Numerical ranges of companion matrices: flat portions on the boundary

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Abstract. Criterion for a companion matrix to have a certain number of flat portions on the boundary of its numerical range is given. The criterion is specialized to the cases of 3×3 and 4×4 matrices. In the latter case, it is proved that a 4×4 unitarily irreducible companion matrix cannot have 3 flat portions on the boundary of its numerical range. Numerical examples are given to illustrate the main results.

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1. Introduction

The numerical range $W(A)$ of an $n \times n$ matrix A is a subset of the complex plane $\mathbb{C}$ defined as

$$W(A) = \{\langle Ax, x\rangle \colon x \in \mathbb{C}^n,\ \|x\| = 1\},$$

where $\langle \cdot, \cdot \rangle$ stands for the standard inner product in $\mathbb{C}^n$. This set first appeared in classical works by Toeplitz [15] and Hausdorff [9], and since then has been studied intensively. Among standard contemporary references are [7] and [10, Chapter I], and all properties of the numerical range we will be using without proof can be found, e.g., in these two monographs.

Among other things, it is of interest to locate flat portions (if any) on the boundary $\partial W(A)$ of the numerical range, and in particular to establish a bound for the number $f(A)$ of such portions for various matrix classes. If A is *unitarily reducible*, that is, unitarily similar to a block diagonal matrix with at least two diagonal blocks A_j, then $W(A)$ is the convex hull of $W(A_j)$. The flat portions on $\partial W(A)$ are then bound to emerge, unless one of $W(A_j)$ contains all others. In particular, for normal A the blocks A_j can be made one-dimensional and $W(A)$ is nothing but the convex hull of the spectrum $\sigma(A)$. It is easy to see therefore that $f(A)$ is at most n for normal A.

The picture is trivial for $n = 2$: $f(A) = 0$ if A is not normal, since $W(A)$ is then an elliptical disk, $f(A) = 1$ for normal A different from a scalar multiple of the identity, since $W(A)$ is then a line segment, and $f(\lambda I) = 0$. For $n = 3$, the classification of possible shapes of $W(A)$ was given by Kippenhahn ([12], see also more accessibly [13]). From this classification it easily follows that the maximal possible $f(A)$ is actually attained by normal A, while $f(A)$ is at most two for non-normal unitarily reducible matrices, and at most one for unitarily irreducible ones. Constructive descriptions of 3×3 matrices matrices A with flat portions on $\partial W(A)$ were obtained in [11, 14].

The case $n = 4$ was undertaken in [2, Theorem 37], where it was established the bound 4 is then sharp, while for unitarily irreducible 4×4 matrix A, the number $f(A)$ is at most 3. On the other hand, for any n there exist $n \times n$ unitarily reducible matrices A for which $f(A) = 2(n-2)$, see Example 38 in [2] (suggested to the authors by C.-K. Li). It is not known, when $n > 4$, (i) whether this delivers the sharp upper bound for $f(A)$ (note that $2(n-2) = 4$ for $n = 4$) and (ii) what is the upper bound for unitarily irreducible A.

In this paper we focus on the case when A is a *companion matrix*, that is,

$$A = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_0 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix}. \tag{1.1}$$

It is well known that the elements of the last row of (1.1) coincide, up to the sign, with the coefficients of its characteristic polynomial:

$$\det(A - \lambda I) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0. \tag{1.2}$$

These matrices were treated in [6], where in particular it was established that for a companion $n \times n$ matrix A, $f(A) \le n$ and all matrices A with $f(A) = n$ were described. They happen to be unitarily reducible, and the question of the maximal number of flat portions for unitarily irreducible companion matrices also remains open.

In our paper, we further tackle the issue of flat portions on $\partial W(A)$ for companion matrices A. Necessary and sufficient conditions for such portions to exist are described in Section 2. For arbitrary n they are rather cumbersome, and (at least in their sufficient part) not easy to check. However, for $n = 3, 4$ they can be recast into constructively verifiable criteria, allowing in particular to describe all possible values of $f(A)$. The cases $n = 3$ and $n = 4$ are treated in Sections 3 and 4, respectively.

2. Conditions for flat portions existence

For convenience of reference, we start with two statements applicable to arbitrary $n \times n$ matrices A. Recall that $\operatorname{Re} A = \frac{1}{2}(A + A^*)$ and $\operatorname{Im} A = \frac{1}{2i}(A - A^*)$.

Lemma 2.1. *Let A be an $n \times n$ matrix A. Then $\partial W(A)$ contains a vertical flat portion to the right of $W(A)$ if and only if*
(i) *the maximal eigenvalue $\lambda_{\max}$ of $\operatorname{Re} A$ is not simple, and*
(ii) *the compression of $\operatorname{Im} A$ (equivalently, of A) onto the eigenspace $\mathfrak{L}$ of $\operatorname{Re} A$ corresponding to $\lambda_{\max}$ is not a scalar multiple of the identity.*

This lemma is well known and was used, e.g., in [11, 2].

Formally speaking, (i) follows from (ii), but we prefer (i) to be stated explicitly since it is the condition addressed in Theorem 2.5 below.

