Simple Lie algebras arising from Leavitt path algebras

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Abstract

For a field K and directed graph E, we analyze those elements of the Leavitt path algebra $L_K(E)$ which lie in the commutator subspace $[L_K(E), L_K(E)]$. This analysis allows us to give easily computable necessary and sufficient conditions to determine which Lie algebras of the form $[L_K(E), L_K(E)]$ are simple, when E is row-finite and $L_K(E)$ is simple.

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Within the past few years, the Leavitt path algebra $L_K(E)$ of a graph E with coefficients in the field K has received much attention throughout both the algebra and analysis communities. As it turns out, quite often the algebraic properties of $L_K(E)$ (for example: simplicity, chain conditions, primeness, primitivity, stable rank) depend solely on the structure of the graph E, and not at all on the structure of the field K (to wit, neither on the cardinality of K, nor on the characteristic of K).

With each associative K-algebra R one may construct the *Lie K-algebra* (or *commutator K-algebra*) $[R, R]$ *of* R, consisting of all K-linear combinations of elements of the form $xy - yx$ where $x, y \in R$. Then $[R, R]$ becomes a (non-associative) Lie algebra under the operation $[x, y] = xy - yx$ for $x, y \in R$. In particular, when $R = L_K(E)$, one may construct and subsequently investigate the Lie algebra $[L_K(E), L_K(E)]$. Such an analysis was carried out in [1] in the case where E is a graph having one vertex and $n \geq 2$ loops. In [1, Theorem 3.4] necessary and sufficient conditions on n and the characteristic of K are given which determine the simplicity of the Lie algebra $[L_K(E), L_K(E)]$ in this situation. In light of the comments made above, it is of interest to note that the characteristic of K does indeed play a role in this result.

There are two main contributions made in the current article. First, we analyze various elements of $L_K(E)$ which lie in the subspace $[L_K(E), L_K(E)]$, and in particular give in Theorem 14 necessary and sufficient conditions for when an arbitrary linear combination of vertices of E (for instance, $1_{L_K(E)}$) is such. Second, we extend [1, Theorem 3.4] to all simple Leavitt path algebras arising from row-finite graphs by giving, in Corollary 21 and Theorem 23, necessary and sufficient conditions on E and K which determine the simplicity of the Lie K-algebra $[L_K(E), L_K(E)]$.

In addition, we achieve a number of supporting results which are of independent interest. In Proposition 6 we give necessary and sufficient conditions which determine when a matrix ring over a simple unital algebra has a simple associated Lie algebra. In Example 30 we present, for each prime p, an infinite class of nonisomorphic simple Leavitt path algebras whose associated Lie algebras are simple. Moreover, these Leavitt path algebras are not isomorphic to the examples presented in [1], showing that the current investigation does indeed extend previously known results. In Theorem 36 we recast Theorem 23 in the context of K-theory. As a result, we observe in Proposition 39 that for two purely infinite simple Leavitt path algebras whose Grothendieck groups correspond appropriately, the Lie algebras associated to these two algebras are either both simple or both non-simple.

1 Lie rings of associative rings

Throughout, the letters R and S will denote associative (but not necessarily unital) rings, and K will denote a field. The center of the ring R will be denoted by $Z(R)$. Given a ring R and two elements $x, y \in R$, we let $[x, y]$ denote the commutator $xy - yx$, and let $[R, R]$ denote the additive subgroup of R generated by the commutators. Then $[R, R]$ is a (non-associative) Lie ring, with operation $x * y = [x, y] = xy - yx$, which we call the *Lie ring associated to* R. If R is in addition an algebra over a field K, then $[R, R]$ is a K-subspace of R (since $k[x, y] = [kx, y]$), and in this way becomes a (non-associative) Lie K-algebra, which we call the *Lie K-algebra associated to* R. Clearly $[R, R] = \{0\}$ if and only if R is commutative.

For a $d \times d$ matrix $A \in \mathbb{M}_d(R)$, trace(A) denotes as usual the sum of the diagonal entries of A. We will utilize the following fact about traces.

Proposition 1 (Corollary 17 from [2])**.** *Let R be a unital ring, d a positive integer, and $A \in \mathbb{M}_d(R)$. Then $A \in [\mathbb{M}_d(R), \mathbb{M}_d(R)]$ if and only if* trace$(A) \in [R, R]$. (*In particular, any $A \in \mathbb{M}_d(R)$ of trace zero is necessarily in* $[\mathbb{M}_d(R), \mathbb{M}_d(R)]$.)

Let L denote a Lie ring (respectively, Lie K-algebra). A subset I of L is called a *Lie ideal* if I is an additive subgroup (respectively, K-subspace) of L and $[L, I] \subseteq I$. The Lie ring (respectively, Lie K-algebra) L is called *simple* if $[L, L] \neq 0$ and the only Lie ideals of L are 0 and L.

While the following fact is well known, for completeness we include a proof, since we were unable to find one in the literature.

Lemma 2 (see page 34 of [3])**.** *Let K be a field and L a Lie K-algebra. Then L is simple as a Lie ring if and only if L is simple as Lie K-algebra.*

Proof. We only show that simplicity as a Lie K-algebra implies simplicity as a Lie ring, since the other direction is trivial. So suppose that I is a nonzero ideal of L, in the Lie ring sense (i.e., we do not assume that I is a K-subspace of L). We seek to show that $I = L$. Since $[L, I] \subseteq I$, it is easy to see that the additive subgroup $[L, I]$ of L is a Lie ideal (in the Lie ring sense) of L. Since L is simple, the center of L is zero, which yields that $[L, I] \neq \{0\}$. But for any $k \in K$, $i \in I$, and $\ell \in L$ we have $k[\ell, i] = [k\ell, i] \in [L, I]$, showing that $[L, I]$ is a K-subspace of L. By the simplicity of L as a Lie algebra, this gives $[L, I] = L$, and since $[L, I] \subseteq I$, we have $I = L$, as desired. $\square$

As a consequence of Lemma 2, throughout the article we will often use the concise phrase "L is simple" to indicate that the Lie K-algebra L is simple either as a Lie ring or as a Lie K-algebra. The following result of Herstein will play a pivotal role in our analysis.

Theorem 3 (Theorem 1.13 from [4])**.** *Let S be a simple ring. Assume either that* $\mathrm{char}(S) \neq 2$*, or that S is not 4-dimensional over $Z(S)$, where $Z(S)$ is a field. Then $U \subseteq Z(S)$ for any proper Lie ideal U of the Lie ring $[S, S]$.*

Corollary 4. *Let R be a simple ring, d a positive integer, and $S = \mathbb{M}_d(R)$. If $Z(R) = 0$, then either the Lie ring $[S, S]$ is simple, or $[[S, S], [S, S]] = 0$.*

Proof. If R is a simple ring, then so is S. The result now follows from Theorem 3 upon noting that if $0 = Z(R) = Z(S)$ (where we identify R with its diagonal embedding in S), then S cannot be 4-dimensional over $Z(S)$. $\square$

Lemma 5. *Let R be a ring, $d \geq 2$ an integer, and $S = \mathbb{M}_d(R)$. If $Z(R) \cap [R, R] \neq 0$, then the Lie ring $[S, S]$ is not simple.*

Proof. Let $a \in Z(R) \cap [R, R]$ be any nonzero element, and let $A \in S$ be the matrix $\mathrm{diag}(a)$ (having a as each entry on the main diagonal and zeros elsewhere). Write $a = \sum_{i=1}^{n} [b_i, c_i]$ for some $b_i, c_i \in R$, and set $B_i = \mathrm{diag}(b_i)$ and $C_i = \mathrm{diag}(c_i)$. Then $A = \sum_{i=1}^{n} [B_i, C_i]$ is a nonzero element of $[S, S]$. Since $A \in Z(S)$, the additive subgroup generated by A is a nonzero Lie ideal of $[S, S]$. This Lie ideal is proper, since it consists of diagonal matrices, while by Proposition 1, $[S, S]$ contains all matrices having trace zero, and since $d \geq 2$, some such matrices must be non-diagonal. Hence $[S, S]$ is not simple. $\square$

