J. Marine. Sci. Appl. (2010) 9: 208-212
DOI: 10.1007/s11804-010-9072-2

Chaotic Roll Motions of Ships in Regular Longitudinal Waves

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Abstract: Parametric resonance can lead to dangerously large rolling motions, endangering the ship, cargo and crew. The ***QR***-factorization method for calculating (LCEs) Lyapunov Characteristic Exponents was introduced; parametric resonance stability of ships in longitudinal waves was then analyzed using LCEs. Then the safe and unsafe regions of target ships were then identified. The results showed that this method can be used to analyze ship stability and to accurately identify safe and unsafe operating conditions for a ship in longitudinal waves.
Keywords: parametric resonance; Lyapunov characteristic exponents; longitudinal waves.
Article ID: 1671-9433(2010)02-0208-05

1 Introduction

Large amplitude rolling motions and capsizing of ships are reportedly most likely occur in longitudinal waves (Belenky *et al.,*2003, France *et al.,*2003). Although a ship exhibits little or no direct roll excitation under these conditions, strong roll response can be induced by parametric resonance, and loss of roll restoring capacity may lead to capsizing of the ship on a wave crest.

Parametric resonance is considered as the induced roll motion of a ship due to the periodic change of the restoring characteristics as the ship advances through the waves (Paulling and Rosenberg, 1959; Paulling, 1961). Parametric roll is expected to occur when the wave encounter frequency is close to double (principal parametric resonance) or single (fundamental parametric resonance) of the natural roll frequency of the vessel (Sanchez and Nayfeh, 1990; Zhang Dong, 1998).

Over the last two decades a substantial effort has been made to better understand nonlinear roll motion in capsizing via theoretical analysis. Sanchez and Nayfeh (1990) carried out an analysis of a mathematical model that takes account of the non-linear coupling of the pitch and roll modes of ship motions in regular longitudinal waves. They demonstrated that saturation phenomenon occurs when the wave amplitude is large enough as a consequence of the interaction between pitch and roll motions. The kinematic energy delivered by the waves exciting the pitch motion of the ship is partly transferred to the roll motion so that the roll amplitude grows gradually while the pitch amplitude remains constant. Hang *et al*.(1999) constructed an equivalent linearised differential equation of the rolling motion, and used Laplace transformation to find the safe region and unsafe region of the ship in longitudinal waves, Umeda and Hamamoto (2000); Umeda and petrer (2002) classified the ship capsizing in following and quartering seas into four modes: broaching, low cycle resonance, stability loss on a wave crest, and bow diving, and used nonlinear dynamics method to reveal the qualitative and quantitative characteristics of ships in low cycle resonance and broaching. Their research also find that before a ship capsizes it will have a chaotic motion, and in these conditions the ship's motion becomes unstable and unsafe, easy to capsize.

So it is important to know whether the ship is in chaotic motion or not, the method of Lyapunov characteristic exponents (LCEs) is a very useful tool (Nayfeh and Mook, 1979; Zhang *et al*.,2004). Lyapunov exponents measure the rate of convergence or divergence of nearby trajectories. A positive Lyapunov exponent indicates an exponential divergence or a strong sensitivity to initial conditions. This sensitivity is a standard sign of chaotic behavior. The aim of this research is to detect this sort of rapid divergence of the roll/roll-velocity trajectory. Therefore, danger can be foreseen by measuring rates of divergence or convergence of neighboring trajectories while tracking trends of instabilities.

Using Lyapunov exponents to study capsize has been touched on in the literature for both naval architecture and nonlinear dynamics (Falzarano, 1990). In recent years, the asymptotic Lyapunov exponent has been calculated from equations of motion for the mooring problem (Papoulias, 1987), 1-DOF capsize models (McCue,2005), 1-DOF flooded ship models (Murashige *et al*.,2000), works studying the effects of rudder angle while surf riding as it leads to capsize (Spyrou,1996), and investigation into the use of the tool for validating numerical simulation to experimental data (McCue and Troesch,2004).