Lemma 2.2. *Let A be an $n\times n$ matrix A. Suppose that $\operatorname{Re} A$ has an eigenvalue λ of multiplicity bigger that $\lfloor n/2 \rfloor$ while the compression of $\operatorname{Im} A$ onto the corresponding eigenspace $\mathfrak{L}$ of $\operatorname{Re} A$ is a scalar multiple of the identity. Then A is unitarily reducible.*

Proof. Passing from A to $A - zI$ with an appropriate choice of $z \in \mathbb{C}$, we may without loss of generality suppose that $\lambda = 0$ and the compression of $\operatorname{Im} A$ onto $\mathfrak{L} := \operatorname{Ker} \operatorname{Re} A$ is the zero operator. But the latter condition means simply that $(\operatorname{Im} A)\mathfrak{L} \perp \mathfrak{L}$. Since $2 \dim \mathfrak{L} > n$, this is only possible if $\operatorname{Im} A$ is not injective on $\mathfrak{L}$, that is, $\mathfrak{L}$ contains a non-zero vector x from $\operatorname{Ker} \operatorname{Im} A$. Then x is an eigenvector for both $\operatorname{Re} A$ and $\operatorname{Im} A$ (equivalently, for both A and A^*), which makes it a normally splitting eigenvector for A, and A itself — unitarily reducible into 1×1 and $(n-1) \times (n-1)$ blocks. □

Remark 2.3. For $n = 3$ condition on λ in Lemma 2.2 merely means that this is not a simple eigenvalue. Consequently, for unitarily irreducible 3×3 matrices condition (ii) of Lemma 2.1 can be dropped. This observation was also used in [11, 2].

For companion matrices, a constructive criterion of unitary reducibility is known. It was obtained in [5, Section 1] and can be summarized as follows.

Lemma 2.4. *An $n\times n$ companion matrix is unitarily reducible if and only if $\sigma(A) = \{\eta\omega_j : j \in J_1\} \cup \{\overline{\eta}^{-1}\omega_j : j \in J_2\}$ for some $\eta \in \mathbb{C} \setminus \{0\}$ and partition $J_1 \cup J_2$ of $\{1, \dots, n\}$, where both J_1 and J_2 are non-empty; $\omega_1, \dots, \omega_n$ being the set of all nth roots of 1. If this condition holds, then A is unitarily similar to $A_1 \oplus A_2$, with $\sigma(A_1) = \{\eta\omega_j : j \in J_1\}$, $\sigma(A_2) = \{\overline{\eta}^{-1}\omega_j : j \in J_2\}$. The matrix A is unitary if $|\eta| = 1$, and A_1, A_2 are unitarily irreducible otherwise.*

We are now ready to state the necessary condition for the flat portion existence.

Theorem 2.5. *Let A be given by (1.1). Then for $W(A)$ to have a flat portion on the boundary it is necessary that*

$$\sum_{j=0}^{n-2} a_j \omega^{n-j} \sin \frac{\pi(j+1)}{n} = \sin \frac{\pi}{n} \tag{2.1}$$

and

$$\operatorname{Re}(a_{n-1}\omega) = \sum_{j=2}^{n-1} \frac{|\gamma_j|^2}{\cos\frac{\pi}{n} - \cos\frac{\pi j}{n}} - \cos\frac{\pi}{n} \tag{2.2}$$

for some ω *with* $|\omega| = 1$ *and*

$$\gamma_j = \frac{1}{\sqrt{2n}} \left(\sin\frac{\pi j(n-1)}{n} - \sum_{k=0}^{n-2} a_k \omega^{n-k} \sin\frac{\pi j(k+1)}{n} \right). \tag{2.3}$$

If conditions (2.1), (2.2) *hold, then the potential flat portion passes through the point* $\overline{\omega}\cos\frac{\pi}{n}$ *and has the slope* $\frac{\pi}{2} - \arg\omega$.

Proof. Observe first of all that for any ω with absolute value one the matrix ωA, while not being companion itself (for $\omega \neq 1$), is nevertheless unitarily similar to a companion matrix

$$B = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -b_0 & \dots & -b_{n-2} & -b_{n-1} \end{bmatrix},$$

where $b_j = a_j \omega^{n-j}$: $\omega A = \Omega^{-1} B \Omega$ with

$$\Omega = \operatorname{diag}[1, \omega, \dots, \omega^{n-1}]. \tag{2.4}$$

Consequently,

$$W(A) = \overline{\omega} W(\omega A) = \overline{\omega} W(B).$$

It therefore suffices to show that conditions (2.1), (2.2) with $\omega = 1$ are necessary for $\partial W(A)$ to contain a vertical line segment located to the right of $W(A)$, and that this line segment (if it exists) passes through the real point $\cos\frac{\pi}{n}$.

Let us show that (2.1), (2.2) can be interpreted as condition (i) of Lemma 2.1 for A given by (1.1).

Due to the interlacing property of eigenvalues of hermitian matrices, $\lambda_{\max}$ will be the maximal eigenvalue of all $(n-1)\times(n-1)$ principal submatrices of A. For A given by (1.1),

$$\operatorname{Re} A = \frac{1}{2} \left[\begin{array}{cccc|c} & & & & -\overline{a_0} \\ & & T & & \vdots \\ & & & & -\overline{a_{n-3}} \\ & & & & 1 - \overline{a_{n-2}} \\ \hline -a_0 & \dots & -a_{n-3} & 1 - a_{n-2} & -2\operatorname{Re} a_{n-1} \end{array} \right], \tag{2.5}$$

where T is the $(n-1)\times(n-1)$ tridiagonal matrix with zeros on the main diagonal and ones on two side diagonals:

$$T = \begin{bmatrix} 0 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{bmatrix}.$$

The eigenvalues and the eigenvectors of T are well known. Namely (see, e.g., [8] or [1, Section 2.2]),

$$Tv_j = 2\cos\frac{\pi j}{n} v_j, \quad j = 1, \dots, n-1,$$

where

$$v_j = \left[\sin\frac{\pi j}{n}, \dots, \sin\frac{\pi j(n-1)}{n}\right]^T. \tag{2.6}$$

So, the abscissa of the potential vertical flat portion is indeed $\cos\frac{\pi}{n}$. On the other hand, the left upper $(n-1)\times(n-1)$ block of A is the Jordan cell J_{n-1}, so that $W(A) \supset W(J_{n-1})$. In its turn, $W(J_{n-1}) = \{z\colon\ |z| \le \cos\frac{\pi}{n}\}$ (see, e.g., [8]), so that the above mentioned flat portion should be passing through the real point $\cos\frac{\pi}{n}$.