Proposition 6. *Let be R a simple unital ring, $d \geq 2$ an integer, and $S = \mathbb{M}_d(R)$. Then the Lie ring $[S, S]$ is simple if and only if the following conditions hold:*

(1) $1 \notin [R, R]$,

(2) $\mathrm{char}(R)$ *does not divide d.*

Proof. Suppose that $[S, S]$ is simple as a Lie ring. By Lemma 5, we have $Z(R) \cap [R, R] = 0$, and hence (1) holds. Now, suppose that $\mathrm{char}(R)$ divides d. Then I (the identity) is a nonzero matrix in $Z(S)$ with $\mathrm{trace}(I) = 0$. By Proposition 1, $I \in [S, S]$, and hence the additive subgroup generated by I is a nonzero Lie ideal of $[S, S]$, which is proper (as in the proof of Lemma 5, this ideal consists of diagonal matrices, whereas $[S, S]$ does not), contradicting the simplicity of $[S, S]$. Thus, if $[S, S]$ is simple, then (1) and (2) must hold.

For the converse, suppose that (1) and (2) hold. It is well-known that $Z(R)$ is a field for any simple unital ring R. We first note that it could not be the case that $2 = \mathrm{char}(S)$ ($= \mathrm{char}(R)$), $Z(S)$ is a field, and S has dimension 4 over $Z(S)$. For in that case, since $d \geq 2$, we necessarily have $d = 2$ and $R = Z(R) = Z(S)$ (where R is identified with its diagonal embedding in S). But, this would violate (2). Thus, by Theorem 3, given a proper Lie ideal $U \subseteq [S, S]$, we have $U \subseteq Z(S) = Z(R)$. Now, let $A \in U$ be any matrix. Since, $A \in Z(S)$, we have $A = \mathrm{diag}(a)$ for some $a \in Z(R)$. By Proposition 1, $\mathrm{trace}(A) = da \in [R, R] \cap Z(R)$, which, by (2), implies that $a \in [R, R] \cap Z(R)$ (since d is a nonzero element of the field $Z(R)$). By (1), this can only happen if $a = 0$. Hence $A = 0$, and therefore also $U = 0$, showing that $[S, S]$ contains no nontrivial ideals. It remains only to show that $[[S, S], [S, S]] \neq 0$. But, by Proposition 1, the matrix units e_{12} and e_{21} are elements of $[S, S]$, and hence $0 \neq e_{11} - e_{22} = [e_{12}, e_{21}] \in [[S, S], [S, S]]$. $\square$

2 Commutators in Leavitt path algebras

We now take up the first of our two main goals: to describe various elements of a Leavitt path algebra $L_K(E)$ which may be written as sums of commutators. The main result of this section is Theorem 14, where we give (among other things) necessary and sufficient conditions for the specific element $1_{L_K(E)}$ to be so written.

We start by defining the relevant algebraic and graph-theoretic structures. A *directed graph* $E = (E^0, E^1, r, s)$ consists of two sets E^0, E^1 and functions $r, s : E^1 \to E^0$. The word *graph* will always mean *directed graph*. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. The sets E^0 and E^1 are allowed to be of arbitrary cardinality. A *path* μ in E is a finite sequence of edges $\mu = e_1 \dots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$; in this case, $s(\mu) := s(e_1)$ is the *source* of μ, $r(\mu) := r(e_n)$ is the *range* of μ, and n is the *length* of μ. We view the elements of E^0 as paths of length 0. We denote by $\mathrm{Path}(E)$ the set of all paths in E (including paths of length 0). If $\mu = e_1 \dots e_n$ is a path in E, and if $v = s(\mu) = r(\mu)$ and $s(e_i) \neq s(e_j)$ for every $i \neq j$, then μ is called a *cycle based at* v. A cycle consisting of one edge is called a *loop*. A graph which contains no cycles is called *acyclic*; a graph for which $|s^{-1}(v)|$ is finite for all $v \in E^0$ is called *row-finite*; a graph for which both E^0 and E^1 are finite sets is called a *finite* graph. A vertex v for which $|s^{-1}(v)| = 0$ is called a *sink*, while a vertex v for which $1 \leq |s^{-1}(v)| < \infty$ is called a *regular* vertex. An edge e is an *exit* for a path $\mu = e_1 \dots e_n$ if there exists i $(1 \leq i \leq n)$ such that $s(e) = s(e_i)$ and $e \neq e_i$. We say that a vertex v *connects* to a vertex w in case there is a path $p \in \mathrm{Path}(E)$ for which $s(p) = v$ and $r(p) = w$.

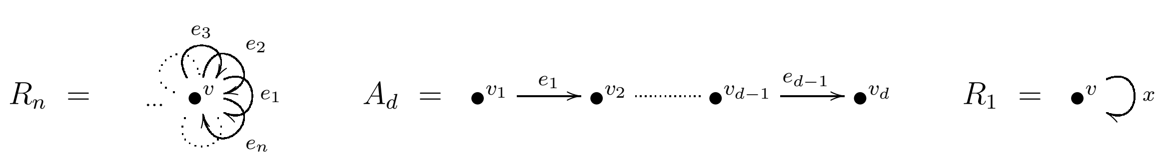
Of central focus in this article are Leavitt path algebras.

Definition 7. Let K be a field, and let E be a graph. The *Leavitt path K-algebra $L_K(E)$ of E with coefficients in K* is the K-algebra generated by a set $\{v \mid v \in E^0\}$, together with a set of variables $\{e, e^* \mid e \in E^1\}$, which satisfy the following relations:

(V) $vw = \delta_{v,w} v$ for all $v, w \in E^0$ (i.e., $\{v \mid v \in E^0\}$ is a set of orthogonal idempotents),
(E1) $s(e)e = er(e) = e$ for all $e \in E^1$,
(E2) $r(e)e^* = e^* s(e) = e^*$ for all $e \in E^1$,
(CK1) $e^* e' = \delta_{e,e'} r(e)$ for all $e, e' \in E^1$,
(CK2) $v = \sum_{\{e \in E^1 | s(e) = v\}} ee^*$ for every regular vertex $v \in E^0$. $\square$

We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. If $\mu = e_1 \dots e_n \in \mathrm{Path}(E)$, then we denote by μ^* the element $e_n^* \dots e_1^*$ of $L_K(E)$. An expression of this form is called a *ghost path*.

Many well-known algebras arise as the Leavitt path algebra of a graph. For example, the classical Leavitt K-algebra $L_K(n)$ for $n \geq 2$; the full $d \times d$ matrix algebra $\mathbb{M}_d(K)$ over K; and the Laurent polynomial algebra $K[x, x^{-1}]$ arise, respectively, as the Leavitt path algebra of the "rose with n petals" graph R_n $(n \geq 2)$; the oriented line graph A_d having d vertices; and the "one vertex, one loop" graph R_1 pictured here.



Along the way we will utilize the Cohn path algebra of a graph, defined here.

Definition 8. Let K be a field, and let E be a graph. The *Cohn path K-algebra $C_K(E)$ of E with coefficients in K* is the K-algebra generated by a set $\{v \mid v \in E^0\}$, together with a set of variables $\{e, e^* \mid e \in E^1\}$, which satisfy the relations (V), (E1), (E2), and (CK1) of Definition 7.

