^*$ is the ("adjoint") Hermite operator with the point spectrum [4, p. 48]

$$\sigma(\mathbf{B}^*) = \big\{\lambda_k = -\tfrac{k}{2}, \quad k = |\beta| = 0, 1, 2, ...\big\} \quad (\beta \text{ is a multiindex}), \tag{2.2}$$

where each λ_k has the multiplicity $\frac{(k+1)(k+2)}{2}$ (for $N = 3$). The corresponding complete and closed set of eigenfunctions $\Phi^* = \{\psi^*_\beta(y)\}$ is composed from separable Hermite polynomials. Note another important property of Hermite polynomials:

$$\forall\, \psi^*_\beta, \quad \text{any derivative } D^\gamma \psi^*_\beta \quad \text{is also an eigenfunction with } k = |\beta| - |\gamma| \geq 0. \tag{2.3}$$

[2]Of course, using $\mathbb{P}$ in (1.6) emphasizes an unpleasant fact that the NSEs are a *nonlocal* parabolic problem, so that a somehow full use of order-preserving properties of the semigroup is illusive; though some "remnants" of the Maximum Principle for such *second-order* flows may remain and actually appear from time to time in some results (but these are completely illusive for more difficult fluid models (1.3)).

Recall that [4]

$$\text{polynomial set } \Phi^* \text{ is complete and closed in } L^2_{\rho^*}(\mathbb{R}^3). \tag{2.4}$$

Further spectral properties are convenient to demonstrate using the linear operator $\mathbf{B}$,

$$\mathbf{B} = \Delta + \tfrac{1}{2}\, y \cdot \nabla + \tfrac{3}{2}\, I \quad \text{in} \quad L^2_\rho(\mathbb{R}^3), \quad \text{where} \quad \rho = \tfrac{1}{\rho^*}, \tag{2.5}$$

which is adjoint to $\mathbf{B}^*$ in the dual L^2-metric. It has the same point spectrum and the corresponding eigenfunctions are multiple of the same Hermite polynomials according to the well-known *generating formula*:

$$\psi_\beta(y) = \tfrac{(-1)^{|\beta|}}{\sqrt{\beta!}}\, D^\beta F(y) \equiv \psi^*_\beta(y) F(y), \quad \text{where} \quad F(y) = \tfrac{1}{(4\pi)^{3/2}}\, \mathrm{e}^{-|y|^2/4} \tag{2.6}$$

is the rescaled kernel of the fundamental solutions of $D_t - \Delta$ in $\mathbb{R}^3 \times \mathbb{R}_+$. Then, the bi-orthonormality holds:

$$\langle \psi^*_\beta, \psi_\gamma \rangle = \delta_{\beta\gamma} \quad \text{for any} \quad \beta,\, \gamma, \tag{2.7}$$

where $\langle \cdot, \dot{\rangle}$ is the scalar product in $L^2(\mathbb{R}^3)$. As is well known, this dual L^2-metric can be also treated as a weighted one in $L^2_{\rho^*}(\mathbb{R}^3)$ (where $\mathbf{B}^*$ becomes symmetric):

$$\langle \psi^*_\beta, \psi_\gamma \rangle \equiv \textstyle\int_{\mathbb{R}^3} F(y)\, \psi^*_\beta \psi^*_\gamma \, \mathrm{d}y \sim \langle \psi^*_\beta, \psi^*_\gamma \rangle_{\rho^*},$$

since $F(y) \sim \rho^*(y)$, up to a constant multiplier. However, we prefer to keep the bi-orthonormality in the non-symmetric form (2.7), since a similar condition occurs in the principally non-symmetric Burnett cases; see (B.20).

Obviously, one needs to consider eigenfunction expansions in the solenoidal restriction

$$\hat{L}^2_{\rho^*}(\mathbb{R}^3) = L^2_{\rho^*}(\mathbb{R}^3)^3 \cap \{\operatorname{div} \mathbf{v} = 0\}. \tag{2.8}$$

Indeed, among the polynomials $\Phi^* = \{\psi^*_\beta\}$, there are many that well-suit the solenoidal fields. Namely, introducing the eigenspaces

$$\Phi^*_k = \operatorname{Span}\{\psi^*_\beta,\ |\beta| = k\}, \quad k \ge 1,$$

in view of (2.3), div plays a role of a "shift operator" in the sense that

$$\operatorname{div} : \Phi^{*3}_k \to \Phi^*_{k-1}. \tag{2.9}$$

We next define the corresponding solenoidal eigenspaces as follows:

$$\mathcal{S}^*_k = \{\mathbf{v}^* = [v^*_1, v^*_2, v^*_3]^T : \quad \operatorname{div} \mathbf{v}^* = 0,\ v^*_i \in \Phi^*_k\}, \quad \text{where} \quad \dim \mathcal{S}^*_k = k(k+2); \tag{2.10}$$

see [22, 23, 24] and further references therein.

The study [22] deals with *global* asymptotics as $t \to +\infty$, where the adjoint operator $\mathbf{B}$ given in (2.5) occurs. Since $\mathbf{B}$ is self-adjoint in $L^2_\rho(\mathbb{R}^3)$, almost all the results from [23, Append. A] are applied to $\mathbf{B}^*$. For a full collection, see [5, 6] for further large-time asymptotic expansions and self-similar solutions. In particular, this made it possible to construct therein fast decaying solutions of the NSEs on each 1D stable manifolds with the asymptotic behaviour[3]

$$\mathbf{u}_\beta(x,t) \sim t^{\lambda_k - \frac{1}{2}}\, \mathbf{v}_\beta\big(\tfrac{x}{\sqrt{t}}\big) + ... \quad \text{as} \quad t \to \infty, \;\; \text{where} \;\; \mathbf{v}_\beta = \mathbf{v}^*_\beta F \in \mathcal{S}_k \tag{2.11}$$

are solenoidal eigenfunctions of $\mathbf{B}$. Namely, taking

$$\mathbf{v} = [v_1, v_2, v_3]^T \in \mathcal{S}_k, \;\; v_i \in \Phi_k = \mathrm{Span}\,\big\{\psi_\beta = \tfrac{(-1)^{|\beta|}}{\sqrt{\beta!}}\, D^\beta F(y),\, |\beta| = k\big\}, \tag{2.12}$$

where F stands for the rescaled Gaussian in (2.6), we have that

$$\mathrm{div}\,\mathbf{v} = (v_1)_{y_1} + (v_2)_{y_2} + (v_3)_{y_3} = \mathrm{div}\,(\mathbf{v}^* F) \equiv (\mathrm{div}\,\mathbf{v}^*)F - \tfrac{1}{2}\, y \cdot \mathbf{v}^*\, F. \tag{2.13}$$