As it is stated in [8] (and can also be checked via a routine trigonometrical calculation), $\|v_j\|^2 = n/2$ for all j. Therefore, the matrix

$$V = \sqrt{\frac{2}{n}}\left[\sin\frac{\pi jk}{n}\right]_{k,j=1}^{n-1} \tag{2.7}$$

is an hermitian (actually, real symmetric) involution which diagonalizes T:

$$T = 2V \operatorname{diag}\left[\cos\frac{\pi}{n}, \dots, \cos\frac{\pi(n-1)}{n}\right] V.$$

Consequently, matrix (2.5) is unitarily similar to

$$H = \left[\begin{array}{ccc|c} \cos\frac{\pi}{n} & & & \overline{\gamma_1} \\ & \ddots & & \vdots \\ & & \cos\frac{\pi(n-1)}{n} & \overline{\gamma_{n-1}} \\ \hline \gamma_1 & \dots & \gamma_{n-1} & -\operatorname{Re} a_{n-1} \end{array}\right], \tag{2.8}$$

where

$$\gamma_j = \frac{1}{\sqrt{2n}}\left(\sin\frac{\pi j(n-1)}{n} - \sum_{k=0}^{n-2} a_k \sin\frac{\pi j(k+1)}{n}\right)$$

(observe that the latter formula is the particular case of (2.3) for $\omega = 1$).

From (2.8) it is easily seen that

$$\det\left(H - \cos\frac{\pi}{n} I\right) = -\left|\gamma_1\right|^2 \prod_{j=2}^{n-1}\left(\cos\frac{\pi j}{n} - \cos\frac{\pi}{n}\right).$$

Thus, $\cos\frac{\pi}{n}$ is an eigenvalue of H (and therefore of $\operatorname{Re} A$) if and only if $\gamma_1 = 0$. This coincides with (2.1) in which $\omega = 1$.

The multiplicity of $\cos\frac{\pi}{n}$ as an eigenvalue of $\operatorname{Re} A$ cannot exceed 2, since the matrix $H - \cos\frac{\pi}{n} I$ contains a non-singular $(n-2)\times(n-2)$ submatrix

$$\operatorname{diag}\left[\cos\frac{\pi j}{n} - \cos\frac{\pi}{n}\right]_{j=2}^{n-1}.$$

In order for this multiplicity to equal 2 it is necessary and sufficient that, in addition to $\gamma_1 = 0$, the right lower $(n-1)\times(n-1)$ submatrix of $H - \cos\frac{\pi}{n} I$,

$$\left[\begin{array}{ccc|c} \cos\frac{2\pi}{n} - \cos\frac{\pi}{n} & & & \overline{\gamma_2} \\ & \ddots & & \vdots \\ & & \cos\frac{\pi(n-1)}{n} - \cos\frac{\pi}{n} & \overline{\gamma_{n-1}} \\ \hline \gamma_2 & \dots & \gamma_{n-1} & -\operatorname{Re} a_{n-1} - \cos\frac{\pi}{n} \end{array}\right],$$

is singular. This is an arrowhead matrix, the determinant of which can be computed by an easy induction and equals

$$\left(\sum_{j=2}^{n-1} \frac{|\gamma_j|^2}{\cos\frac{\pi}{n} - \cos\frac{\pi j}{n}} - \operatorname{Re} a_{n-1} - \cos\frac{\pi}{n}\right) \cdot \prod_{j=2}^{n-1} \left(\cos\frac{\pi j}{n} - \cos\frac{\pi}{n}\right).$$

Thus, it equals zero if and only if (2.2) holds (once again, with $\omega = 1$). □

Note that necessity of condition (2.1) in a slightly different way was established in [6], see Lemma 3 there.

It follows from Lemma 2.4 and Theorem 2.5 that in the generic case matrices (1.1) are unitarily irreducible and have no flat portions on the boundary. Namely, the set of companion matrices for which (2.1) has no unimodular solutions is open and dense within the set of all companion matrices. The openness of this set is clear from continuity of roots of algebraic equations as functions of the equations' coefficients. As for denseness, assume $a_0 \neq 0$, and let

$$\left(\sum_{j=0}^{n-2} a_j \omega^{n-j} \sin\frac{\pi(j+1)}{n}\right) - \sin\frac{\pi}{n} = a_0(\omega - \omega_1)\cdots(\omega - \omega_n),$$

where $\omega_1, \dots, \omega_n$ are all the roots of (2.1) counted with multiplicities. Note that $\omega_1 \cdots \omega_n = (-1)^{n+1} a_0^{-1} \sin\frac{\pi}{n}$ and $\sum_{j=1}^{n} \omega_j^{-1} = 0$. We now perturb $\omega_1, \dots, \omega_n$ slightly resulting in $\omega_1', \dots \omega_n'$ respectively such that none of the ω_j's is unimodular and the equality $\sum_{j=1}^{n} (\omega_j')^{-1} = 0$ holds. Clearly such perturbation is possible. Now define $a_0', \dots, a_n'$ by the equalities

$$\omega_1' \cdots \omega_n' = (-1)^{n+1} (a_0')^{-1} \sin\frac{\pi}{n}$$

and

$$\left(\sum_{j=0}^{n-2} a_j' \omega^{n-j} \sin \frac{\pi(j+1)}{n}\right) - \sin \frac{\pi}{n} = a_0'(\omega - \omega_1') \cdots (\omega - \omega_n').$$

As a result, a companion matrix is obtained, as close as we wish to A, for which the corresponding equation (2.1) has no unimodular solutions.