We let $N \subseteq C_K(E)$ denote the ideal of $C_K(E)$ generated by elements of the form $v - \sum_{\{e \in E^1 : s(e)=v\}} ee^*$, where $v \in E^0$ is a regular vertex. □

In particular, we may view the Leavitt path algebra $L_K(E)$ as the quotient algebra

$$L_K(E) \cong C_K(E)/N.$$

If E is a graph for which E^0 is finite, then $\sum_{v \in E^0} v$ is the multiplicative identity, viewed either as an element of $L_K(E)$ or $C_K(E)$. If E^0 is infinite, then both $L_K(E)$ and $C_K(E)$ are nonunital. Identifying v with v^* for each $v \in E^0$, one can show that

$$\{pq^* \mid p, q \in \mathrm{Path}(E) \text{ such that } r(p) = r(q)\}$$

is a basis for $C_K(E)$.

Lemma 9. *Let K be a field, and let E be a graph. Let $y = v - \sum_{\{e \in E^1 : s(e)=v\}} ee^* \in N \subseteq C_K(E)$, where $v \in E^0$ is a regular vertex.*

(1) *If $p \in \mathrm{Path}(E) \setminus E^0$, then $yp = 0$.*

(2) *If $q \in \mathrm{Path}(E) \setminus E^0$, then $q^*y = 0$.*

Proof. (1) Write $p = fp'$ for some $f \in E^1$ and $p' \in \mathrm{Path}(E)$. If $s(f) \neq v$ then $yp = 0$ immediately. On the other hand, if $s(f) = v$ then $f \in \{e \in E^1 : s(e) = v\}$, in which case, by (CK1), we get

$$yp = (v - \sum_{\{e \in E^1 : s(e)=v\}} ee^*)fp' = fp' - ff^*fp' = fp' - fp' = 0.$$

The proof of (2) is similar. □

Definition 10. Let K be a field, let E be a graph, and write $E^0 = \{v_i \mid i \in I\}$. Let $K^{(I)}$ denote the direct sum of copies of K indexed by I. For each $i \in I$, let $\epsilon_i \in K^{(I)}$ denote the element with $1 \in K$ as the i-th coordinate and zeros elsewhere. Let $T : C_K(E) \to K^{(I)}$ be the K-linear map which acts as

$$T(pq^*) = \begin{cases} \epsilon_i & \text{if } q^*p = v_i \\ 0 & \text{otherwise} \end{cases}$$

on the aforementioned basis of $C_K(E)$. □

We note that $T(v_i) = \epsilon_i$ for all $i \in I$, and for any $p \in \mathrm{Path}(E) \setminus E^0$, $T(p) = 0 = T(p^*)$.

Lemma 11. *Let K be a field, let E be graph, and write $E^0 = \{v_i \mid i \in I\}$. Let T denote the K-linear transformation given in Definition 10. Then for all $x, y \in C_K(E)$ we have $T(xy) = T(yx)$. In particular, $T(z) = 0$ for every $z \in [C_K(E), C_K(E)]$.*

Proof. Since T is K-linear, it is enough to establish the result for x and y that are elements of the basis for $C_K(E)$ described above. That is, we may assume that $x = pq^*$ and $y = tz^*$, for some $p, q, t, z \in \text{Path}(E)$ with $r(p) = r(q) = v_i \in E^0$ and $r(t) = r(z) = v_j \in E^0$. Now, $pq^*tz^* = 0$ unless either $t = qh$ or $q = th$ for some $h \in \text{Path}(E)$. Also, $tz^*pq^* = 0$ unless either $p = zg$ or $z = pg$ for some $g \in \text{Path}(E)$. Let us consider the various resulting cases separately.

Suppose that $t = qh$ for some $h \in \text{Path}(E)$ but $z \neq pg$ for all $g \in \text{Path}(E)$. Then $pq^*tz^* = pq^*qhz^* = phz^*$ and $T(pq^*tz^*) = T(phz^*) = 0$, since $z \neq ph$. Also, as mentioned above, $tz^*pq^* = 0$ unless $p = zg$ for some $g \in \text{Path}(E)$. If $tz^*pq^* = 0$, then we have $T(tz^*pq^*) = 0 = T(pq^*tz^*)$. Therefore, let us suppose that $p = zg$ for some $g \in \text{Path}(E)$. Then $tz^*pq^* = tz^*zgq^* = tgq^* = qhgq^*$, and hence $T(tz^*pq^*) = T(qhgq^*) = 0$ unless $hg \in E^0$. But, $hg \in E^0$ can happen only if $h = g \in E^0$, in which case $p = z$ (since $p \neq 0$), contradicting our assumption. Therefore, $p \neq zg$ for all $g \in \text{Path}(E)$, and we have $T(pq^*tz^*) = 0 = T(tz^*pq^*)$.

Let us next suppose that $t = qh$ and $z = pg$ for some $g, h \in \text{Path}(E)$. Then $pq^*tz^* = pq^*qhg^*p^* = phg^*p^*$, and hence $T(pq^*tz^*) = \epsilon_j$ if $g = h$ and 0 otherwise. Also, $tz^*pq^* = qhg^*p^*pq^* = qhg^*q^*$, and so $T(tz^*pq^*) = \epsilon_j$ if $g = h$ and 0 otherwise. Thus, in either case we have $T(pq^*tz^*) = T(tz^*pq^*)$.

Now suppose that $t \neq qh$ for all $h \in \text{Path}(E)$ but $z = pg$ for some $g \in \text{Path}(E)$. Then $pq^*tz^* = pq^*tg^*p^* \neq 0$ only if $q = th$ for some $h \in \text{Path}(E)$. Hence $T(pq^*tz^*) \neq 0$ only if $z = p$ and $q = t$, which is not the case, by hypothesis. Similarly, $tz^*pq^* = tg^*p^*pq^* = tg^*q^*$, and hence $T(tz^*pq^*) \neq 0$ only if $t = qg$, which is not the case. Thus, $T(tz^*pq^*) = 0 = T(pq^*tz^*)$.

Finally, suppose that $t \neq qh$ and $z \neq pg$ for all $g, h \in \text{Path}(E)$. Then $pq^*tz^* = 0$ unless $q = th$ for some $h \in \text{Path}(E)$, and $T(pq^*tz^*) = 0$ unless $q = th$ and $p = zh$ for some $h \in \text{Path}(E)$. Similarly, $T(tz^*pq^*) = 0$ unless $q = th$ and $p = zh$ for some $h \in \text{Path}(E)$. Thus, let us suppose that $q = th$ and $p = zh$ for some $h \in \text{Path}(E)$. In this case,

$$T(pq^*tz^*) = T(zhh^*t^*tz^*) = T(zhh^*z^*) = v_i = T(thh^*t^*) = T(tz^*zhh^*t^*) = T(tz^*pq^*),$$

as desired.

Therefore, in all cases $T(pq^*tz^*) = T(tz^*pq^*)$, proving the first claim of the lemma. The second follows trivially. $\square$

Definition 12. Let E be a graph, and write $E^0 = \{v_i \mid i \in I\}$. If v_i is a regular vertex, for all $j \in I$ let a_{ij} denote the number of edges $e \in E^1$ such that $s(e) = v_i$ and $r(e) = v_j$. In this situation, define

$$B_i = (a_{ij})_{j \in I} - \epsilon_i \in \mathbb{Z}^{(I)}.$$

On the other hand, let

$$B_i = (0)_{j \in I} \in \mathbb{Z}^{(I)},$$

if v_i is not a regular vertex. $\square$

Lemma 13. *Let K be a field, let E be a graph, and write $E^0 = \{v_i \mid i \in I\}$. Then for all $w \in N \subseteq C_K(E)$ we have $T(w) \in \text{span}_K\{B_i \mid i \in I\} \subseteq K^{(I)}$.*