This establishes a one-to-one correspondence between solenoidal eigenfunction classes $\mathcal{S}^*_k$ in (2.10) for $\mathbf{B}^*$ and $\mathcal{S}_k$ in (2.11) for $\mathbf{B}$; see (2.14)–(2.16) below for the first eigenfunctions $\mathbf{v}_\beta = \mathbf{v}^*_\beta F$. Therefore, $\dim \mathcal{S}_k = k(k+2)$, etc.; see details and rather involved proofs of the asymptotics (2.11) for $k = 1$ and 2 in [22].

In particular, those solenoidal Hermite polynomial eigenfunctions of $\mathbf{B}^*$ can be chosen as follows [23, p. 2166-69] (the choice is obviously not unique, normalization constants are omitted):

$$\underline{\lambda_0 = 0:} \quad \mathbf{v}^*_0 = [1,1,1]^T = \mathbf{e} \quad \text{(the first solenoidal Hermite polynomial)}, \tag{2.14}$$

$$\underline{\lambda_1 = -\tfrac{1}{2}:} \quad \mathbf{v}^*_{11} = \begin{bmatrix} 0 \\ -y_3 \\ y_2 \end{bmatrix}, \quad \mathbf{v}^*_{12} = \begin{bmatrix} y_3 \\ 0 \\ -y_1 \end{bmatrix}, \quad \mathbf{v}^*_{13} = \begin{bmatrix} -y_2 \\ y_1 \\ 0 \end{bmatrix} \;\; (\dim \mathcal{S}^*_1 = 3); \tag{2.15}$$

$$\begin{gathered}
\underline{\lambda_2 = -1:} \;\; \mathbf{v}^*_{21} = \begin{bmatrix} 4 - y_2^2 - y_3^2 \\ y_1 y_2 \\ -y_1 y_3 \end{bmatrix}, \; \mathbf{v}^*_{22} = \begin{bmatrix} y_1 y_2 \\ 4 - y_1^2 - y_3^2 \\ -y_2 y_3 \end{bmatrix}, \; \mathbf{v}^*_{23} = \begin{bmatrix} y_1 y_3 \\ -y_2 y_3 \\ 4 - y_1^2 - y_2^2 \end{bmatrix}, \\
\mathbf{v}^*_{24} = - \begin{bmatrix} 0 \\ -y_1 y_3 \\ y_1 y_2 \end{bmatrix}, \quad \mathbf{v}^*_{25} = - \begin{bmatrix} y_2 y_3 \\ 0 \\ -y_2 y_1 \end{bmatrix}, \\
\mathbf{v}^*_{26} = \begin{bmatrix} -y_2 y_3 \\ y_2 y_3 \\ y_1^2 - y_2^2 \end{bmatrix}, \; \mathbf{v}^*_{27} = \begin{bmatrix} y_1 y_2 \\ y_3^2 - y_1^2 \\ -y_2 y_3 \end{bmatrix}, \; \mathbf{v}^*_{28} = \begin{bmatrix} y_2^2 - y_3^2 \\ -y_1 y_2 \\ y_1 y_3 \end{bmatrix} \;\; (\dim \mathcal{S}^*_2 = 8), \quad \text{etc.}
\end{gathered} \tag{2.16}$$

We need the following final conclusion. By (2.4), the set of vectors Φ^{*3} is complete and closed[4] in $L^2_{\rho^*}(\mathbb{R}^3)^3$, so that

$$\forall\, \mathbf{v} \in L^2_{\rho^*}(\mathbb{R}^3)^3 \quad \Longrightarrow \quad \mathbf{v} = \textstyle\sum_{(\beta)} \mathbf{c}_\beta \mathbf{v}^*_\beta, \quad \mathbf{v}^*_\beta \in \Phi^{*3}_k, \;\; k = |\beta| \geq 0, \tag{2.17}$$

[3]We present here only the first term of expansion; as usual in dynamical system theory, other terms in the case of "resonance" can contain $\ln t$-factors (*q.v.* [1] for a typical PDE application); this phenomenon was shown to exist for the NSEs in $\mathbb{R}^2$ [23, p. 236].

[4]Note a standard result of functional analysis: all reasonable polynomials are complete in any weighted L^p-space with an exponentially decaying weight; see the analyticity argument in Kolmogorov–Fomin [25, p. 431].

where we use the vector notation

$$\mathbf{c}_\beta(\tau) = [c_\beta^1(\tau), c_\beta^2(\tau), c_\beta^3(\tau)]^T \quad \Longrightarrow \quad \mathbf{c}_\beta \mathbf{v}_\beta^* \equiv [c_\beta^1(\tau) v_{\beta 1}^*, c_\beta^2(\tau) v_{\beta 2}^*, c_\beta^3(\tau) v_{\beta 3}^*]^T. \tag{2.18}$$

It then follows from (2.7)–(2.9) that

$$\text{polynomial set } \hat{\Phi}^* = \Phi^{*3} \cap \{\operatorname{div} \mathbf{v} = 0\} \text{ is complete and closed in } \hat{L}^2_{\rho^*}(\mathbb{R}^3). \tag{2.19}$$

In what follows, we always assume that we deal with "solenoidal" asymptotics involving eigenfunctions as in (2.10).

As was promised to go in parallel, for Burnett equations in (1.3), the blow-up rescaling and elements of linear solenoidal spectral theory are given in Appendices A and B.