A specific subclass of unitarily irreducible companion matrices with no flat portions on the boundary of their numerical ranges is delivered by the following

Corollary 2.6. *Let A be given by* (1.1) *with*

$$a_0 = \cdots = a_{n-2} = 0. \tag{2.9}$$

Then A is unitarily irreducible and $W(A)$ has no flat portions on the boundary.

Indeed, such A are singular, and therefore (as follows from Lemma 2.4) unitarily irreducible. On the other hand, equation (2.1) takes the form $0 = \sin \frac{\pi}{n}$ and thus has no solutions.

Note that if, in addition to (2.9), also $a_{n-1} = 0$, then A is simply a nilpotent Jordan block, with $W(A)$ being a circular disk. If $a_{n-1} \neq 0$, the numerical range of A cannot be circular according to [6, Theorem 1], but still there will be no flat portions on $\partial W(A)$. An example illustrating this, more interesting, situation when $n = 4$, will be given in Section 4.

Of course, a criterion for the flat portion existence can be formulated by imposing condition (ii) of Lemma 2.1 (interpreted for the case of companion matrices) on matrices satisfying Theorem 2.5.

Theorem 2.7. *Let conditions* (2.1), (2.2) *hold for some matrix A given by* (1.1) *and ω having absolute value 1. Then $\partial W(A)$ has a flat portion passing through $\overline{\omega} \cos \frac{\pi}{n}$ if and only if at least one of the scalar products $\langle \mathrm{Im}(\omega A)x_1, x_2\rangle$ and $\langle \mathrm{Im}(\omega A)x_2, x_2\rangle$ differs from zero. Here*

$$x_1 = \Omega^{-1} \begin{bmatrix} v_1 \\ 0 \end{bmatrix}, \quad x_2 = \Omega^{-1} \begin{bmatrix} V & 0 \\ 0 & 1 \end{bmatrix} \xi, \tag{2.10}$$

with Ω, v_1 and V given by (2.4), (2.6) *and* (2.7) *respectively,*

$$\xi = [0, \xi_2, \ldots, \xi_{n-1}, 1]^T \text{ and } \xi_j = \frac{\overline{\gamma_j}}{\cos \frac{\pi}{n} - \cos \frac{\pi j}{n}}, \quad j = 2, \ldots, n-1.$$

Proof. Under conditions of Theorem 2.5 (and in the notation of its proof), vectors $[1, 0, \ldots, 0]^T (\in \mathbb{C}^n)$ and ξ form a basis of $\mathrm{Ker}\left(H - \cos \frac{\pi}{n} I\right)$. Consequently, (2.10) delivers a basis for $\mathfrak{L} = \mathrm{Ker}\left(\mathrm{Re}\, A - \cos \frac{\pi}{n} I\right)$. Since

$$\langle \mathrm{Im}(\omega A)x_1, x_1\rangle = \langle (\mathrm{Im}\, J_{n-1})v_1, v_1\rangle = 0,$$

the compression of $\mathrm{Im}(\omega A)$ onto $\mathfrak{L}$ is a scalar multiple of the identity if and only if it equals zero. This is equivalent to $\langle \mathrm{Im}(\omega A)x_1, x_2\rangle = \langle \mathrm{Im}(\omega A)x_2, x_2\rangle = 0$. It remains to invoke Lemma 2.1. □

Thus the number of flat portions on the boundary of the numerical range of the matrix (1.1) coincides with the number of distinct solutions ω of (2.1), (2.2) for which $|\omega| = 1$ and the "if and only if" conditions of Theorem 2.7 are satisfied.

3. 3×3 matrices

As was mentioned in the Introduction, the case $n = 2$ is trivial, and there is no need to consider companion 2×2 matrices

$$\begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \tag{3.1}$$

separately. Nevertheless note that conditions (2.1), (2.2) in this case amount to

$$a_0\omega^2 = 0, \quad \operatorname{Re}(a_1\omega) = 0.$$

They hold if and only if $|a_0| = 1$ and $2\arg a_1 - \arg a_0 = \pi$ (the latter condition being redundant when $a_1 = 0$). These are exactly the requirements for (3.1) to be normal, as it should be.

We move therefore to the case $n = 3$.

Theorem 3.1. *Let A be a 3×3 companion matrix:*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}.$$

Then $\partial W(A)$ contains a flat portion if and only if the equation

$$a_0\omega^3 + a_1\omega^2 = 1 \tag{3.2}$$

has a solution ω with $|\omega| = 1$ in addition satisfying

$$2\operatorname{Re}(a_2\omega) = |a_0|^2 - 1, \tag{3.3}$$

and the triple a_0, a_1, a_2 differs from

$$a_0 = -2\zeta^3, \ a_1 = 3\zeta^2\overline{w}, \ a_2 = \frac{3}{2}\zeta w, \tag{3.4}$$

where $|\zeta| = 1$ and w is any cube root of 1.

Proof. Necessity of (3.2), (3.3) follows directly from Theorem 2.5. Indeed, (2.1) for $n = 3$ takes the form (3.2), while (2.3) for $n = 3$ and $j = 2$ yields

$$\gamma_2 = \frac{1}{2\sqrt{2}}\left(-1 - a_0\omega^3 + a_1\omega^2\right).$$

Taking (3.2) into consideration, we conclude further that $\gamma_2 = -a_0\omega^3/\sqrt{2}$. Based on this observation, (2.2) with $n = 3$ turns into (3.3).

