arXiv:1107.3136v2 [math.AP] 20 Jul 2011

H^2 REGULARITY FOR THE $p(x)-$LAPLACIAN IN TWO-DIMENSIONAL CONVEX DOMAINS

LEANDRO M. DEL PEZZO AND SANDRA MARTÍNEZ

ABSTRACT. In this paper we study the H^2 global regularity for solutions of the $p(x)-$Laplacian in two dimensional convex domains with Dirichlet boundary conditions. Here $p : \Omega \to [p_1, \infty)$ with $p \in \mathrm{Lip}(\overline{\Omega})$ and $p_1 > 1$.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^2$ and let $p : \Omega \to (1, +\infty)$ be a measurable function. In this work, we study the H^2 global regularity of the weak solution of the following problem

$$\begin{cases} -\Delta_{p(x)} u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_{p(x)} u = \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)-$Laplacian. The hypothesis over p, f and g will be specified later.

Note that, the $p(x)-$Laplacian extends the classical Laplacian ($p(x) \equiv 2$) and the $p-$Laplacian ($p(x) \equiv p$ with $1 < p < +\infty$). This operator has been recently used in image processing and in the modeling of electrorheological fluids, see [3, 5, 24].

Motivate by the applications to image processing problem, in [8], the authors study two numerical methods to approximate solutions of the type of (1.1). In Theorem 5.1, the authors prove the convergence in $W^{1,p(\cdot)}(\Omega)$ of the conformal Galerking finite element method. It is of our interest to study, in a future work, the rate of this convergence. In general, all the error bounds depend on the global regularity of the second derivatives of the solutions, see for example [6, 22]. However, there appear to be no existing regularity results in the literature that can be applied here, since all the results have either a first order or local character.

The H^2 global regularity for solutions of the $p-$Laplacian is studied in [22]. There the authors prove the following: Let $1 < p \leq 2$, $g \in H^2(\Omega)$, $f \in L^q(\Omega)$ $(q > 2)$ and u be the unique weak solution of (1.1). Then

- If $\partial\Omega \in C^2$ then $u \in H^2(\Omega)$;
- If Ω is convex and $g = 0$ then $u \in H^2(\Omega)$;

Key words and phrases. Variable exponent spaces. Elliptic Equations. H^2 regularity.
2010 *Mathematics Subject Classification.* 35B65, 35J60, 35J70.
Supported by UBA X117, UBA 20020090300113, CONICET PIP 2009 845/10 and PIP 11220090100625.

- If Ω is convex with a polygonal boundary and $g \equiv 0$ then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$.

Regarding the regularity of the weak solution of (1.1) when $f = 0$, in [2, 7], the authors prove the $C^{1,\alpha}_{loc}$ regularity (in the scalar case and also in the vectorial case). Then, in the paper [15] the authors study the case where the functional has the so called $(p,q)-$ growth conditions. Following these ideas, in [17], the author proves that the solutions of (1.1) are in $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha > 0$ if Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with $C^{1,\gamma}$ boundary, $p(x)$ is a Hölder function, $f \in L^\infty(\Omega)$ and $g \in C^{1,\gamma}(\overline{\Omega})$. While in [4], the authors prove that the solutions are in $H^2_{loc}(\{x \in \Omega \colon p(x) \leq 2\})$ if $p(x)$ is uniformly Lipschitz $(\mathrm{Lip}(\Omega))$ and $f \in W^{1,q(\cdot)}_{loc}(\Omega) \cap L^\infty(\Omega)$.

Our aim, it is to generalized the results of [22] in the case where $p(x)$ is a measurable function. To this end, we will need some hypothesis over the regularity of $p(x)$. Moreover, in all our result we can avoid the restriction $g = 0$, assuming some regularity of $g(x)$.

On the other hand, to prove our results, we can assume weaker conditions over the function f than the ones on [4]. Since, we only assume that $f \in L^{q(\cdot)}(\Omega)$, we do not have a priori that the solutions are in $C^{1,\alpha}(\Omega)$. Then we can not use it to prove the H^2 global regularity. Nevertheless, we can prove that the solutions are in $C^{1,\alpha}(\overline{\Omega})$, after proving the H^2 global regularity.