3. First application of Hermitian spectral theory: Sturmian local structure of zero sets of bounded solutions and unique continuation

3.1. **A dynamical system for Fourier coefficients.** Consider the NSEs (1.2). We assume that, in a neighbourhood of the point $(x,t) = (0,0^-)$, the solution $\mathbf{u}(x,t)$ is uniformly bounded and is such that the eigenfunction expansion of the corresponding rescaled function

$$\hat{\mathbf{u}}(y,\tau) = \textstyle\sum_{(\beta)} \mathbf{c}_\beta(\tau) \mathbf{v}_\beta^*(y), \quad \text{where} \quad \mathbf{c}_\beta \mathbf{v}_\beta^* = (c_1 v_{\beta 1}^*, c_2 v_{\beta 2}^*, c_3 v_{\beta 3}^*)^T \in \hat{L}^2_{\rho^*}(\mathbb{R}^3), \tag{3.1}$$

converges in $\hat{L}^2_{\rho^*}(\mathbb{R}^3)$, and moreover, uniformly on compact subsets. These convergence questions of polynomial series are standard [4]; see also [11, 16], where generalized Hermite polynomials occur and further references and details are given. In particular, if $\mathbf{u}(x,t)$ remains bounded for all $t \in [-1,0]$, then, obviously, for such bounded data $\mathbf{u}_0$, the convergence in (3.1) always takes place.

Then, the expansion coefficients satisfy the following dynamical system (DS):

$$\begin{cases} \dot{\mathbf{c}}_\beta = \big(\lambda_\beta - \frac{1}{2}\big)\mathbf{c}_\beta + \sum_{(\alpha,\gamma)} d_{\alpha\gamma\beta} \mathbf{c}_\alpha \mathbf{c}_\gamma \quad \text{for any } |\beta| \ge 0, \\ \text{where} \quad d_{\alpha\gamma\beta} = -\langle \mathbb{P}\, (\hat{\mathbf{v}}_\alpha^* \cdot \nabla) \hat{\mathbf{v}}_\gamma^*, \mathbf{v}_\beta \rangle \quad \text{for all} \quad \alpha,\, \gamma. \end{cases} \tag{3.2}$$

Since (3.1) is a standard eigenfunction expansion via Hermite polynomials of a given bounded smooth rescaled solution $\hat{\mathbf{u}}(y,\tau) \in H^2_{\rho^*}(\mathbb{R}^3)$, the quadratic sum on the right-hand side converges. Recall that, moreover, according to the blow-up scaling (1.4), we actually deal with bounded and uniformly exponentially small rescaled solutions satisfying

$$|\hat{\mathbf{u}}(y,\tau)| \le C\, \mathrm{e}^{-\frac{\tau}{2}} \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}_+. \tag{3.3}$$

The DS (3.2) is difficult for a general study. For instance, it is supposed to contain the answer to the existence/nonexistence of the L^∞-blow-up question (The Millennium Prize Problem, [14]), i.e., whether there exists a Type-II blow-up at the internal point $(0,0^-)$ (see a discussion in [20, § 5]).

For regular points, the DS (3.2) can provide us with a typical classification of multiple zeros and nodal sets of solutions. Note again that this kind of study was first performed by Sturm in 1836 for linear 1D parabolic equations [30]; see historical and other details in [16, Ch. 1].

Thus, following these lines, we clarify local zero sets of solutions of the NSEs at regular points. Assume that

$$\mathbf{u}(0,0) = \mathbf{0}. \tag{3.4}$$

In this connection, recall that the first eigenfunction of $\mathbf{B}^*$ with $\lambda_\beta = 0$

$$\mathbf{v}_0^*(y) = (1,1,1)^T, \tag{3.5}$$

is the only ones that have an empty nodal set. Then, bearing in mind the blow-up scaling term $(1-t)^{-\frac{1}{2}} \equiv \mathrm{e}^{\frac{\tau}{2}}$ in (1.4), we have to assume that

$$\mathbf{c}_0(\tau) = \mathbf{0} \;\text{ or }\; \mathbf{c}_0(\tau) \to \mathbf{0} \quad \text{as} \quad \tau \to +\infty \;\text{ exponentially faster than }\; \mathrm{e}^{-\frac{\tau}{2}}. \tag{3.6}$$

3.2. **Polynomial nodal sets for the Stokes equations.** A first clue to a correct understanding of the DS (3.2) is given by the Stokes equations (1.1), i.e., without the quadratic convection term. Then (3.2) becomes a linear diagonal system and is easily solved:

$$\dot{\mathbf{c}}_\beta = \big(\lambda_\beta - \tfrac{1}{2}\big)\mathbf{c}_\beta \quad \Longrightarrow \quad \mathbf{c}_\beta(\tau) = \mathbf{c}_\beta(0)\mathrm{e}^{-\frac{(1+|\beta|)\tau}{2}} \quad \text{for any} \quad |\beta| \ge 0. \tag{3.7}$$

Therefore, according to (3.1) (and bearing in mind the completeness-closure of the Hermite polynomials), all possible multiple zero asymptotics for the Stokes problem (its local "micro-scale turbulence") are described by finite solenoidal Hermite polynomials, and the zero sets of rescaled velocity components also asymptotically, as $\tau \to +\infty$ (i.e., $t \to 0^-$) obey the nodal Hermite structures.

3.3. **Nodal sets for the Navier–Stokes equations.** Consider the full nonlinear dynamical system (3.2), which on integration is

$$\mathbf{c}_\beta(\tau) = \mathbf{c}_\beta(0)\mathrm{e}^{-\frac{(1+|\beta|)\tau}{2}} - \mathrm{e}^{-\frac{(1+|\beta|)\tau}{2}} \int\limits_0^\tau \textstyle\sum_{(\alpha,\gamma)} d_{\alpha\gamma\beta}(\mathbf{c}_\alpha \mathbf{c}_\gamma)(s)\mathrm{e}^{\frac{(1+|\beta|)s}{2}}\,\mathrm{d}s. \tag{3.8}$$

It follows that the nonlinear quadratic terms in (3.8), under certain assumptions, can affect the rate of decay of solutions near the multiple zero. As usual in calculus, the indeterminacies in this integral quadratic term can be tackled by L'Hospital rule, but technically this is very difficult.