In this paper, Lyapunov exponents are considered with

Received date: 2009-12-19
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respect to an analytical 1-DOF model for parametrically excited roll motion in regular longitudinal waves. We present a simple and high efficiency algorithm to calculate Lyapunov characteristic exponents, using Lyapunov characteristic exponents as the criteria for chaos to find the safe region and unsafe region of the ship in longitudinal waves.

2 The method of Lyapunov characteristic exponents

The method of Lyapunov characteristic exponents serves as a useful tool to quantify chaos. Specifically, Lyapunov exponents measure the rate of convergence or divergence of nearby trajectories. Negative Lyapunov exponents indicate convergence, while positive Lyapunov exponents demonstrate divergence and chaos. The magnitude of the Lyapunov exponents is an indicator of the time scale on which chaotic behavior can be predicted or transients decay for the positive exponent cases respectively.

There are many algorithms that can calculate LCEs (Wolf *et al*., 1985; Rangarajan *et al*., 1998; Udwadia and Bremen, 2001), in this paper, a simple and high-efficiency algorithm is presented here to calculate Lyapunov characteristic exponents. In this method, $\boldsymbol{QR}$ -factorization is the fundamental solution of the system. The fundamental solution $\boldsymbol{Y}=(y_1(t),y_2(t))^{\mathrm{T}}$ is expressed as $\boldsymbol{Y}=\boldsymbol{QR}$, where $\boldsymbol{Q}$ is an orthogonal matrix and $\boldsymbol{R}$ is an upper triangular matrix.

Consider a continuous dynamical system:

$$\dot{y}(t)=f(y(t)),\quad y(0)=y_0 \tag{1}$$

where $y\in R^n$ and $t\in R$. The variational equation associated with the dynamical system described in Eq.(1) is given by

$$\dot{\boldsymbol{Y}}=\boldsymbol{JY} \qquad \boldsymbol{Y}(y;0)=\boldsymbol{I} \tag{2}$$

where $y\in R^{n\times n}$. Here $\boldsymbol{I}$ is the n by n identity matrix and $\boldsymbol{J}$ is the n by n Jacobian matrix of $\dot{y}(t)=f(y)$ at $\boldsymbol{Y}(y;0)$. The n Lyapunov exponents λ_i are the logarithms of the eigenvalues of the matrix $\boldsymbol{\varLambda_y}$ given by

$$\boldsymbol{\varLambda_y}=\lim_{t\to\infty}\left[Y(y;t)^{\mathrm{T}}Y(y;t)\right]^{1/(2t)} \tag{3}$$

The $\boldsymbol{QR}$ decomposition $\boldsymbol{Y}$ as $\boldsymbol{Y}=\boldsymbol{QR}$, then Eq.(2) is changed as follows:

$$\dot{\boldsymbol{Q}}\boldsymbol{R}+\boldsymbol{Q}\dot{\boldsymbol{R}}=\boldsymbol{JQR},\quad \boldsymbol{Q(0)R(0)}=\boldsymbol{I_n} \tag{4}$$

Premultiplication of Eq.(4) by $\boldsymbol{Q}^{\mathrm{T}}$ and postmultiplication by $\boldsymbol{R}^{-1}$ together with $\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q}=\boldsymbol{I}$ gives:

$$\boldsymbol{Q}^{\mathrm{T}}\dot{\boldsymbol{Q}}+\dot{\boldsymbol{R}}\boldsymbol{R}^{-1}=\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{JQ},\quad \boldsymbol{Q(0)}=\boldsymbol{I_n},\quad \boldsymbol{R(0)}=\boldsymbol{I_n} \tag{5}$$