Sufficiency. Conditions (3.2), (3.3), being the 3×3 version of (2.1), (2.2), guarantee that the maximal eigenvalue of $\operatorname{Re}(\omega A)$ is not simple. By Remark 2.3, for 3×3 matrices this implies the existence of a flat portion on $\partial W(A)$ (with a slope $\frac{\pi}{2} - \arg\omega$) provided that A is unitarily irreducible. It remains therefore to consider the case of unitarily reducible A.

According to Lemma 2.4 in the case $n = 3$, the eigenvalues of a unitarily reducible A are $\lambda_1 = \eta\omega_1$, $\lambda_2 = \eta\omega_2$, $\lambda_3 = \omega_3/\overline{\eta}$, with ω_j corresponding to the three cube roots of unity (in no particular order) and some non-zero η. Moreover, A is then unitarily similar to the orthogonal sum of a 2×2 block A_2 with the eigenvalues λ_1, λ_2 and the 1×1 block $A_1 = \{\lambda_3\}$.

Letting $|\eta| = r$ and $\arg\eta = \theta$, we therefore conclude from Vieta's formulas that

$$\begin{aligned}
a_0 &= & -\lambda_1\lambda_2\lambda_3 &= & -\eta^2/\overline{\eta} &= & -re^{3i\theta}&,\\
a_1 &= & \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 &= & \eta^2\omega_1\omega_2 + \eta(\omega_1+\omega_2)\omega_3/\overline{\eta} &= & (r^2-1)e^{2i\theta}\overline{\omega_3}&,\\
a_2 &= & -(\lambda_1+\lambda_2+\lambda_3) &= & -\eta(\omega_1+\omega_2) - \omega_3/\overline{\eta} &= & \frac{r^2-1}{r}e^{i\theta}\omega_3&.
\end{aligned}$$

If $r = 1$, then $a_1 = a_2 = 0$, $|a_0| = 1$, so that (3.3) is a tautology while (3.2) has three equidistant solutions ω on the unit circle. The matrix A is in this case unitary, and $W(A)$ has three flat portions on the boundary.

On the other hand, (3.3) implies that $2\,|a_2| \geq |a_0|^2 - 1$, that is,

$$2\frac{\left|r^2-1\right|}{r} \geq \left|r^2-1\right|.$$

If $r \neq 1$, this is only possible when $r \leq 2$.

Due to the unitary similarity of A and $A_1 \oplus A_2$, the numerical range $W(A)$ is the convex hull of λ_3 and $W(A_2)$. The latter, in its turn, is the ellipse with the foci at λ_1, λ_2 and the major axis of the length

$$\begin{aligned}
&\sqrt{\operatorname{tr}(A_2^*A_2) - |\lambda_1|^2 - |\lambda_2|^2 + |\lambda_1-\lambda_2|^2} = \\
&\qquad\sqrt{\operatorname{tr}(A^*A) - |\lambda_1|^2 - |\lambda_2|^2 - |\lambda_3|^2 + |\lambda_1-\lambda_2|^2} = \\
&\qquad\sqrt{2 + |a_0|^2 + |a_1|^2 + |a_2|^2 - 2r^2 - r^{-2} + 3r^2} = \sqrt{1+r^2+r^4},
\end{aligned}$$

while

$$|\lambda_1 - \lambda_3| + |\lambda_2 - \lambda_3| = \left|r\omega_1 - r^{-1}\omega_3\right| + \left|r\omega_2 - r^{-1}\omega_3\right| = 2\sqrt{1+r^2+r^{-2}}.$$

But

$$2\sqrt{1+r^2+r^{-2}} > \sqrt{1+r^2+r^4} \text{ for } 0 < r < 2,$$

while for $r = 2$ the equality is attained. Consequently, the point λ_3 lies outside the ellipse $W(A_2)$, and their convex hull has two flat portions on the boundary, unless $r = 2$. It remains to observe that the case $r = 2$ corresponds exactly to the exception (3.4). □

Note that Example 6 in [6] is a particular case of (3.4) corresponding to $\zeta = w = 1$.

Companion 3×3 matrices with elliptical numerical ranges were treated in [3], based on the tests proposed in [11]. According to Kippenhahn's classification (see [13]), for irreducible 3×3 companion matrices A not satisfying conditions of [3] or our Theorem 3.1, $W(A)$ has ovular shape.

We remark that (3.3) is a tautology if $a_2 = 0$, $|a_0| = 1$, it has no unimodular solutions if $2|a_0| < \left||a_0|^2 - 1\right|$, and its (automatically unimodular) solutions are given by

$$\frac{|a_0|^2 - 1 \pm i\sqrt{4|a_2|^2 - (|a_0|^2 - 1)^2}}{2a_2}$$

in the remaining case $0 \neq 2|a_2| \geq \left||a_0|^2 - 1\right|$. So, conditions (3.2), (3.3) can be recast as follows: either

$$a_2 = 0, \ |a_0| = 1, \text{ and } a_0\omega^3 + a_1\omega^2 = 1 \text{ for some unimodular } \omega, \tag{3.5}$$

or

$$a_2 \neq 0, \qquad 2|a_2| \geq \left||a_0|^2 - 1\right| \tag{3.6}$$

and

$$a_0\left(|a_0|^2 - 1 + i\kappa\sqrt{4|a_2|^2 - (|a_0|^2 - 1)^2}\right)^3 + 2a_1a_2\left(|a_0|^2 - 1 + i\kappa\sqrt{4|a_2|^2 - (|a_0|^2 - 1)^2}\right)^2 = 8a_2^3 \tag{3.7}$$

for some choice of $\kappa = \pm 1$.