Since we are mainly interested in the study of nodal structures of solutions by using the eigenfunction expansion (3.1), we naturally need to assume that it is possible to choose the leading decaying term (or a linear combination of terms) in this sum as $\tau \to +\infty$. Then obviously these leading terms will asymptotically describe "polynomial-like" structures of nodal sets as $t \to 0^-$. For PDEs with local nonlinearities, this is done in a standard manner as in [16, § 4]; in the nonlocal case, this can cause technical difficulties. However, the DS (3.2) looks (but illusionary) as being obtained from a problem with local nonlinearities. In other words, the nonlocal nature of the NSEs is hidden in (3.2) in the structure of the quadratic sum coefficients $\{d_{\alpha\gamma\beta}\}$, and this does not affect the nodal set behaviour for some classes of multiple zeros. We will check this as follows:

Resonance zeros. We consider a "resonance class" of multiple zeros. Namely, let us assume there exist a multiindex subset $\mathcal{B}$ and a function $\mathbf{h}(\tau) \to \mathbf{0}$ such that

$$\begin{aligned} &\mathbf{c}_\beta(\tau) \sim \mathbf{h}(\tau) \quad \text{as} \quad \tau \to +\infty \quad \text{for any } \beta \in \mathcal{B}, \\ &|\mathbf{c}_\beta(\tau)| \ll |\mathbf{h}(\tau)| \quad \text{as} \quad \tau \to +\infty \quad \text{for any } \beta \notin \mathcal{B}. \end{aligned} \tag{3.9}$$

In other words, only the coefficients $\{\mathbf{c}_\beta(\tau),\, \beta \in \mathcal{B}\}$ are assumed to define the nodal set via (3.1), and other terms are negligible as $\tau \to +\infty$. Under the natural assumption of a strong enough convergence of the quadratic sums in (3.2) (this is expected not to be valid in the case of singular blow-up points only), taking the ODEs from (3.2) for each $\beta \in \mathcal{B}$ yields, for $\tau \gg 1$,

$$\dot{\mathbf{c}}_\beta = \big(\lambda_\beta - \tfrac{1}{2}\big)\mathbf{c}_\beta + o(\mathbf{c}_\beta), \quad \text{where} \quad \mathbf{c}_\beta(\tau) \sim \mathbf{h}(\tau). \tag{3.10}$$

Hence, the asymptotic balancing of these equations must assume that, as $\tau \to +\infty$,

$$\dot{\mathbf{h}} \sim \big(\lambda_\beta - \tfrac{1}{2}\big)\mathbf{h} \quad \Longrightarrow \quad \mathbf{c}_\beta(\tau) \sim \mathbf{h}(\tau) \sim \mathrm{e}^{(-\frac{k}{2} - \frac{1}{2})\tau} \quad \text{and} \quad |\beta| = k, \tag{3.11}$$

where we may omit lower-order multipliers. Thus, there exists a $k \ge 1$ such that $|\beta| = k$ for any $\beta \in \mathcal{B}$. One can see that, for such "resonance" multiple zeros, the nonlocal quadratic term in (3.2) is not important. Thus, in the resonance zero class prescribed by (3.9), as $\tau \to +\infty$, on compact subsets in y, similar to Stokes' problem,

$$\boxed{\text{the nodal set of } \hat{\mathbf{u}}(y,\tau) \text{ is governed by some solenoidal } \textbf{Hermite} \text{ polynomials.}} \tag{3.12}$$

Polynomial structure of multiple zeros is universal. Note first that the conclusion that, locally, for any zero of finite order at $(0,0)$,

$$\text{nodal sets of } \mathbf{u}(x,t) \text{ are governed by finite-degree polynomials} \tag{3.13}$$

is trivially true for any sufficiently smooth solution. Indeed, this follows from the Taylor expansion of such solutions

$$\mathbf{u}(x,t) = \sum_{(|\mu|,|\nu| \le K)} C_{\mu\nu}\, x^\mu\, (-t)^\nu + \mathbf{R}_K(x,t), \quad \text{where} \quad C_{\mu\nu} = \tfrac{(-1)^\nu}{\mu!\,\nu!}\big(D^{\mu;\nu}_{x,t}\mathbf{u}\big)(0,0) \tag{3.14}$$

and $\mathbf{R}_K = o(|x|^K(-t)^K)$ is a higher-order remainder. Translating (3.14) via (1.4) into the expansion for $\hat{\mathbf{u}}(y,\tau)$ yields some polynomial structure, so (3.13) is obviously true. Thus, the principal feature of (3.12) is that the Hermite polynomials count only therein.

Non-resonance zeros: a general classification. For the nonlocal problem (1.6), there exist other non-resonance zeros. Indeed, let $(0,0)$ be a zero of $\mathbf{u}(x,t)$ of a finite order $M \ge 1$, i.e.,

$$\mathbf{u}(x,0) = \textstyle\sum_{(|\sigma|=M)} a_\sigma x^\sigma (1 + o(1)) \sim x^\sigma \quad \text{as} \quad x \to 0 \quad \big(\sum_{(|\sigma|=M)} |a_\sigma| \ne 0\big). \tag{3.15}$$

We now use the following expansion:

$$\mathbf{u}(x,t) = \mathbf{u}(x,0) - \mathbf{u}_t(x,0)(-t) + \tfrac{1}{2!}\,\mathbf{u}_{tt}(x,0)(-t)^2 + \ldots, \tag{3.16}$$

where, by (1.6), all the time-derivatives $D_t^\mu \mathbf{u}(x,0)$ can be calculated:

$$\mathbf{u}_t(x,0) = \Delta\mathbf{u}(x,0) + (\mathbb{P}(\mathbf{u}\cdot\nabla)\mathbf{u})(x,0) \sim x^{\sigma-2} + (\mathbb{P}(\mathbf{u}\cdot\nabla)\mathbf{u})(x,0), \tag{3.17}$$

with a natural meaning of $\Delta x^\sigma \sim x^{\sigma-2}$. If the nonlocal term is negligible here and for other time-derivatives, i.e.,

$$\mathbf{u}_t(x,0) \sim x^{\sigma-2}, \quad \mathbf{u}_{tt}(x,0) \sim x^{\sigma-4}, \dots,$$

then according to (3.16) this leads to a Hermitian structure of nodal sets. In fact, this repeats the pioneering zero-set calculations performed by Sturm (1836); see his original computations in [15, p. 3].