We now employ an explicit representation of the orthogonal matrix $\boldsymbol{Q}$ representing it as a product of $n(n-1)/2$ orthogonal matrices, each of which corresponds to a simple rotation on the (i,j)th plane $(i<j)$. Denoting the matrix corresponding to this rotation by $\boldsymbol{O^{(i,j)}}$, where $\boldsymbol{O}$ is an n by n orthogonal matrix, its matrix elements are given by

$$\boldsymbol{Q}_{kl}^{(ij)}(\boldsymbol{t})=\begin{cases}1 & if\ k=l\neq i,j\\ \cos\phi(t) & if\ k=l=i \text{ or } j\\ \sin\phi(t) & if\ k=i,l=j\\ -\sin\phi(t) & if\ k=j,l=i\\ 0 & \text{otherwise}\end{cases} \tag{6}$$

here ϕ denotes an angle variable. Thus the $n\times n$ matrix $\boldsymbol{Q}$ is represented by

$$\boldsymbol{Q}=O^{(12)}O^{(13)}\cdots O^{(1n)}O^{(23)}\cdots O^{(n-1,n)} \tag{7}$$

here $\boldsymbol{Q}$ is parametrized by $n(n-1)/2$ angles which are denoted by $\theta_i[i=1,2,\cdots,n(n-1)/2]$. These angles will be collectively denoted by θ.

Since the upper-triangular matrix $\boldsymbol{R}$ has positive diagonal entries, it can be represented as follows:

$$\boldsymbol{R(t)}=\begin{bmatrix}e^{\lambda_1(t)} & r_{12}(t) & \cdots & \cdots & r_{1n}(t)\\ & e^{\lambda_2(t)} & r_{23}(t) & \cdots & r_{2n}(t)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & e^{\lambda_n(t)}\end{bmatrix} \tag{8}$$

The quantities λ_i will be shown to be intimately related to the Lyapunov exponents. Our final equations will be in terms of λ_i which already appear in the exponent, thus removing the need for rescaling. The quantities λ_{ij} represent the supradiagonal terms in $\boldsymbol{R}$.

Using Eq.(7) and Eq.(8), we can get

$$\boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{t})\dot{\boldsymbol{Q}}(\boldsymbol{t})=\begin{bmatrix}0 & -f_1(\dot{\theta}(t)) & \cdots & -f_{n-1}(\dot{\theta}(t))\\ f_1(\dot{\theta}(t)) & 0 & \cdots & -f_{2n-3}(\dot{\theta}(t))\\ \vdots & \vdots & \vdots & \vdots\\ f_{n-1}(\dot{\theta}(t)) & \cdots & f_{n(n-1)/2}(\dot{\theta}(t)) & 0\end{bmatrix} \tag{9}$$

and

$$\dot{\boldsymbol{R}}(\boldsymbol{t})\boldsymbol{R}^{-1}(\boldsymbol{t})=\begin{bmatrix}\dot{\lambda}_1(t) & r'_{12}(t) & \cdots & \cdots & r'_{1n}(t)\\ 0 & \dot{\lambda}_2(t) & r'_{23}(t) & \cdots & r'_{2n}(t)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \dot{\lambda}_n(t)\end{bmatrix} \tag{10}$$

here, each of the $n(n-1)/2$ functions f_i depends (in principle) on the time derivatives $\dot{\theta}_i$ of all the angles used to represent $\boldsymbol{Q}$. In fact, they actually depend only on a subset of the angles. The quantities r'_{ij} are of no concern since they are not present in the final equations.