Proof. It is sufficient to show that for any generator

$$y_i = v_i - \sum_{\{e\in E^1 : s(e)=v_i\}} ee^*$$

of N and any two elements c, c' of $C_K(E)$, we have $T(cy_ic') \in \mathrm{span}_K\{B_i \mid i \in I\} \subseteq K^{(I)}$. But, by Lemma 11, $T(cy_ic') = T(c'cy_i)$, and hence we only need to show that $T(cy_i) \in \mathrm{span}_K\{B_i \mid i \in I\}$ for any $c \in C_K(E)$. Further, since T is K-linear, we may assume that $c = pq^*$ belongs to the basis for $C_K(E)$ described above; in particular, $p, q \in \mathrm{Path}(E)$. Again using Lemma 11, we have $T(cy_i) = T(pq^*y_i) = T(q^*y_ip)$. But, by Lemma 9, the expression q^*y_ip is zero unless $q^* = v_i = p$. So the only nonzero term of the form $T(cy_i)$ is

$$T(cy_i) = T(y_i) = T(v_i - \sum_{\{e\in E^1 : s(e)=v_i\}} ee^*) = \epsilon_i - (a_{ij})_{j\in I} = -B_i,$$

since for each $e \in E^1$ with $s(e) = v_i$ and $r(e) = v_j$, we have $T(ee^*) = \epsilon_j$. Clearly $-B_i \in \mathrm{span}_K\{B_i \mid i \in I\}$, and we are done. □

Here now is our first goal, achieved.

Theorem 14. *Let K be a field, let E be a graph, and write $E^0 = \{v_i \mid i \in I\}$. For each $i \in I$ let B_i denote the element of $K^{(I)}$ given in Definition 12, and let $\{k_i \mid i \in I\} \subseteq K$ be a set of scalars where $k_i = 0$ for all but finitely many $i \in I$. Then*

$$\sum_{i\in I} k_iv_i \in [L_K(E), L_K(E)] \quad \text{if and only if} \quad (k_i)_{i\in I} \in \mathrm{span}_K\{B_i \mid i \in I\}.$$

In particular, if E^0 is finite (so that $L_K(E)$ is unital), then

$$1_{L_K(E)} \in [L_K(E), L_K(E)] \quad \text{if and only if} \quad (1, \dots, 1) \in \mathrm{span}_K\{B_i \mid i \in I\} \subseteq K^{(I)}.$$

Proof. First, suppose that $(k_i)_{i\in I} \in \mathrm{span}_K\{B_i \mid i \in I\}$. For all $i, j \in I$ such that v_i is regular, let $e_1^{ij}, \dots, e_{a_{ij}}^{ij}$ be all the edges $e \in E^1$ satisfying $s(e) = v_i$ and $r(e) = v_j$. (We note that there are only finitely many such elements.) Then for each regular v_i we have

$$\sum_{j\in I}\sum_{l=1}^{a_{ij}} [e_l^{ij}, (e_l^{ij})^*] = \sum_{j\in I}\sum_{l=1}^{a_{ij}} e_l^{ij}(e_l^{ij})^* - \sum_{j\in I}\sum_{l=1}^{a_{ij}} (e_l^{ij})^*e_l^{ij}$$

$$= \sum_{\{e\in E^1 : s(e)=v_i\}} ee^* - \sum_{j\in I}\sum_{l=1}^{a_{ij}} (e_l^{ij})^*e_l^{ij} = v_i - \sum_{j\in I} a_{ij}v_j.$$

By hypothesis we can write $(k_i)_{i\in I} = \sum_{i\in I} t_iB_i$ for some $t_i \in K$ (all but finitely many of which are 0). We may assume that $t_i = 0$ whenever v_i is not regular, since in that case B_i is zero. Thus,

$$\sum_{i\in I} k_iv_i = -\sum_{i\in I} t_i(v_i - \sum_{j\in I} a_{ij}v_j),$$

which is an element of $[L_K(E), L_K(E)]$, by the above computation.

For the converse, viewing $L_K(E)$ as $C_K(E)/N$, we shall show that if $\sum_{i\in I} k_i v_i + N \in [L_K(E), L_K(E)]$ for $v_i \in E^0$ and $k_i \in K$ satisfying the hypotheses in the statement, then $(k_i)_{i\in I} \in \mathrm{span}_K\{B_i \mid i \in I\}$. Now, if $\sum_{i\in I} k_i v_i + N \in [L_K(E), L_K(E)]$, then there are elements $x_j, y_j \in C_K(E)$ such that $\sum_{i\in I} k_i v_i = \sum_j [x_j, y_j] + w$ for some $w \in N$. Using Lemma 11 we get

$$(k_i)_{i\in I} = T(\sum_{i\in I} k_i v_i) = T(\sum_j [x_j, y_j]) + T(w) = 0 + T(w) = T(w).$$

Lemma 13 then gives the desired result.

To prove the final claim, write $E^0 = \{v_1, \dots, v_m\}$ and use the previously noted fact that $1_{L_K(E)} = v_1 + \cdots + v_m$. ◻

We conclude this section by identifying additional elements of $[L_K(E), L_K(E)]$.

Lemma 15. *Let K be a field, E a graph, and $p, q \in \mathrm{Path}(E) \setminus E^0$ any paths.*

(1) *If $s(p) \neq r(p)$, then $p, p^* \in [L_K(E), L_K(E)]$.*

(2) *If $p \neq qx$ and $q \neq px$ for all $x \in \mathrm{Path}(E)$ with $s(x) = r(x)$, then $pq^* \in [L_K(E), L_K(E)]$.*

(3) *We have $pp^* \in [L_K(E), L_K(E)]$ if and only if $r(p) \in [L_K(E), L_K(E)]$.*

Proof. (1) If $s(p) \neq r(p)$, then $r(p)p = 0 = p^* r(p)$, and hence $p = [p, r(p)]$ and $p^* = [r(p), p^*]$.

(2) We have $[p, q^*] = pq^* - q^*p$. If $p \neq qx$ and $q \neq px$ for all $x \in \mathrm{Path}(E)$, then $q^*p = 0$, and hence $pq^* \in [L_K(E), L_K(E)]$. Let us therefore suppose that either $p = qx$ or $q = px$ for some $x \in \mathrm{Path}(E)$ such that $s(x) \neq r(x)$. Thus $[p, q^*] = pq^* - x$ in the first case, and $[p, q^*] = pq^* - x^*$ in the second. In either situation, (1) implies that $pq^* \in [L_K(E), L_K(E)]$.

(3) We note that $[p, p^*] = pp^* - p^*p = pp^* - r(p)$, from which the desired conclusion follows. ◻

Question 16. *Do there exist a graph E, a field K, and a cycle $p \in \mathrm{Path}(E)$ for which $p \in [L_K(E), L_K(E)]$?*

3 Simple Leavitt path algebras and associated Lie algebras

In this section we apply the results proved in Section 2 together with Herstein's result (Theorem 3) in order to achieve our second main goal, namely, to identify the fields K and row-finite graphs E for which the simple Leavitt path algebra $L_K(E)$ yields a simple Lie algebra $[L_K(E), L_K(E)]$. We begin by recording two basic facts about Leavitt path algebras and recalling two previously known results.

Lemma 17. (1) *There is up to isomorphism exactly one simple commutative Leavitt path algebra, specifically the algebra $K \cong L_K(\bullet)$.*

(2) *The only K-division algebra of the form $L_K(E)$ for some graph E is $K \cong L_K(\bullet)$.*

Proof. (1) We establish the contrapositive: we show that a simple Leavitt path algebra $L_K(E)$ corresponding to any graph E other than $\bullet$ is not commutative, by showing that

$[L_K(E), L_K(E)] \neq \{0\}$. If E were to contain no edges then E would consist of (at least two) isolated vertices, and thus would not be simple. So we may assume E contains at least one edge. If E contains an edge e for which $s(e) \neq r(e)$ then by Lemma 15(1) $0 \neq e \in [L_K(E), L_K(E)]$. On the other hand, if E contains no such edges, then all edges in E are loops. In this situation, by the simplicity of $L_K(E)$, there can only be one vertex v in E. If there are at least two distinct loops p, q based at v then by Lemma 15(2) $0 \neq pq^* \in [L_K(E), L_K(E)]$. In case there is only one loop p at v, we have $L_K(E) \cong K[x, x^{-1}]$, which is not simple, completing the proof.