In general, the nonlocal term in (3.17) is not specified by a local structure of the zero under consideration, so, obviously, it can essentially affect the zero evolution. For instance, as a hint, we can have the following zero:

$$\text{(3.18)} \quad \mathbf{u}_t(0,0) = \mathbf{C} \ne 0 \;\Rightarrow\; \mathbf{u}(x,t) \sim \textstyle\sum_{(|\sigma|=M)} a_\sigma x^\sigma - (-t) = \mathrm{e}^{-\tau} \big(\sum_{(|\sigma|=M)} a_\sigma z^\sigma - 1 \big),$$

where $z = \frac{x}{(-t)^{1/m}}$. Hence, this nodal set is governed by the rescaled variable z, which is different from the standard similarity one y in (1.4). Of course, due to the nonlocality of the equation, many other types of zeros can be described. Actually, such non-resonance zeros can be governed by sufficiently arbitrary finite polynomials as the general expansion (3.14) suggests. However, there exists a countable family of "admissible" rescaled variables. Recalling that bounded smooth solutions of the NSEs are analytic in both x and t (see references below), we have to have that there exists a finite $K \ge 1$ such that

$$\text{(3.19)} \qquad D_t^K \mathbf{u}(0,0) \ne 0 \quad \text{and} \quad D_t^s \mathbf{u}(0,0) = 0 \quad \text{for} \quad s = 1, 2, \dots, K-1.$$

Therefore, close to $(0,0)$, the structure of such a multiple zero is given by

$$\text{(3.20)} \quad \mathbf{u}(x,t) \sim \textstyle\sum_{(|\sigma|=M)} a_\sigma x^\sigma - (-t)^K = \mathrm{e}^{-K\tau} \big(\sum_{(|\sigma|=M)} a_\sigma z^\sigma - 1 \big), \; z = \frac{x}{(-t)^\gamma}, \; \gamma = \frac{K}{M}.$$

Since K and $M = |\sigma|$ are arbitrary positive integers, the exponent γ in the expansion (3.20) can be an arbitrary positive rational number. Thus, *the rescaled functions and variables in* (3.20) *exhaust* ALL *types of zero surfaces (points) focusing as x, $t \to 0$ for the NSEs in $\mathbb{R}^3$*.

Finally, the proof that zeros of infinite order are not possible for smooth non-analytic PDEs (and, as usual in such Carleman and Agmon-type uniqueness results, this occurs for $\mathbf{u} \equiv 0$ only) is a difficult technical problem; see an example in [16, § 6.2]. For analytic in y solutions of the NSEs (see references and results in [8, 9, 32]), this problem is nonexistent, and then in (3.12) the degree of the solenoidal vector Hermite polynomials is always finite, though can be arbitrarily large.

3.4. **An application: a unique continuation theorem.** Note another straightforward consequence of this analysis that gives the following conventional *unique continuations result*: *let* (3.4) *hold,* $(0,0)$ *be a resonance zero*[5]*, and at least one component of the nodal set of* $\hat{\mathbf{u}}(y,\tau)$ *does not obey* (3.12). *Then*

$$\text{(3.21)} \qquad \mathbf{u} \equiv \mathbf{0} \quad \textit{everywhere.}$$

[5]Indeed, this is hard to check. However, for the Stokes equations (1.1), as well as for any sufficiently smooth PDEs with local nonlinearities (see [16]), this assumption is not needed, so that such a unique continuation theorem makes a full sense.

Of course, this is not that surprising, since the result is just included in the existing and properly converging eigenfunction expansion (3.1) under the assumption (3.9). According to (3.14), there exists another "funny version" of the unique continuation result: (3.21) *holds if a multiple zero is formed in a non-polynomial self-focusing of zero surfaces, or via a rescaled variable not available in* (3.20), but this is indeed trivial.

For elliptic equations $P(x, D)u = 0$, this has the natural counterpart on *strong unique continuation property* saying that nontrivial solutions cannot have zeros of infinite order; a result first proved by Carleman in 1939 for $P = -\Delta + V$, $V \in L^\infty_{\rm loc}$, in $\mathbb{R}^2$ [7]; see [10, 31] for further references and modern extensions.

Thus, we are finishing a discussion of a first application of solenoidal Hermitian polynomial vector fields for regular solutions of the NSEs. We expect that, due to the DS (3.2), some "traces" of such an analysis and Hermite polynomials should be seen in the fully nonlinear study of $\hat{\mathbf{u}}(y, \tau)$ at the singular blow-up point $(0, 0)$, where, instead of (3.4), we have to assume that, in the sense of $\limsup_{x,t}$,

$$|\mathbf{u}(0, 0)| = +\infty. \tag{3.22}$$

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Appendix A: Multiple zeros for Burnett equations

A.1. **Burnett equations.** For both the systems (1.3), the blow-up scaling (1.4) is replaced by

$$\mathbf{u}(x,t) = (-t)^{-\frac{3}{4}}\, \hat{\mathbf{u}}(y,\tau), \quad y = \tfrac{x}{(-t)^{1/4}}, \tag{A.1}$$

so that the rescaled system (1.6) takes a similar form

$$\hat{\mathbf{u}}_\tau = \mathbf{H}(\hat{\mathbf{u}}) \equiv \big(\mathbf{B}^* - \tfrac{3}{4}\, I\big)\hat{\mathbf{u}} - \mathbb{P}\,(\hat{\mathbf{u}} \cdot \nabla)\hat{\mathbf{u}} \quad \text{in} \quad \mathbb{R}^3 \times \mathbb{R}_+. \tag{A.2}$$

The spectral theory of the given here "adjoint" operator

$$\mathbf{B}^* = -\Delta^2 - \tfrac{1}{4}\, y \cdot \nabla, \quad \text{where} \quad \sigma(\mathbf{B}^*) = \big\{\lambda_\beta = -\tfrac{|\beta|}{4},\, |\beta| = 0, 1, 2, ...\big\} \tag{A.3}$$

with eigenfunctions being generalized Hermite polynomials is available in [11]; a solenoidal extension in the same lines is needed. Necessary spectral theory of the operator pair $\{\mathbf{B}, \mathbf{B}^*\}$ is developed below, in Appendix B.