Substituting the above two expressions in Eq. (6), we obtain

$$\begin{bmatrix} \dot{\lambda}_1(t) & r''_{12}(t) & \cdots & \cdots & r''_{1n}(t) \\ f_1(\theta) & \dot{\lambda}_2(t) & r''_{23}(t) & \cdots & r''_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{n-1}(\theta) & \cdots & \cdots & f_{n(n-1)/2}(\theta) & \dot{\lambda}_1(t) \end{bmatrix} = \boldsymbol{Q}^{\mathrm{T}}(t)\boldsymbol{J}(t)\boldsymbol{Q}(t) \quad (11)$$

Denoting the matrix $\boldsymbol{Q}^{\mathrm{T}}(t)\boldsymbol{J}(t)\boldsymbol{Q}(t)$ by $\boldsymbol{S}$ and comparing diagonal elements on both sides of Eq. (11), we get

$$\dot{\lambda}_i(t) = \boldsymbol{S}_{ii} \quad , \quad i = 1,2,\cdots,n \quad (12)$$

Differential equations for the angles can be obtained by comparing the subdiagonal elements in Eq.(11). This gives

$$\dot{\theta}_i(t) = g_i(\theta), \quad i = 1,2,\cdots n(n-1)/2 \quad (13)$$

where the equations for θ_i are decoupled from the equations for λ_i. From Eq.(12) and Eq.(13), we can get the Lyapunov characteristic exponents:

$$\lambda_i = \lim_{t\to\infty} \frac{\lambda_i(t)}{t} \qquad i = 1,2,3,\cdots,n \quad (14)$$

If the system is 2-DOF, then $\boldsymbol{Q}$ is parametrized as follows:

$$\boldsymbol{Q(t)} = \begin{bmatrix} \cos\theta(t) & \sin\theta(t) \\ -\sin\theta(t) & \cos\theta(t) \end{bmatrix} \quad (15)$$

and upper triangular matrix $\boldsymbol{R}$ may be written as

$$\boldsymbol{R(t)} = \begin{bmatrix} e^{\lambda_1(t)} & r_{12} \\ 0 & e^{\lambda_2(t)} \end{bmatrix} \quad (16)$$

The Jacobian matrix $\boldsymbol{J}$ is parametrized as follows:

$$\boldsymbol{J} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \quad (17)$$

Substituting the above into Eq.(11), we obtain the desired equation for $\lambda_1(t)$, $\lambda_2(t)$ and $\theta(t)$

$$\begin{cases} \dfrac{d\lambda_1(t)}{dt} = J_{11}\cos^2\theta + J_{22}\sin^2\theta - \dfrac{1}{2}(J_{12}+J_{21})\sin 2\theta \\ \dfrac{d\lambda_2(t)}{dt} = J_{11}\sin^2\theta + J_{22}\cos^2\theta + \dfrac{1}{2}(J_{12}+J_{21})\sin 2\theta \\ \dfrac{d\theta(t)}{dt} = -\dfrac{1}{2}(J_{11}-J_{22})\sin 2\theta + J_{12}\sin^2\theta - J_{21}\cos^2\theta \end{cases} \quad (18)$$

The above differential equations are numerically integrated forward in time until the desired convergence for the exponents $\frac{\lambda_1}{t}$ and $\frac{\lambda_1}{t}$ are achieved.

The method to calculate Lypunov characteristic exponents without rescaling and reorthogonalization has been introduced above, now we use this method to analyse the stability of ship in regular longitudinal wave. The ship's parametric roll equation can be expressed as follows:

$$\ddot{\varphi} + 2\mu\dot{\varphi} + \mu_3\dot{\varphi}^3 + \varphi + \alpha_3\varphi^3 + \alpha_5\varphi^5 + h\cdot\cos(\Omega t)\varphi = 0 \quad (19)$$

Eq.(19) is a dimensionless equation, μ and μ_3 are linear and nonlinear damping moment coefficients, α_3 and α_5 are nonlinear restoring moment coefficients, $\Omega = \frac{\omega_e}{\omega_0}$ is the ratio of frequency of encounter to natural frequency of roll, h is the amplitude of parametric excitation.