(2) Since a division algebra has no zero divisors, in order for $L_K(E)$ to be such a ring, the graph E must have exactly one vertex and at most one loop at that vertex. But as noted previously, the Leavitt path algebra of the graph with one vertex and one loop is isomorphic to $K[x, x^{-1}]$, and thus is not a division ring. The result follows. □

We call a simple Leavitt path algebra $L_K(E)$ *nontrivial* in case $L_K(E) \ncong K$.

Lemma 18. *Let K be a field, E a graph, and $R = L_K(E)$ a Leavitt path algebra. If $[R, R] \neq 0$, then $[[R, R], [R, R]] \neq 0$. In particular, if R is a nontrivial simple Leavitt path algebra, then $[[R, R], [R, R]] \neq 0$.*

Proof. First, suppose there is an edge $e \in E^1$ that is not a loop. Then $r(e) \neq s(e)$, implying that $e^* r(e) = 0$ and $r(e)e = 0$. Thus

$$[[r(e), e^*], [e, r(e)]] = [e^*, e] = r(e) - ee^* \in [[R, R], [R, R]]$$

is nonzero, since $(r(e) - ee^*)r(e) = r(e) \neq 0$. Next, suppose that v is a vertex at which two distinct loops e and f are based. Then

$$[[e, e^*], [e, f]] = [ee^* - v, ef - fe] = ef - efee^* + fe^2e^* \in [[R, R], [R, R]]$$

is nonzero, since multiplying this element on the left by f^* and on the right by e yields the nonzero element e^2. Thus the only remaining configuration for E not covered by these two cases is that E is a disjoint union of isolated vertices together with vertices at which there is exactly one loop. But in this case $L_K(E)$ is a direct sum of copies of K with copies of $K[x, x^{-1}]$, so is commutative, and hence $[R, R] = 0$.

The second statement follows immediately from Lemma 17(1). □

We note that the first statement of Lemma 18 does not hold for an arbitrary ring R. For instance, let R be the associative (unital or otherwise) ring generated by the following generators and relations

$$\langle x, y : x^3 = y^3 = xy^2 = yx^2 = x^2y = y^2x = xyx = yxy = 0 \rangle.$$

Then $[x, y] \neq 0$, and hence $[R, R] \neq 0$. But, all the nonzero commutators in R are integer multiples of $[x, y]$, and hence $[[R, R], [R, R]] = 0$.

A description of the row-finite graphs E and fields K for which $L_K(E)$ is simple is given in [5, Theorem 3.11]. Using [6, Lemma 2.8] to streamline the statement of this result, we have

Theorem 19 (The Simplicity Theorem)**.** *Let K be a field and E a row-finite graph. Then $L_K(E)$ is simple if and only if E has the following two properties.*

(1) *Every vertex v of E connects to every sink and every cycle of E.*

(2) *Every cycle of E has an exit.*

Specifically, we note that the simplicity of the algebra $L_K(E)$ is independent of K. (This is intriguing, especially in light of the fact that we will show below that the simplicity of the corresponding Lie algebra $[L_K(E), L_K(E)]$ does indeed depend on K.) The following is due to Aranda Pino and Crow.

Theorem 20 (Theorem 4.2 from [7])**.** *Let K be a field, and let E be a row-finite graph for which $L_K(E)$ is a simple Leavitt path algebra.*

(1) *If $L_K(E)$ is unital, then $Z(L_K(E)) = K$.*

(2) *If $L_K(E)$ is not unital, then $Z(L_K(E)) = 0$.*

This result immediately allows us to identify simple Lie algebras arising from graphs having infinitely many vertices.

Corollary 21. *Let K be a field, and let E be a row-finite graph, having infinitely many vertices, for which $L_K(E)$ is a simple Leavitt path algebra. Then $[L_K(E), L_K(E)]$ is a simple Lie K-algebra.*

Proof. This follows by combining Theorem 20(2) with Corollary 4 and Lemma 18, since if E has infinitely many vertices, then $L_K(E)$ is not unital. □

On the other hand, we get the following result for graphs having finitely many vertices.

Corollary 22. *Let K be a field, and let E be a finite graph for which $L_K(E)$ is a nontrivial simple Leavitt path algebra. Then the Lie K-algebra $[L_K(E), L_K(E)]$ is simple if and only if $1 = 1_{L_K(E)} \notin [L_K(E), L_K(E)]$.*

Proof. If $1 \in [L_K(E), L_K(E)]$, then the K-subspace $\langle 1 \rangle$ it generates is a nonzero Lie ideal of $[L_K(E), L_K(E)]$. Since $\langle 1 \rangle$ is a commutative subalgebra of $L_K(E)$, by Lemma 18 we have that $\langle 1 \rangle$ is proper. Thus, $[L_K(E), L_K(E)]$ is not simple.

Conversely, if $1 \notin [L_K(E), L_K(E)]$, then we have $Z(L_K(E)) \cap [L_K(E), L_K(E)] = 0$, by Theorem 20(1). Since $L_K(E)$ is nontrivial simple, $[[L_K(E), L_K(E)], [L_K(E), L_K(E)]] \neq 0$, by Lemma 18. Further, it cannot be the case that $\mathrm{char}(K) = 2$ and $L_K(E)$ is 4-dimensional over $Z(L_K(E)) = K$, for then we would have $L_K(E) \cong M_2(K)$ (it is well-known that a 4-dimensional central simple K-algebra that is not a division ring must be of this form, and $L_K(E)$ is not a division ring by Lemma 17(2)). But, if $\mathrm{char}(K) = 2$, then $1 \in [M_2(K), M_2(K)]$ by Proposition 1, contradicting our assumption. Thus, the desired conclusion now follows from Theorem 3. □

Now combining Theorem 14 with Corollary 22, we have achieved our second main goal.

Theorem 23. *Let K be a field, and let E be a finite graph for which $L_K(E)$ is a nontrivial simple Leavitt path algebra. Write $E^0 = \{v_1, \dots, v_m\}$, and for each $1 \leq i \leq m$ let B_i be as in Definition 12. Then the Lie K-algebra $[L_K(E), L_K(E)]$ is simple if and only if $(1, \dots, 1) \notin \mathrm{span}_K\{B_1, \dots, B_m\}$.*

Here is the first of many consequences of Theorem 23.

Corollary 24. *Let K be a field, let E be a finite graph for which $L_K(E)$ is a nontrivial simple Leavitt path algebra, and let d be a positive integer. Write $E^0 = \{v_1, \ldots, v_m\}$, and for each $1 \le i \le m$ let B_i be as in Definition 12. Then the Lie K-algebra $[\mathbb{M}_d(L_K(E)), \mathbb{M}_d(L_K(E))]$ is simple if and only if $(1, \ldots, 1) \notin \operatorname{span}\{B_1, \ldots, B_m\}$ and $\operatorname{char}(K)$ does not divide d.*

Proof. The $d = 1$ case is precisely Theorem 23 (noting of course that $\operatorname{char}(K)$ never divides 1), while the $d \ge 2$ case follows by applying Proposition 6 (and Lemma 2) to Theorem 14. $\square$

Since for any positive integer d and any graph E, the K-algebra $\mathbb{M}_d(L_K(E))$ is isomorphic to a Leavitt path algebra with coefficients in K (see e.g. [8, Proposition 9.3]), the result of previous corollary can in fact be established using Theorem 23 directly. In particular, we get as a consequence of Corollary 24 a second, more efficient proof of the aforementioned previously-established result for matrix rings over Leavitt algebras.