Therefore, under the same assumptions, the polynomial structure of nodal sets is guaranteed for the corresponding Stokes-like and Burnett equations (resonance zeros) in (1.3); and, moreover, for an arbitrary $2m$th-order viscosity operator $-(-\Delta)^m \mathbf{u}$ therein.

Remark: Burnett equations in a hierarchy of hydrodynamic models. The Burnett equations in (1.3) appear as the *second approximation* (the NSEs (1.2) being the *first one*) of the corresponding kinetic equations on the basis of Grad's method in Chapman–Enskog expansions for hydrodynamics. Namely, Grad's method applied to kinetic equations, by expanding the kernel of the integral operators involved via those with pointwise supports, yields, in addition to the classic operators of the Euler equations, other viscosity parts as follows:

$$D_t \mathbf{u} \equiv \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \sum_{n=0}^{\infty} \varepsilon^{2n+1} \Delta^n (\mu_n \Delta \mathbf{u}) + ... = \varepsilon\big(\mu_0 \Delta \mathbf{u} + \varepsilon^2 \mu_1 \Delta^2 \mathbf{u} + ...\big) + ... ,$$

where $\varepsilon > 0$ is the *Knudsen number* Kn; see details in Rosenau's regularization approach, [29]. In a full model, truncating such series at $n = 0$ leads to the Navier–Stokes equations (1.2) (with $\mu_0 > 0$), while $n = 1$ is associated with the Burnett equations in (1.3).

Recall also that Burnett-type equations, with a small parameter, appeared as higher-order viscosity approximations of the Navier–Stokes equations, represent an effective tool for proving existence of their weak ("turbulent" in Leray's sense) solutions; see Lions' monograph [27, § 6, Ch. 1]. Note that the "Problem on blow-up/non-blow-up for Burnett equations in (1.3) at an inner point" starts from dimensions $N = 7$; for $N \le 6$, there exists a unique global smooth L^2-solution, [21, § 6]. It is expected that this open problem in $\mathbb{R}^7$ is not easier at all than the classic *Millennium Prize One* for the NSEs in $\mathbb{R}^3$ (1.2). In both cases, a construction (or proving its nonexistence) of a Type-II blow-up singularity is necessary, since, most plausibly, a Type-I self-similar blow-up solutions are nonexistent, [20, App. B].

Appendix B: Solenoidal Hermitian spectral theory for $2m$th-order operators

We describe the necessary spectral properties of the linear $2m$th-order differential operator in $\mathbb{R}^N$ ($m=2$ for the Burnett equations in (1.3))

$$\mathbf{B}^* = (-1)^{m+1}\Delta_y^m - \tfrac{1}{2m}\, y\cdot\nabla_y, \tag{B.1}$$

and of its L^2-adjoint $\mathbf{B}$ given by

$$\mathbf{B} = (-1)^{m+1}\Delta_y^m + \tfrac{1}{2m}\, y\cdot\nabla_y + \tfrac{N}{2m}\, I. \tag{B.2}$$

As we have seen, for $m=1$, (B.1) and (B.2) are classic Hermite self-adjoint operators with completely known spectral properties, [4, p. 48]. For any $m \ge 2$, both operators (B.1) and (B.2), though looking very similar to those for $m=1$, *are not symmetric* and do not admit a self-adjoint extension, so we follow [11] in presenting spectral theory.

B.1. **Fundamental solution, rescaled kernel, and first estimates.** The fundamental solution $b(x,t)$ of the linear poly-harmonic parabolic equation

$$u_t = -(-\Delta)^m u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+ \tag{B.3}$$

takes the standard similarity form

$$b(x,t) = t^{-\frac{N}{2m}} F(y), \quad y = \tfrac{x}{t^{1/2m}}. \tag{B.4}$$

The rescaled kernel F is the unique radial solution of the elliptic equation with the operator (B.2), i.e.,

$$\mathbf{B}F \equiv -(-\Delta)^m F + \tfrac{1}{2m}\, y\cdot\nabla F + \tfrac{N}{2m}\, F = 0 \ \text{ in } \mathbb{R}^N, \quad \text{with } \textstyle\int F = 1. \tag{B.5}$$

For $m \ge 2$, the rescaled kernel function $F(|y|)$ is oscillatory as $|y| \to \infty$ and satisfies [12, 13]

$$|F(y)| < D\, \mathrm{e}^{-d_0 |y|^\alpha} \ \text{ in } \mathbb{R}^N, \quad \text{where } \alpha = \tfrac{2m}{2m-1} \in (1,2), \tag{B.6}$$

for some positive constants D and d_0 depending on m and N.

B.2. **Some constants.** As we have seen, the rescaled kernel $F(y)$ satisfies (B.6), where d_0 admits an explicit expression; see below. Such optimal exponential estimates of the fundamental solutions of higher-order parabolic equations are well-known and were first obtained by Evgrafov–Postnikov (1970) and Tintarev (1982); see Barbatis [2, 3] for key references.