Let $x = \varphi, y = \dot{\varphi}$ and rewrite Eq. (1) as follows:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -2\mu y - \mu_3 y^3 - x - \alpha_3 x^3 - \alpha_5 x^5 - h\cdot\cos(\Omega t)x \end{cases} \quad (20)$$

Eq.(20) is a 2-D nonautonomous system, using substituting Eq.(21):

$$y_1 = x, \quad y_2 = \dot{x}, \quad y_3 = t \quad (21)$$

Eq.(20) becomes a 3-D autonomous system:

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -2\mu y_2 - \mu_3 y_2^3 - y_1 - \alpha_3 y_1^3 - \alpha_5 y_1^5 - h\cdot\cos(\Omega t)x \\ \dot{y}_3 = 1 \end{cases} \quad (22)$$

where, $y_1(0) = y_{10}$, $y_2(0) = y_{20}$, $y_3(0) = 0$.

To calculate the LCEs of Eq.(22), 2-DOF subsystem is considered only:

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -2\mu y_2 - \mu_3 y_2^3 - y_1 - \alpha_3 y_1^3 - \alpha_5 y_1^5 - h\cdot\cos(\Omega t)x \end{cases} \quad (23)$$

Eq. (23) is a 2-DOF system, so we can get $\boldsymbol{Q(t)}$ and $\boldsymbol{R(t)}$ by Eqs.(15) and (16). Eq.(23) can be written as Eq.(24) for short:

$$\dot{\boldsymbol{Y}} = \boldsymbol{JY} \;, \quad \boldsymbol{Y}\ (y;0) = \boldsymbol{I} \quad (24)$$

where $\boldsymbol{J}$ is defined by Eq.(17)

$$\boldsymbol{J} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 - 3\alpha_3 y_1^2 - 5\alpha_5 y_1^4 & -2\mu - 3\mu_3 y_2^2 \end{pmatrix} \tag{25}$$

Substituting Eq.(25) into Eq.(18) and integrating them together with Eq.(23), the Lypunov characteristic exponents of the system (19) can be got.

3 Examples

Take an ordinary low freeboard ship as an example (Sanchez and Nayfeh, 1990), the parameters of the ship are shown in Table 1.

Fig.1 and Fig.3 show the time history of roll motion when frequency ratio Ω is equal to 1.4 and 1.75 respectively, while h is equal to 0.5. Fig.2 and Fig.4 show the time evolution of the LCEs when Ω is equal to 1.4 and 1.75 respectively.

Table 1 Parameters of model ship

ω_0	α_3	α_5	μ	μ_3
5.278	-1.402	0.271	0.086	0.108

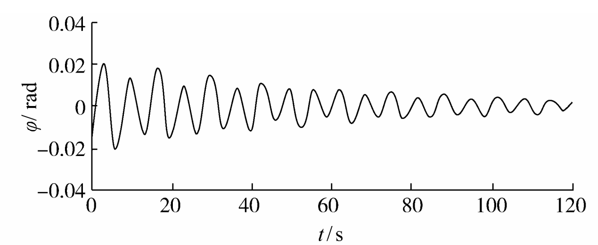


Fig.1 Time history of ship rolling ($\Omega = 1.4$)

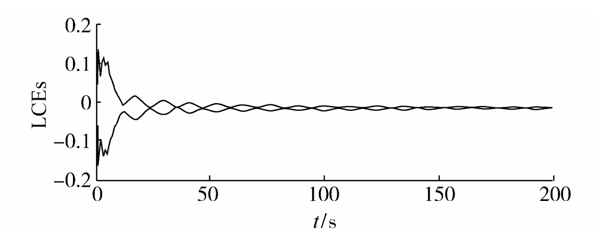


Fig.2 Time evolution of the LCEs ($\Omega = 1.4$)

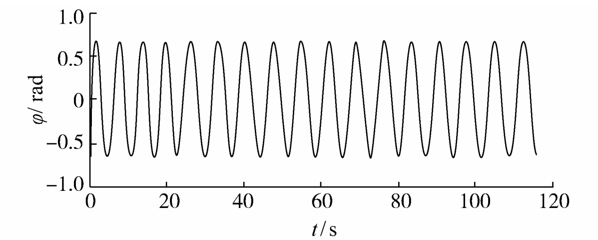


Fig.3 Time history of ship rolling ($\Omega = 1.75$)