Corollary 25 (Theorem 3.4 from [1]). *Let K be a field, let $n \ge 2$ and $d \ge 1$ be integers, and let $L_K(n)$ be the Leavitt K-algebra. Then the Lie K-algebra $[\mathbb{M}_d(L_K(n)), \mathbb{M}_d(L_K(n))]$ is simple if and only if $\operatorname{char}(K)$ divides $n - 1$ and does not divide d.*

Proof. Let E be the graph having one vertex v_1 and n loops. Then $L_K(n) \cong L_K(E)$. We have $B_1 = n - 1 \in K$, and hence $1 \notin \operatorname{span}\{B_1\} = (n-1)K$ if and only if $\operatorname{char}(K)$ divides $n - 1$. The result now follows from Corollary 24. $\square$

Throughout the remainder of the article, in a standard pictorial description of a directed graph E, a (n) written on an edge connecting two vertices indicates that there are n edges connecting those two vertices in E. We now recall (the germane portion of) [9, Lemma 5.1].

Lemma 26. *For integers $d \ge 2$ and $n \ge 2$ we denote by E_n^d the following graph.*

$$\bullet^{v_1} \xrightarrow{(d-1)} \bullet^{v_2} \circlearrowright (n)$$

Then for any field K, we have $\mathbb{M}_d(L_K(n)) \cong L_K(E_n^d)$.

We now present a number of examples which highlight the computational nature of Theorem 23. We start by offering an additional proof of the $d \ge 2$ case of Corollary 25, one which makes direct use of the Leavitt path algebra structure of $\mathbb{M}_d(L_K(n))$.

Additional proof of the $d \ge 2$ case of Corollary 25: We establish the contrapositive. Using the isomorphism $L_K(E_n^d) \cong \mathbb{M}_d(L_K(n))$ of Lemma 26, it is clear that the graph E_n^d yields $B_1 = (-1, d-1)$ and $B_2 = (0, n-1)$. So by Theorem 23 we seek properties of the integers n, d and field K for which $(1, 1) \in \operatorname{span}\{B_1, B_2\}$, i.e., for which the equation $k_1(-1, d-1) + k_2(0, n-1) = (1, 1)$ has solutions in $K \times K$. Equating coordinates, we seek to solve

$$\begin{aligned} -k_1 \quad &= \quad 1 \\ (d-1)k_1 + (n-1)k_2 \quad &= \quad 1 \end{aligned}$$

with $k_1, k_2 \in K$. So $k_1 = -1$, which gives $-(d-1) + k_2(n-1) = 1$, and thus $d = k_2(n-1)$. In case $n - 1 \ne 0$ in K (i.e., $\operatorname{char}(K)$ does not divide $n - 1$), this obviously has a solution, while in case $n - 1 = 0$ in K, the equation has a solution precisely when $d = 0$ in K, i.e., when $\operatorname{char}(K)$ divides d. $\square$

Remark 27. The following observations follow directly from Corollary 25.

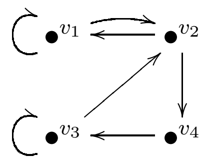
(1) The Lie K-algebra $[\mathbb{M}_d(L_K(n)), \mathbb{M}_d(L_K(n))]$ is not simple when $\mathrm{char}(K) = 0$.

(2) Let $\mathcal{P} = \{p_1, p_2, \ldots, p_t\}$ be a finite set of primes, and let $q = p_1 p_2 \cdots p_t \in \mathbb{N}$. Then the Lie K-algebra $[L_K(q+1), L_K(q+1)]$ is simple if and only if $\mathrm{char}(K) \in \mathcal{P}$.

(3) The Lie K-algebra $[L_K(2), L_K(2)]$ is not simple for all fields K. $\square$

The observations made in Remark 27 naturally suggest the following question: are there graphs E for which $[L_K(E), L_K(E)]$ is a simple Lie K-algebra for *all* fields K? We construct such an example now.

Example 28. Let E be the graph pictured below.



By Theorem 19, we see that $L_K(E)$ is simple for any field K.

For this graph E we have $B_1 = (0,1,0,0), B_2 = (1,-1,0,1), B_3 = (0,1,0,0)$, and $B_4 = (0,0,1,-1)$. We determine whether or not $(1,1,1,1)$ is in $\mathrm{span}_K\{B_1, B_2, B_3, B_4\}$. Upon building the appropriate augmented matrix of the resulting linear system, and using one row-swap and two add-an-integer-multiple-of-one-row-to-another operations, we are led to the matrix

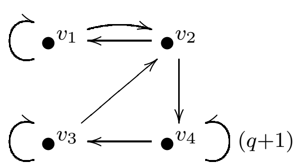
$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Clearly the final row indicates that the system has no solution, regardless of the characteristic of K. So by Theorem 23, the Lie algebra $[L_K(E), L_K(E)]$ is simple for any field K. $\square$

In particular, Example 28 together with Remark 27(1) show that Theorem 23 indeed enlarges the previously-known class of Leavitt path algebras for which the associated Lie algebra is simple.

We consider a complementary question arising from Remark 27(2). Specifically, for a given set of primes we produce a graph for which the Lie algebras corresponding to the associated Leavitt path algebras over specified fields are *not* simple.

Example 29. Let $\mathcal{P} = \{p_1, p_2, \ldots, p_t\}$ be a finite set of primes, let $q = p_1 p_2 \cdots p_t \in \mathbb{N}$, and let E_q be the graph pictured below.



By Theorem 19, we see that $L_K(E_q)$ is simple for any integer q and any field K.

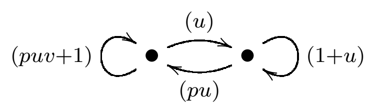
For this graph E_q we have $B_1 = (0,1,0,0), B_2 = (1,-1,0,1), B_3 = (0,1,0,0)$, and $B_4 = (0,0,1,q)$. We determine whether or not $(1,1,1,1)$ is in $\text{span}_K\{B_1, B_2, B_3, B_4\}$. Upon building the appropriate augmented matrix of the resulting linear system, and using a sequence of row-operations analogous to the sequence used in Example 28, we are led to the matrix

$$\begin{pmatrix} 1 & -1 & 1 & 0 & \vdots & 1 \\ 0 & 1 & 0 & 0 & \vdots & 1 \\ 0 & 0 & 0 & 1 & \vdots & 1 \\ 0 & 0 & 0 & 0 & \vdots & -q \end{pmatrix}.$$

Clearly the final row indicates that the system has solutions precisely when $\text{char}(K)$ divides q, i.e., when $\text{char}(K) \in \mathcal{P}$. So by Theorem 23, the Lie K-algebra $[L_K(E_q), L_K(E_q)]$ is not simple if and only if $\text{char}(K) \in \mathcal{P}$. $\square$

We finish this section by presenting, for each prime p, an infinite collection of graphs E for which the Lie K-algebra $[L_K(E), L_K(E)]$ is simple, where K is any field of characteristic p.

Example 30. For any prime p, and any pair of integers $u \geq 2, v \geq 2$, consider the graph $E = E_{u,v,p}$ pictured below.



By Theorem 19, $L_K(E)$ is a simple algebra for any field K. Here we have $B_1 = (puv, u)$ and $B_2 = (pu, u)$. Then $(1,1) \in \text{span}\{B_1, B_2\}$ precisely when we can solve the system

$$\begin{aligned} puvk_1 + puk_2 &= 1 \\ uk_1 + \quad uk_2 &= 1 \end{aligned}$$

for $k_1, k_2 \in K$. But clearly the first equation has no solutions in any field of characteristic p. Thus, by Theorem 23, the Lie algebra $[L_K(E), L_K(E)]$ is simple when $\text{char}(K) = p$, as desired. $\square$

In the next section we will show that the Leavitt path algebras associated to the graphs in Example 30 are pairwise non-isomorphic, as well as show that none of these algebras is isomorphic to an algebra of the form $\mathbb{M}_d(L_K(n))$.