As a crucial issue for the further boundary point regularity study, we will need a sharper, than given by (B.6), asymptotic behaviour of the rescaled kernel $F(y)$ as $y \to +\infty$. To get that, we keep four leading terms in (B.5) and obtain, in terms of the radial variable $y \mapsto |y| > 0$:

$$(-1)^{m+1}\big[F^{(2m)} + m\tfrac{N-1}{y}\, F^{(2m-1)} + ...\big] + \tfrac{1}{2m}\, yF' + \tfrac{N}{2m}\, F = 0 \quad \text{for} \quad y \gg 1. \tag{B.7}$$

Using standard classic WKBJ asymptotics, we substitute into (B.7) the function

$$F(y) = y^{-\delta_0}\, \mathrm{e}^{a y^\alpha} + ... \quad \text{as} \quad y \to +\infty, \tag{B.8}$$

exhibiting two scales. Balancing two leading terms gives the algebraic equation for a and δ_0:

$$(-1)^m (\alpha a)^{2m-1} = \tfrac{1}{2m} \quad \text{and} \quad \delta_0 = \tfrac{m(2N-1)-N}{2m-1} > 0\,. \tag{B.9}$$

By construction, one needs to get the root a of (B.9) with the maximal $\mathrm{Re}\, a < 0$. This yields (see e.g., [2, 3])

$$a = \tfrac{2m-1}{(2m)^\alpha}\big[-\sin\big(\tfrac{\pi}{2(2m-1)}\big) + \mathrm{i}\,\cos\big(\tfrac{\pi}{2(2m-1)}\big)\big] \equiv -d_0 + \mathrm{i}\, b_0 \quad (d_0>0). \tag{B.10}$$

Finally, this gives the following double-scale asymptotic of the kernel:

$$F(y) = y^{-\delta_0}\,\mathrm{e}^{-d_0 y^\alpha}\big[C_1 \sin(b_0 y^\alpha) + C_2 \cos(b_0 y^\alpha)\big] + ... \quad \text{as} \quad y = |y| \to +\infty, \tag{B.11}$$

where $C_{1,2}$ are real constants, $|C_1| + |C_2| \ne 0$. In (B.11), we present the first two leading terms from the m-dimensional bundle of exponentially decaying asymptotics.

In particular, for the Burnett equations in (1.3) in $\mathbb{R}^3$, we have

$$m=2,\ N=3: \quad \alpha = \tfrac{4}{3}, \quad d_0 = 3\cdot 2^{-\frac{11}{3}}, \quad b_0 = 3^{\frac{3}{2}}\cdot 2^{-\frac{11}{3}}, \quad \text{and} \quad \delta_0 = \tfrac{7}{3}. \tag{B.12}$$

B.3. **The discrete real spectrum and eigenfunctions of B.** For $m \ge 2$, $\mathbf{B}$ is considered in the weighted space $L^2_\rho(\mathbb{R}^N)$ with the exponentially growing weight function

$$\rho(y) = \mathrm{e}^{a|y|^\alpha} > 0 \quad \text{in } \mathbb{R}^N, \tag{B.13}$$

where $a \in (0, 2d_0)$ is a fixed constant. We next introduce a standard Hilbert (a weighted Sobolev) space of functions $H^{2m}_\rho(\mathbb{R}^N)$ with the inner product and the induced norm

$$\langle v, w\rangle_\rho = \int\limits_{\mathbb{R}^N} \rho(y) \sum_{k=0}^{2m} D^k_y v(y)\,\overline{D^k_y w(y)}\,\mathrm{d}y \quad \text{and} \quad \|v\|^2_\rho = \int\limits_{\mathbb{R}^N} \rho(y) \sum_{k=0}^{2m} |D^k_y v(y)|^2\,\mathrm{d}y.$$

Then $H^{2m}_\rho(\mathbb{R}^N) \subset L^2_\rho(\mathbb{R}^N) \subset L^2(\mathbb{R}^N)$, and $\mathbf{B}$ is a bounded linear operator from $H^{2m}_\rho(\mathbb{R}^N)$ to $L^2_\rho(\mathbb{R}^N)$. Key spectral properties of the operator $\mathbf{B}$ are as follows [11]:

Lemma B.1. (i) *The spectrum of* $\mathbf{B}$ *comprises real simple eigenvalues only,*

$$\sigma(\mathbf{B}) = \big\{\lambda_\beta = -\tfrac{k}{2m},\ k = |\beta| = 0, 1, 2, ...\big\}. \tag{B.14}$$

(ii) *The eigenfunctions* $\psi_\beta(y)$ *are given by*

$$\psi_\beta(y) = \tfrac{(-1)^{|\beta|}}{\sqrt{\beta!}}\, D^\beta F(y), \quad \textit{for any } |\beta| = k. \tag{B.15}$$

(iii) *Eigenfunction subset* (B.14) *is complete in* $L^2(\mathbb{R})$ *and in* $L^2_\rho(\mathbb{R})$.
(iv) *The resolvent* $(\mathbf{B} - \lambda I)^{-1}$ *for* $\lambda \notin \sigma(\mathbf{B})$ *is a compact integral operator in* $L^2_\rho(\mathbb{R}^N)$.

By Lemma B.1, the centre and stable subspaces of $\mathbf{B}$ are given by

$$E^c = \mathrm{Span}\{\psi_0 = F\} \quad \text{and} \quad E^s = \mathrm{Span}\{\psi_\beta,\ |\beta| \ge 1\}. \tag{B.16}$$

B.4. **Polynomial eigenfunctions of the operator B*.** Consider the operator (B.1) in the weighted space $L^2_{\rho^*}(\mathbb{R}^N)$, where $\langle\cdot,\cdot\rangle_{\rho^*}$ and $\|\cdot\|_{\rho^*}$ are the inner product and the norm, with the "adjoint" exponentially decaying weight function

$$\rho^*(y) \equiv \tfrac{1}{\rho(y)} = \mathrm{e}^{-a|y|^\alpha} > 0. \tag{B.17}$$