Example 1. Let $a_0 = 2 + i$, $a_1 = -1 - i$, and $a_2 = 2 + 3i$ so that we have:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 - i & 1 + i & -2 - 3i \end{bmatrix}. \tag{3.8}$$

Then (3.3) holds with $\omega = 1$, the exception (3.4) does not hold, and (3.2) has only one unimodular solution: $\omega_1 = 1$. Thus, the matrix A given by (3.8) has one (vertical) flat portion on the boundary of its numerical range. As such, this A is automatically unitarily irreducible. The respective $W(A)$ is pictured in Figure 1 below[1]:

[1] All numerical ranges are plotted using the program by C. Cowen and E. Harel, available at `http://www.math.iupui.edu/~ccowen/Downloads/33NumRange.html`.

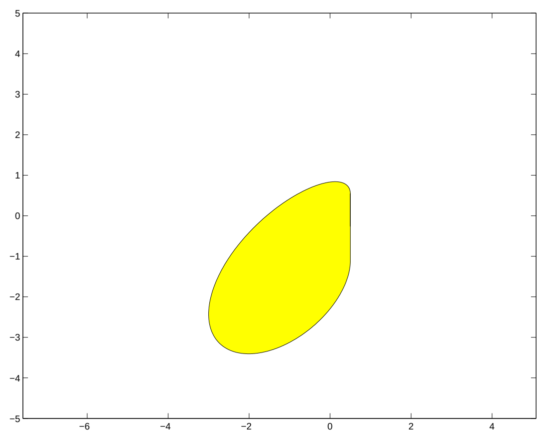

FIGURE 1. Numerically calculated plot of the numerical range of A, as given by (3.8).

4. 4×4 matrices

In this section, we consider the case $n = 4$, that is,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix}, \tag{4.1}$$

After some simple algebra, equations (2.1) and (2.2) take the form

$$a_0\omega^4 + \sqrt{2}a_1\omega^3 + a_2\omega^2 = 1 \tag{4.2}$$

and

$$\sqrt{2}\,\mathrm{Re}(a_3\omega) = 2|\gamma_2|^2 + |\gamma_3|^2 - 1, \tag{4.3}$$

respectively. On the other hand, a computation shows that the γ_j's defined by (2.3) with $n = 4$ are given by the formulas

$$\gamma_2 = \frac{1}{2\sqrt{2}}(-1 - a_0\omega^4 + a_2\omega^2), \quad \gamma_3 = \frac{1}{4}(1 - a_0\omega^4 + \sqrt{2}a_1\omega_3 - a_2\omega^2),$$

which simplify further by using (4.2) to

$$\gamma_2 = -\frac{a_1\omega^3}{2} - \frac{a_0\omega^4}{\sqrt{2}}, \qquad \gamma_3 = \frac{1}{\sqrt{2}}a_1\omega^3. \tag{4.4}$$

Substitute (4.4) for γ_2 and γ_3 in the right hand side of (4.3) to yield

$$\sqrt{2}\,\mathrm{Re}((a_3 - a_0\overline{a_1})\omega) = |a_0|^2 + |a_1|^2 - 1. \tag{4.5}$$

So, the system of equations (2.1), (2.2) is equivalent to the system (4.2), (4.5).

Similarly to the situation for $n = 3$, (4.5) is a tautology if

$$a_3 = a_0\overline{a_1}, \quad |a_0|^2 + |a_1|^2 = 1, \tag{4.6}$$

it has no unimodular solutions if $\sqrt{2}\,|a_3 - a_0\overline{a_1}| < \left||a_0|^2 + |a_1|^2 - 1\right|$, and its (automatically unimodular) solutions are given by

$$\frac{|a_0|^2 + |a_1|^2 - 1 \pm i\sqrt{2\,|a_3 - a_0\overline{a_1}|^2 - (|a_0|^2 + |a_1|^2 - 1)^2}}{\sqrt{2}(a_3 - a_0\overline{a_1})} \tag{4.7}$$

in the remaining case

$$0 \neq |a_3 - a_0\overline{a_1}| \geq \frac{1}{\sqrt{2}}\left||a_0|^2 + |a_1|^2 - 1\right|.$$

We thus obtain the following:

Corollary 4.1. *Let A be given by* (4.1)*. Then for $W(A)$ to have a flat portion on the boundary it is necessary that*

$$|a_3 - a_0\overline{a_1}| \geq \frac{1}{\sqrt{2}}\left||a_0|^2 + |a_1|^2 - 1\right|$$

and (4.2) *has a unimodular solution ω. Moreover, this ω must coincide with one of the values given by* (4.7)*, unless* (4.6) *holds, and corresponds to the flat portion (if any) with the slope $\frac{\pi}{2} - \arg\omega$.*

This result is instrumental in establishing a peculiar gap in the number of possible flat portions for 4×4 companion matrices.

Theorem 4.2. *There are no 4×4 companion matrices A with $f(A) = 3$.*

Proof. Let us first address the case when A is unitarily reducible. According to Lemma 2.4, it is then either unitary, with the eigenvalues located in the vertices of a square centered at the origin (in which case $f(A) = 4$), or is unitarily similar to the orthogonal sum of two unitarily irreducible blocks. If these blocks are both 2×2, then $W(A)$ is the convex hull of two ellipses — the construction that can a priori have 0, 2, or 4 flat portions (though the case $f(A) = 4$ does not materialize, as shown in [6]) but not 1 or 3. Finally, if A reduces to the orthogonal sum of a 1×1 and 3×3 block, then the numerical range of the latter has no flat portions on the boundary according to [4, Theorem 2.5], which leaves only options $f(A) = 0, 2$ possible.