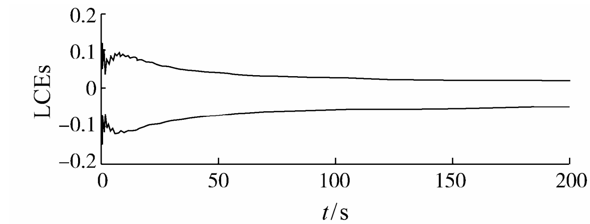


Fig.4 Time evolution of the LCEs ($\Omega = 1.75$)

Fig.1 to Fig.4 show that if the parametric resonance doesn't occur, the amplitude of ship's roll motion is small, the LCEs calculated from Eq.(12) are -0.0154 and -0.0172, all the LCEs are negative, so the ship motion is stable. If the parametric resonance occurs, the amplitude of ship's roll motion becomes large, the results of LCEs are 0.0186 and -0.0512, one of the LCEs is positive, so the ship is in chaotic motion and unsafe at that time.

When the amplitude of parametric resonance $h = 0.5$, choose $\varphi = 0.1$, $\dot{\varphi} = 0.1$ as ship's initial conditions. The frequency response curve of ship motion is shown in Fig.5.

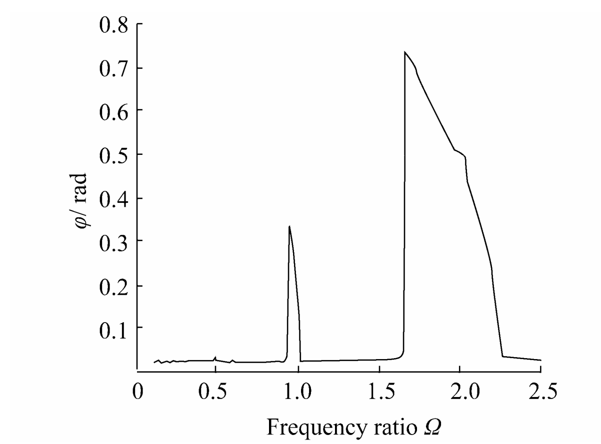


Fig.5 Response-frequency curve

Fig.5 shows that when the ship navigates on the sea the encounter frequency is close to double or single of the natural roll frequency, the parametric resonance could occur and the amplitude of ship's roll motion would increase dramatically. In order to analyze the stability of ship motion, the LCEs method is adopted which is mentioned in section 2 to calculate the system's LCEs in different frequency ratios. When all LCEs are non-positive, the ship is in the stable motion and when one of the LCEs is positive, the ship is in chaotic and unstable motion. According to that, we can find the safe and unsafe regions of the ship in longitudinal wave.

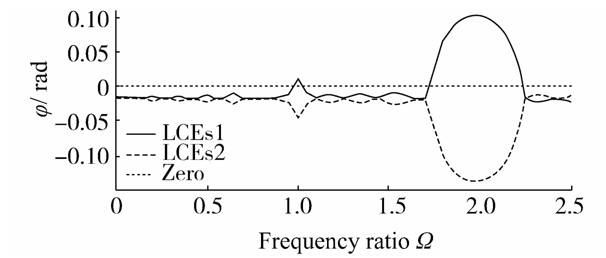


Fig.6 Safe and unsafe regions of the ship in longitudinal waves

Fig.6 indicates the safe and unsafe regions of the ship roll in longitudinal waves. The ship is unsafe in the region where the value of LCEs is larger than 0 and the ship is safe in the region where the value of LCEs is less then 0. This could be helpful to pilot ship correctly and avoid capsizing.