We ascribe to $\mathbf{B}^*$ the domain $H^{2m}_{\rho^*}(\mathbb{R}^N)$, which is dense in $L^2_{\rho^*}(\mathbb{R}^N)$, and then

$$\mathbf{B}^* :\ H^{2m}_{\rho^*}(\mathbb{R}^N) \to L^2_{\rho^*}(\mathbb{R}^N)$$

is a bounded linear operator. $\mathbf{B}$ is adjoint to $\mathbf{B}^*$ in the usual sense: denoting by $\langle \cdot, \cdot \rangle$ the inner product in the dual space $L^2(\mathbb{R}^N)$, we have

$$\langle \mathbf{B}v, w \rangle = \langle v, \mathbf{B}^* w \rangle \quad \text{for any } \; v \in H^{2m}_\rho(\mathbb{R}^N) \quad \text{and} \quad w \in H^{2m}_{\rho^*}(\mathbb{R}^N). \tag{B.18}$$

The eigenfunctions of $\mathbf{B}^*$ take a particularly simple finite polynomial form and are as follows:

Lemma B.2. (i) $\sigma(\mathbf{B}^*) = \sigma(\mathbf{B})$.
(ii) *The eigenfunctions* $\psi^*_\beta(y)$ *of* $\mathbf{B}^*$ *are generalized Hermite polynomials of degree* $|\beta|$ *given by*

$$\psi^*_\beta(y) = \tfrac{1}{\sqrt{\beta!}} \Big[y^\beta + \textstyle\sum_{j=1}^{[|\beta|/2m]} \frac{1}{j!} (-\Delta)^{mj} y^\beta \Big] \quad \textit{for any} \quad \beta. \tag{B.19}$$

(iii) *Eigenfunction subset* (B.19) *is complete in* $L^2_{\rho^*}(\mathbb{R}^N)$.
(iv) $\mathbf{B}^*$ *has a compact resolvent* $(\mathbf{B}^* - \lambda I)^{-1}$ *in* $L^2_{\rho^*}(\mathbb{R}^N)$ *for* $\lambda \notin \sigma(\mathbf{B}^*)$.
(v) *The bi-orthonormality of the bases* $\{\psi_\beta\}$ *and* $\{\psi^*_\gamma\}$ *holds in the dual* L^2*-metric:*

$$\langle \psi_\beta, \psi^*_\gamma \rangle = \delta_{\beta\gamma} \quad \textit{for any} \quad \beta, \, \gamma. \tag{B.20}$$

Remark on closure. This is an important issue for using eigenfunction expansions of solutions. Firstly, in the self-adjoint case $m = 1$, the sets of eigenfunctions are closed in the corresponding spaces, [4] (and we have used this in our previous NSEs study).

Secondly, for $m \geq 2$, one needs some extra details. Namely, using (B.20), we can introduce the subspaces of eigenfunction expansions and begin with the operator $\mathbf{B}$. We denote by $\tilde{L}^2_\rho$ the subspace of eigenfunction expansions $v = \sum c_\beta \psi_\beta$ with coefficients $c_\beta = \langle v, \psi^* \rangle$ defined as the closure of the finite sums $\{\sum_{|\beta| \leq M} c_\beta \psi_\beta\}$ in the norm of L^2_ρ. Similarly, for the adjoint operator $\mathbf{B}^*$, we define the subspace $\tilde{L}^2_{\rho^*} \subseteq L^2_{\rho^*}$. Note that since the operators are not self-adjoint and the eigenfunction subsets are not orthonormal, in general, these subspaces can be different from L^2_ρ and $L^2_{\rho^*}$, and particularly the equality is guaranteed in the self-adjoint case $m = 1$, $a = \frac{1}{4}$.

Thus, for $m \geq 2$, in the above subspaces obtained via a suitable closure, we can apply standard eigenfunction expansion techniques as in the classic self-adjoint case $m = 1$.

B.5. **Solenoidal Hermite polynomials.** The vector solenoidal Hermite polynomials are constructed from (B.19) in a manner similar to that for $m = 1$; cf (2.14)–(2.16). Namely, given a vector polynomial

$$\mathbf{v}^*_\beta = [\psi^*_{\beta_1}, \psi^*_{\beta_2}, ..., \psi^*_{\beta_N}]^T, \quad \text{where} \quad |\beta_1| = |\beta_2| = ... = |\beta_N| = |\beta|, \tag{B.21}$$

it gets solenoidal provided that

$$\operatorname{div} \mathbf{v}^*_\beta \equiv \sum_{i=1}^N (\psi_{\beta_i})_{y_i} = 0. \tag{B.22}$$

For instance, for the Burnett case $m = 2$ and $N = 3$ (not all linearly independent ones are presented, normalization constants are omitted):

$$\lambda_0 = 0: \quad \mathbf{v}^*_0 = [1, 1, 1]^T, \tag{B.23}$$

$$\lambda_1 = -\tfrac{1}{4}: \quad \mathbf{v}^*_{11} = [y_2, -y_3, y_2]^T, \quad \mathbf{v}^*_{12} = [y_3, y_3, -y_1]^T, \quad \mathbf{v}^*_{13} = [-y_2, y_1, y_1]^T, \tag{B.24}$$

$$\lambda_2 = -\tfrac{1}{2}: \quad \mathbf{v}^*_{21} = [-y_1^2 - y_3^2, y_1 y_2, y_1 y_3]^T, \; \mathbf{v}^*_{22} = [y_1 y_2, -y_2^2 - y_3^2, y_2 y_3]^T, \; \text{etc.} \tag{B.25}$$

(B.26) $\lambda_3 = -\frac{3}{4}: \quad \mathbf{v}_{31}^* = [y_2^3, y_3^3, y_1^3], \quad \mathbf{v}_{32}^* = [y_1 y_2^2, y_2 y_1^2, -y_3(y_1^2 + y_2^2)], \;$ etc.

(B.27) $\lambda_4 = -1: \quad \mathbf{v}_{41}^* = [y_2^4 + 4!, y_3^4 + 4!, y_1^4 + 4!]^T, \;\; \mathbf{v}_{42}^* = [y_1 y_2^3, y_2 y_1^3, -y_3(y_1^3 + y_2^3)]^T, \quad$ etc.

As in the self-adjoint case $m = 1$, some technical efforts are necessary towards completeness/closure of generalized solenoidal Hermite polynomials in suitable spaces. We omit details.

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