4 Lie algebras arising from purely infinite simple Leavitt path algebras

We begin this final section by recasting Theorem 14 in terms of matrix transformations. For a finite graph E having m vertices $\{v_1, ..., v_m\}$ we let A_E denote the *adjacency matrix* of E; this is the $m \times m$ matrix whose (i,j) entry is $a_{i,j}$, the number of edges e for which $s(e) = v_i$ and $r(e) = v_j$. Let $\overline{1}^m$ denote the $m \times 1$ column vector $(1,1,...,1)^t$ (t denotes 'transpose'). Let B_E denote the matrix $A_E^t - I_m$. In case E has no sinks, B_E is the matrix whose i-th column is the element B_i of K^m, as in Definition 12. Let $B_E^{K^m}$ denote the K-linear transformation $K^m \to K^m$ induced by left multiplication by B_E. (For the remainder of the article we view elements of K^m as columns.) Then, using the notation of Theorem 14, it is clear that $(1, 1, ..., 1) \in \mathrm{span}\{B_1, ..., B_m\}$ if and only if $\overline{1}^m \in \mathrm{Im}(B_E^{K^m})$.

Definition 31. For a finite graph E having m vertices we define the matrix M_E by setting

$$M_E = I_m - A_E^t.$$

(In particular, if E has no sinks, then $M_E = -B_E$.) For any field K we let $M_E^{K^m}$ denote the K-linear transformation $K^m \to K^m$ induced by left multiplication by M_E. $\square$

Remark 32. Trivially, when E has no sinks, $\overline{1}^m \in \mathrm{Im}(B_E^{K^m})$ if and only if $\overline{1}^m \in \mathrm{Im}(M_E^{K^m})$. $\square$

Remark 33. Let E be a finite graph without sinks, and write $E^0 = \{v_1, \ldots, v_m\}$. Also, let K be a field with prime subfield k. Then $(1, \ldots, 1) \in \mathrm{span}_K\{B_1, \ldots, B_m\}$ if and only if $(1, \ldots, 1) \in \mathrm{span}_k\{B_1, \ldots, B_m\}$ if and only if $(1, \ldots, 1)^t$ is in the image of $M_E^{k^m} : k^m \to k^m$. This is because solving $M_E\overline{x} = (1, \ldots, 1)^t$ for $\overline{x} \in K^m$ amounts to putting into row-echelon form, via row operations, the matrix resulting from adjoining $(1, \ldots, 1)^t$ as a column to M_E. Since the original matrix M_E is integer-valued, all of the entries in the resulting row-echelon form matrix will come from the prime subfield. Thus in all germane computations we may work over the prime subfield k of K. $\square$

The graphs E for which $L_K(E)$ is a *purely infinite simple* algebra have played a central role in the development of the subject of Leavitt path algebras. (See e.g. [9] for the germane definitions, as well as an overview of the main properties of these algebras.) The key result here is

Theorem 34 (The Purely Infinite Simplicity Theorem)**.** *Let K be a field, and let E be a finite graph. Then $L_K(E)$ is purely infinite simple if and only if E has the following three properties.*

(1) *Every vertex v of E connects to every cycle of E.*

(2) *Every cycle of E has an exit.*

(3) *E has no sinks.*

In other words, using Theorem 19, $L_K(E)$ is purely infinite simple if and only if $L_K(E)$ is simple and E has no sinks.

As the Leavitt algebras $L_K(n)$ (and matrices over them) provide the basic examples of purely infinite simple algebras, it is natural in light of Corollary 25 to investigate the Lie algebras associated to purely infinite simple Leavitt path algebras. We do so for the remainder of this article, and in the process provide a broader context for the results of Section 3. We start with the following interpretation of Theorem 23, which follows from Remark 32.

Corollary 35. *Let K be a field, and let E be a finite graph for which $L_K(E)$ is purely infinite simple. Then the Lie K-algebra $[L_K(E), L_K(E)]$ simple if and only if $\overline{1}^m \notin \mathrm{Im}(M_E^{K^m})$.*

For any positive integer j we denote the cyclic group of order j by $\mathbb{Z}_j$, while for any prime p we denote the field of p elements by F_p.

We assume from now on that E is a finite graph with m vertices, and we often denote M_E simply by M for notational convenience. The matrix M_E has historically played an important role in the structure of purely infinite simple Leavitt path algebras (see [9, Section 3] for more information). For instance, since M is integer-valued, we may view left multiplication by M as a linear transformation from $\mathbb{Z}^m$ to $\mathbb{Z}^m$ (we denote this by $M^{\mathbb{Z}^m}$). Then the Grothendieck group of $L_K(E)$ is given by

$$K_0(L_K(E)) \cong \mathrm{Coker}(M^{\mathbb{Z}^m}) = \mathbb{Z}^m/\mathrm{Im}(M^{\mathbb{Z}^m}).$$

(It is of interest to note that the Grothendieck group of $L_K(E)$ is independent of the field K.) Moreover, under this isomorphism,

$$[1_{L_K(E)}] \text{ in } K_0(L_K(E)) \quad \text{corresponds to} \quad \overline{1}^m + \mathrm{Im}(M^{\mathbb{Z}^m}) \text{ in } \mathrm{Coker}(M^{\mathbb{Z}^m}).$$

For an abelian group G (written additively), an element $g \in G$, and positive integer j we say g is *j-divisible* in case there exists $g' \in G$ for which $g = g' + \cdots + g'$ (j summands). We use the previous discussion to give another interpretation of Theorem 23 in the case of purely infinite simple Leavitt path algebras. We thank Christopher Smith for pointing out this connection.

Theorem 36. *Let K be a field, let E be a finite graph for which $L_K(E)$ is purely infinite simple, and let $M = M_E$ denote the matrix of Definition 31.*

(1) *Suppose that* $\mathrm{char}(K) = 0$. *Then the Lie K-algebra $[L_K(E), L_K(E)]$ is simple if and only if $\overline{1}^m + \mathrm{Im}(M_E^{\mathbb{Z}^m})$ has infinite order in* $\mathrm{Coker}(M_E^{\mathbb{Z}^m})$; *that is, if and only if $[1_{L_K(E)}]$ has infinite order in $K_0(L_K(E))$.*

(2) *Suppose that* $\mathrm{char}(K) = p \neq 0$. *Then the Lie K-algebra $[L_K(E), L_K(E)]$ is simple if and only if $\overline{1}^m + \mathrm{Im}(M_E^{\mathbb{Z}^m})$ is not p-divisible in* $\mathrm{Coker}(M_E^{\mathbb{Z}^m})$; *that is, if and only if $[1_{L_K(E)}]$ is not p-divisible in $K_0(L_K(E))$.*