Now let A be unitarily irreducible. Applying Corollary 4.1 we see that $f(A) = 3$ is only possible when (4.6) holds and, moreover, (4.2) has at least three distinct unimodular solutions, say u, v and w. We consider separately the cases $a_0 = 0$ and $a_0 \neq 0$.

Case 1. $a_0 = 0$. The second equality in (4.6) then implies that $|a_1| = 1$. On the other hand, equation (4.2) in this case has degree 3, and therefore u, v, w are all its roots. By the Vieta theorem,

$$uvw = \frac{1}{\sqrt{2}a_1},$$

which is in contradiction with the unimodularity of u, v, w.

Case 2. $a_0 \neq 0$. Then (4.2) has the fourth root, also different from zero. Since the linear term is missing in (4.2), the inverses of the roots have zero sum. In other words, the fourth root is

$$-\frac{1}{u^{-1}+v^{-1}+w^{-1}} = -1/\overline{z},$$

where we have denoted $z := u + v + w$. Other parts of the Vieta theorem mean that

$$-uvw/\overline{z} = -1/a_0, \quad z - 1/\overline{z} = -\frac{\sqrt{2}a_1}{a_0}.$$

Taking absolute values, we obtain

$$|a_0| = |z|\,, \quad |a_1| = \left||z|^2 - 1\right|/\sqrt{2}.$$

When combined with the second equality in (4.6), this implies $|z| = 1$. Consequently, $a_1 = 0$. Equation (4.2) is therefore biquadratic, its roots come in opposite pairs, and without loss of generality may be relabeled as $\pm u, \pm v$. By the same Vieta theorem,

$$a_0 = -u^{-2}v^{-2}, \quad a_2 = u^{-2} + v^{-2}. \tag{4.8}$$

While we have established that conditions of Corollary 4.1 hold for four distinct unimodular values of ω, this does not necessarily mean that four flat portions actually materialize. So, further reasoning is needed in order to arrive at a contradiction. The first equality in (4.6) and the equality $a_1 = 0$ (proven earlier) imply that $a_3 = 0$ as well. So, the characteristic polynomial (1.2) in our case also is biquadratic, and the eigenvalues of A equal $\pm\lambda_1, \pm\lambda_2$ with λ_1^2, λ_2^2 being the roots of the quadratic equation

$$\mu^2 + a_2\mu + a_0 = 0.$$

The ratio of these roots is obviously a negative real number when $a_2 = 0$. Supposing $a_2 \neq 0$, on the other hand, we obtain

$$\begin{aligned}\frac{\lambda_1^2}{\lambda_2^2} &= \frac{-a_2 + \sqrt{a_2^2 - 4a_0}}{-a_2 - \sqrt{a_2^2 - 4a_0}} = \frac{\left(-a_2 + \sqrt{a_2^2 - 4a_0}\right)^2}{4a_0} \\ &= \frac{2a_2^2 - 4a_0 - 2a_2\sqrt{a_2^2 - 4a_0}}{4a_0} = -1 + \frac{a_2^2 - a_2\sqrt{a_2^2 - 4a_0}}{2a_0} \\ &= -1 + \frac{1 - \sqrt{1 - \frac{4a_0}{a_2^2}}}{2\frac{a_0}{a_2^2}} = -1 + \frac{\sqrt{1 + 2\left(\frac{-2a_0}{a_2^2}\right)} - 1}{-2\frac{a_0}{a_2^2}} = -1 + \frac{\sqrt{1+2x} - 1}{x},\end{aligned} \tag{4.9}$$

where $x = -2a_0/a_2^2$. Using (4.8),

$$x = \frac{2u^{-2}v^{-2}}{(u^{-2}+v^{-2})^2} = \frac{1}{1+\mathrm{Re}(u/v)^2},$$

and is therefore a positive real number. Since for all such x, $\sqrt{1+2x} < 1+x$, expression (4.9) is again negative. So, the eigenvalues $\pm\lambda_1, \pm\lambda_2$ of A are located at the vertices of a rhombus centered at the origin. According to [5], this implies unitary reducibility of A — a contradiction. Therefore, $f(A) = 3$ is an impossibility in this case as well, which concludes the proof. □

Example 2. We provide an explicit example of when A is a unitarily irreducible 4×4 companion matrix and $f(A) = 2$.
Let $a_0 = \frac{9+12i}{25}$, $a_1 = \frac{2\sqrt{2}(7+i)}{25}$, $a_2 = -\frac{4(3+4i)}{25}$, and $a_3 = \frac{6\sqrt{2}(1+i)}{25}$ so that we have:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{9+12i}{25} & -\frac{2\sqrt{2}(7+i)}{25} & \frac{4(3+4i)}{25} & -\frac{6\sqrt{2}(1+i)}{25} \end{bmatrix}. \tag{4.10}$$

The eigenvalues of A, $\{0.6413+0.8475i, 0.6264-0.5578i, -1.0468-0.4290i, -0.5603-0.2000i\}$, each have a different magnitude and therefore (Lemma 2.4 again) A is not unitarily reducible. We also see that for the matrix A given by (4.10) that (4.6) holds and (4.2) has two unimodular solutions, $\omega_1 = 1$ and $\omega_2 = i$, and two non-unimodular solutions, $\omega_3 = -2+i$ and $\omega_4 = -\frac{1}{3} - \frac{2i}{3}$. Moreover, for $A_1 = \omega_1 A = A$, $\mathrm{Re}\, A_1$ has two linearly independent eigenvectors, $f_1 = \left[\frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0\right]^T$ and $f_2 = \left[\frac{\sqrt{2}(-23+14i)}{25}, \frac{-37+16i}{25}, 0, 1\right]^T$, corresponding to the maximal eigenvalue of $\frac{\sqrt{2}}{2}$. Computing the scalar product $\langle(\mathrm{Im}\, A_1)f_1, f_2\rangle = \frac{\sqrt{2}(-7+24i)}{25} \neq 0$ we see that indeed $W(A)$ has a vertical flat portion on the boundary. Similarly, for $A_2 = \omega_2 A = iA$, $\mathrm{Re}\, A_2$ has two linearly independent eigenvectors, $g_1 = \left[-1, \sqrt{2}i, 1, 0\right]^T$ and $g_2 = \left[\frac{\sqrt{2}(-2+11i)}{25}, \frac{13+16i}{25}, 0, 1\right]^T$, corresponding to the maximal eigenvalue of $\frac{\sqrt{2}}{2}$, while $\langle(\mathrm{Im}\, A_2)g_1, g_2\rangle = \frac{36-2i}{25} \neq 0$. Therefore, $W(A)$ also has a horizontal flat portion on its boundary.