4 Conclusions

This work demonstrates that LCEs give indicators of the onset of parametric resonance based on simulation of a 1-DOF analytical model. LCEs method can be used for analyzing the ship's parametric resonance stability in longitudinal waves. The algorithm presented in this paper to calculate LCEs is simple and of high efficiency.

References

Belenky VL, Sevastianov NB (2003). *Stability and safety of ships, Vol. 2 Risk of capsizing*. Elseveir, Amsterdam.

Falzarano JM (1990). Predicting complicated dynamics leading to vessel capsizing. Ph.D. thesis, University of Michigan, Michigan.139-146.

France WM, Levadou M, Treakle TW, Paulling JR, Michel K, Moore C (2003). An Investigation of head-sea parametric rolling and its influence on container lashing systems. *Marine Technology*, **40**(1), 1-19.

Huang YS, Xu H, HU YC (1999). A critical situation leading to capsize of ships due to parametric resonance in astern seas. *Journal of Ship Mechanics*, **3**(5), 27-33. (in Chinese)

McCue LS (2005). Applications of finite-time Lyapunov exponents to the study of capsize in beam seas. *Proceeding of 8th International Ship Stability Workshop*, Istanbul, Turkey,1-9.

McCue LS, Troesch A (2004). Use of Lyapunov Exponents to predict chaotic vessel motions. *STAB*, Bremen, Germany, 156-171.

Murashige S, Yamada T, Aihara K (2000). Nonlinear analyses of roll motion of a flooded ship in waves. *Philosophical Transactions of the Royal Society of London*, **358**(1771), 1793-1812.

Nayfeh AH, Mook DT (1979). *Nonlinear Oscillation*. New York, John Wiley.

Papoulias FA (1987). Dynamic analysis of mooring systems. Ph.D. thesis,University of Michigan, Michigan.

Paulling JR (1961). The transverse stability of a ship in a longitudinal seaway. *Journal of Ship Research*, **4**(4), 37-49.

Paulling JR, Rosenberg RM (1959). On unstable ship motions resulting from nonlinear coupling. *Journal of Ship Research*, 3(1), 36-46.

Rangarajan G, Habib S, Ryne R (1998). Lyapunov exponents without rescaling and reorthogonalization. *Physical Review Letter*, **80**(17), 3747-3750.

Sanchez NE, Nayfeh AH (1990). Nonlinear rolling motions of ships in longitudinal waves. *International Shipbuilding Progress*. **37**(411), 247-272.

Spyrou K (1996). Homoclinic connections and period doublings of a ship advancing in quartering waves.Chaos,**6** (2).

Udwadia FE, Bremen HF (2001). An efficient and stable approach for computation of Lyapunov characteristic exponents of continuous dynamical systems. *Applied Mathematics and Computation*, **121**(2), 219-259.

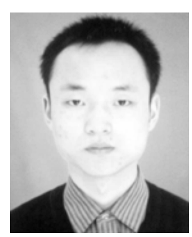
Umeda, N, Hamamoto M (2000). Capsize of ship models in following/quartering waves-physical experiments and non-linear dynamics. *Philosophical Transactions of the Royal Society of London*, **358**, 1883-1904.

Umeda N, Peters A (2002). Recent research progress on intact stability in following/quartering seas. *6th International Ship Stability Workshop,* Webb Institute, Long Island .

Wolf A, Swift JB, Swinney HL, Vastano JA (1985). Determining Lyapunov exponents from a time series. *Physica D*, **16**(3), 285–317.

Zhang B, Li Y, Lu J (2004). Study of the Lyapunov characteristic exponents used as criteria for chaos. *Journal of Jilin University*, **22**(2), 111-114. (in Chinese)

Zhang YF, Dong YQ (1998). The effect of fundamental parametric excitation on the stability of ships in longitudinal seas. *Journal of Ship Mechanics*, **2**(3), 6-12. (in Chinese)



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