Proof. (1) We show that $\overline{1}^m \in \mathrm{Im}(M_E^{K^m})$ if and only if $\overline{1}^m + \mathrm{Im}(M_E^{\mathbb{Z}^m})$ has finite order in $\mathrm{Coker}(M_E^{\mathbb{Z}^m})$, from which the statement follows by Corollary 35. By Remark 33, we need only show that $\overline{1}^m \in \mathrm{Im}(M_E^{\mathbb{Q}^m})$ if and only if $\overline{1}^m + \mathrm{Im}(M_E^{\mathbb{Z}^m})$ has finite order in $\mathrm{Coker}(M_E^{\mathbb{Z}^m})$. If $\overline{1}^m + \mathrm{Im}(M_E^{\mathbb{Z}^m})$ has finite order in $\mathrm{Coker}(M_E^{\mathbb{Z}^m})$, then there exists a positive integer n for which $(n, n, \ldots, n)^t \in \mathrm{Im}(M_E^{\mathbb{Z}^m})$, i.e., there exists $\overline{z} = (z_1, z_2, \ldots, z_m)^t \in \mathbb{Z}^m$ for which $M_E\overline{z} = (n, n \ldots, n)^t$. But then $\overline{q} = (\frac{z_1}{n}, \frac{z_2}{n}, \ldots, \frac{z_m}{n})^t \in \mathbb{Q}^m$ satisfies $M_E\overline{q} = \overline{1}^m$. Conversely, if $\overline{1}^m \in \mathrm{Im}(M_E^{\mathbb{Q}^m})$ then there exists $(\frac{z_1}{n_1}, \frac{z_2}{n_2}, \ldots, \frac{z_m}{n_m})^t \in \mathbb{Q}^m$ with $\overline{1}^m = M_E(\frac{z_1}{n_1}, \frac{z_2}{n_2}, \ldots, \frac{z_m}{n_m})^t$. If $n = n_1 n_2 \cdots n_m$, then $(n, n, \cdots, n)^t = M_E(\frac{z_1 n}{n_1}, \frac{z_2 n}{n_2}, \ldots, \frac{z_m n}{n_m})^t \in \mathrm{Im}(M_E^{\mathbb{Z}^m})$, so that $(1, 1, \cdots, 1)^t + \mathrm{Im}(M_E^{\mathbb{Z}^m})$ has finite order (indeed, order at most n) in $\mathrm{Coker}(M_E^{\mathbb{Z}^m})$.

(2) Analogously to the proof of part (1), we show that $\overline{1}^m \in \mathrm{Im}(M_E^{K^m})$ if and only if $\overline{1}^m + \mathrm{Im}(M_E^{\mathbb{Z}^m})$ is p-divisible in $\mathrm{Coker}(M_E^{\mathbb{Z}^m})$. By Remark 33, we need only show that $\overline{1}^m \in \mathrm{Im}(M_E^{F_p^m})$ if and only if $\overline{1}^m + \mathrm{Im}(M_E^{\mathbb{Z}^m})$ is p-divisible in $\mathrm{Coker}(M_E^{\mathbb{Z}^m})$. If $\overline{1}^m + \mathrm{Im}(M_E^{\mathbb{Z}^m})$ is p-divisible in $\mathrm{Coker}(M_E^{\mathbb{Z}^m})$, then there exists $\overline{z} \in \mathbb{Z}^m$ for which $p\overline{z} + \mathrm{Im}(M_E^{\mathbb{Z}^m}) = \overline{1}^m + \mathrm{Im}(M_E^{\mathbb{Z}^m})$, i.e., $\overline{1}^m - p\overline{z} \in \mathrm{Im}(M_E^{\mathbb{Z}^m})$. Reducing this integer-valued system of equations mod p yields $\overline{1}^m \in \mathrm{Im}(M_E^{F_p^m})$. The converse can be proved by reversing this argument. $\square$

Remark 37. Let K be a field such that $\mathrm{char}(K) = p \neq 0$. For Leavitt path algebras of the form $R = \mathbb{M}_d(L_K(n))$, we have that $K_0(R) \cong \mathbb{Z}_{n-1}$. Moreover, under this isomorphism the element $[1_R]$ in $K_0(R)$ corresponds to the element d in $\mathbb{Z}_{n-1}$. Thus the p-divisibility of $[1_R]$ in $K_0(R)$ is equivalent to the p-divisibility of d in $\mathbb{Z}_{n-1}$, which in turn is equivalent to determining whether or not the linear equation $px \equiv d \pmod{n-1}$ has solutions. It is elementary number theory that this equation has solutions precisely when $g.c.d.(p, n-1)$ divides d. So by Theorem 36(2), we see that $[\mathbb{M}_d(L_K(n)), \mathbb{M}_d(L_K(n))]$ is simple precisely when $g.c.d.(p, n-1)$ does not divide d, which is clearly equivalent to the statement "p divides $n-1$ and p does not divide d". This observation provides a broader framework for Corollary 25. $\square$

Now continuing our description of various connections between the matrix $M = M_E$ and the Grothendieck group of $L_K(E)$, we recall from [9, Section 3] that the matrix M can be utilized to determine the specific structure of $K_0(L_K(E))$ in case $L_K(E)$ is purely infinite simple, as follows. Given any integer-valued $d \times d$ matrix C, we say that a matrix C' is *equivalent* to C in case $C' = PCQ$ for some matrices P, Q which are invertible in $\mathbb{M}_d(\mathbb{Z})$. Computationally, this means C' can be produced from C by a sequence of row swaps and column swaps, by multiplying any row or column by -1, and by using the operation of adding a $\mathbb{Z}$-multiple of one row (respectively, column) to another row (respectively, column). The *Smith normal form* of an integer-valued $d \times d$ matrix C is the diagonal matrix which is equivalent to C, having diagonal entries $\alpha_1, ..., \alpha_d$, such that, for all nonzero α_i $(1 \leq i < d)$, α_i divides α_{i+1}. (The Smith normal form of a matrix always exists. Also, if we agree to write any zero entries last, and to make all α_i nonnegative, then the Smith normal form of a matrix is unique.) By the discussion in [9, Section 3], for a graph E satisfying the properties of Theorem 34, if $\alpha_1, ..., \alpha_d$ are the diagonal entries of the Smith normal form of M_E, then

$$K_0(L_K(E)) \cong \mathbb{Z}_{\alpha_1} \oplus \cdots \oplus \mathbb{Z}_{\alpha_d}.$$

With this observation, we have the tools to justify a statement made in the previous section.

Example 38. Consider again the graphs $E = E_{u,v,p}$ arising in Example 30. Then

$$A_E = \begin{pmatrix} puv+1 & u \\ pu & 1+u \end{pmatrix}, \text{ so that } M_E = I_2 - A_E^t = \begin{pmatrix} -puv & -pu \\ -u & -u \end{pmatrix}.$$

The Smith normal form of M_E is easily computed to be

$$\begin{pmatrix} u & 0 \\ 0 & pu(v-1) \end{pmatrix},$$

implying that $K_0(L_K(E)) \cong \mathbb{Z}_u \oplus \mathbb{Z}_{pu(v-1)}$. Thus for any choices of u, u' and v, v' where $u \neq u'$ we have that $L_K(E_{u,v,p}) \not\cong L_K(E_{u',v',p})$. Furthermore, since $u \geq 2$ and $v \geq 2$, none of these algebras has cyclic K_0, so that none of these algebras is isomorphic to an algebra of the form $\mathbb{M}_d(L_K(n))$, as claimed. $\square$

We conclude the article with an observation about the K-theory of Leavitt path algebras in the context of their associated Lie algebras. An open question in the theory of Leavitt path algebras is the *Algebraic Kirchberg Phillips Question*: If E and F are finite graphs with the property that $L_K(E)$ and $L_K(F)$ are purely infinite simple, and $K_0(L_K(E)) \cong K_0(L_K(F))$ via an isomorphism which takes $[1_{L_K(E)}]$ to $[1_{L_K(F)}]$, are $L_K(E)$ and $L_K(F)$ necessarily isomorphic? (See [10] for more details.) Since the property "the Lie K-algebra $[R, R]$ is simple" is an isomorphism invariant of a K-algebra R, one might look for a possible negative answer to the Algebraic Kirchberg Phillips Question in this context. However, by Theorem 36, we get immediately the following result.

Proposition 39. *Let E and F be finite graphs, and K any field. Suppose that $L_K(E)$ and $L_K(F)$ are purely infinite simple, and that $K_0(L_K(E)) \cong K_0(L_K(F))$ via an isomorphism which takes $[1_{L_K(E)}]$ to $[1_{L_K(F)}]$. Then the Lie K-algebra $[L_K(E), L_K(E)]$ is simple if and only if the Lie K-algebra $[L_K(F), L_K(F)]$ is simple.*

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