Thus, the matrix A given by (4.10) has two flat portions on the boundary of $W(A)$, as shown in Figure 2.

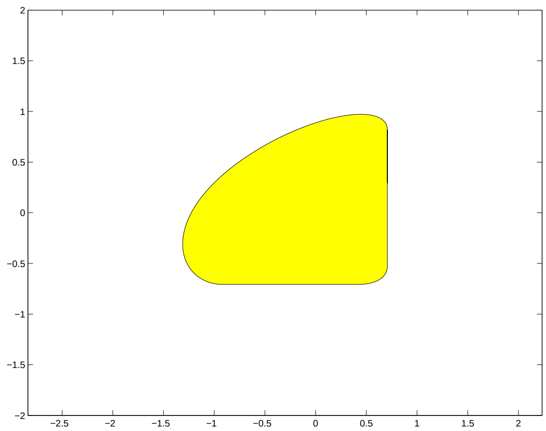

FIGURE 2. Numerically calculated plot of the numerical range of A, as given by (4.10).

*Example 3.*We provide an explicit example of when A is a unitarily irreducible 4×4 companion matrix and $f(A) = 1$. Let $a_0 = 0$, $a_1 = 1$, $a_2 = 1 - \sqrt{2}$, and $a_3 = 0$ so that we have:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -1+\sqrt{2} & 0 \end{bmatrix}. \tag{4.11}$$

Then (4.6) holds and (4.2) has only one unimodular solution: $\omega_1 = 1$. Moreover, for $A_1 = \omega_1 A = A$, $\operatorname{Re} A_1$ has two linearly independent eigenvectors, $f_1 = [1, \sqrt{2}, 1, 0]^T$ and $f_2 = [-1, -\sqrt{2}, 0, 1]^T$, corresponding to the maximal eigenvalue of $\sqrt{2}/2$. Computing the scalar product $\langle (\operatorname{Im} A_1) f_1, f_2 \rangle = \dfrac{2+\sqrt{2}}{2} i$ we see indeed that $W(A)$ has a vertical flat portion on the boundary.

Thus, the matrix A given by (4.11) has one flat portion on the boundary of $W(A)$, as shown in Figure 3.

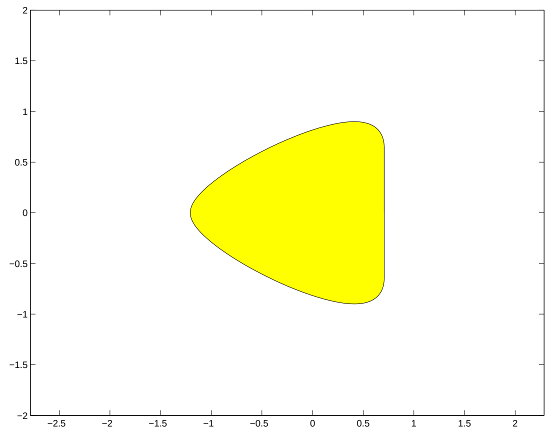

FIGURE 3. Numerically calculated plot of the numerical range of A, as given by (4.11).

Note that having exactly one flat portion on $W(A)$ implies unitary irreducibility of the matrix (4.11), as was shown in the proof of Theorem 4.2 (see the first paragraph there).

Finally, let us provide an example of a 4×4 matrix satisfying conditions of Corollary 2.6, and thus unitarily irreducible with no flat portions on the boundary of its numerical range.

Example 4. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}, \tag{4.12}$$

that is, $a_0 = a_1 = a_2 = 0$ and $a_3 = 2$. The numerical range of this matrix is given in Figure 4.

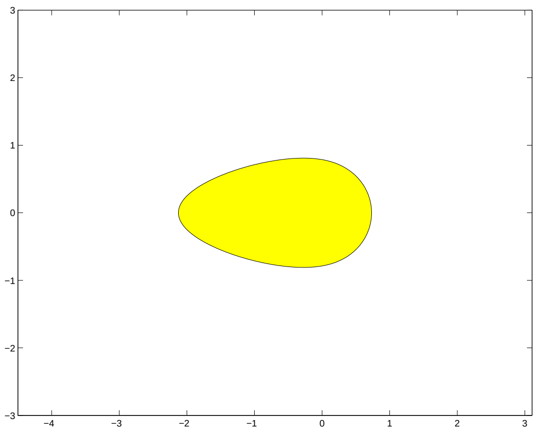

FIGURE 4. Numerically calculated plot of the numerical range of A, as given by (4.12).

Gathering information from Corollary 2.6 (or Example 4), Theorem 4.2 and Examples 2–3 we arrive to our final conclusion.

Theorem 4.3. *For a* 4×4 *unitarily irreducible companion matrix* A *the complete list of admissible values of* $f(A)$ *is* $\{0, 1, 2\}$.

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