The main results of this paper are:

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ with C^2 boundary, $p \in \mathrm{Lip}(\overline{\Omega})$ with $p(x) \geq p_1 > 1$, $g \in H^2(\Omega)$ and u be the weak solution of* (1.1)*. If*

(F1) *$f \in L^{q(x)}(\Omega)$ with $q(x) \geq q_1 > 2$ in the set $\{x \in \Omega \colon p(x) \leq 2\}$;*
(F2) *$f \equiv 0$ in the set $\{x \in \Omega \colon p(x) > 2\}$.*

then $u \in H^2(\Omega)$.

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^2$ with convex boundary, $p \in \mathrm{Lip}(\overline{\Omega})$ with $p(x) \geq p_1 > 1$, $g \in H^2(\Omega)$ and u be the weak solution of* (1.1)*. If f satisfies (F1) and (F2) then $u \in H^2(\Omega)$.*

Using the above theorem we can prove the following,

Corollary 1.3. *Let Ω be a bounded convex domain in $\mathbb{R}^2$ with polygonal boundary, p and f as in the previous theorem, $g \in W^{2,q(x)}(\Omega)$ and u be the weak solution of* (1.1) *then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$.*

Observe that this result extends the one in [17] in the case where Ω is a polygonal domain in $\mathbb{R}^2$.

Organization of the paper. The rest of the paper is organized as follows. After a short Section 2 where we collect some preliminaries results, in Section 3, we study the H^2- regularity for the non-degenerated problem. In Section 4 we prove Theorem 1.1. Then, in Section 5, we study the regularity of the solution u of (1.1) if Ω is convex. In Section 6, we make some comments on the dependence of the H^2-norm of u on p_1. Lastly, in Appendices A and B we give some results related to elliptic linear equation with bounded coefficients and Lipschitz functions, respectively.

2. Preliminaries

We now introduce the space $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ and state some of their properties.

Let Ω be a bounded open set of $\mathbb{R}^n$ and $p\colon \Omega \to [1,+\infty)$ be a measurable bounded function, called a variable exponent on Ω and denote $p_1 := essinf\, p(x)$ and $p_2 := esssup\, p(x)$.

We define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u\colon \Omega \to \mathbb{R}$ for which the modular

$$\varrho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)}\, dx$$

is finite. We define the Luxemburg norm on this space by

$$\|u\|_{L^{p(\cdot)}(\Omega)} := \inf\{k > 0\colon \varrho_{p(\cdot)}(u/k) \le 1\}.$$

This norm makes $L^{p(\cdot)}(\Omega)$ a Banach space.

For the proofs of the following theorems, we refer the reader to [12].

Theorem 2.1 (Hölder's inequality)**.** *Let $p, q, s : \Omega \to [1,+\infty]$ be a measurable functions such that*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = \frac{1}{s(x)} \quad \text{in } \Omega.$$

Then the inequality

$$\|fg\|_{L^{s(\cdot)}(\Omega)} \le 2\|f\|_{L^{p(\cdot)}(\Omega)}\|g\|_{L^{q(\cdot)}(\Omega)}$$

for all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$

Let $W^{1,p(\cdot)}(\Omega)$ denote the space of measurable functions u such that, u and the distributional derivative ∇u are in $L^{p(\cdot)}(\Omega)$. The norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \|u\|_{p(\cdot)} + \||\nabla u|\|_{p(\cdot)}$$

makes $W^{1,p(\cdot)}(\Omega)$ a Banach space.

Theorem 2.2. *Let $p'(x)$ such that, $1/p(x) + 1/p'(x) = 1$. Then $L^{p'(\cdot)}(\Omega)$ is the dual of $L^{p(\cdot)}(\Omega)$. Moreover, if $p_1 > 1$, $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are reflexive.*

We define the space $W_0^{1,p(\cdot)}(\Omega)$ as the closure of the $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. Then we have the following version of Poincare's inequity (see Theorem 3.10 in [21]).

Lemma 2.3 (Poincare's inequity)**.** *If $p : \Omega \to [1,+\infty)$ is continuous in $\overline{\Omega}$, there exists a constant C such that for every $u \in W_0^{1,p(\cdot)}(\Omega)$,*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \le C\|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

In order to have better properties of these spaces, we need more hypotheses on the regularity of $p(x)$.

We say that p is log-*Hölder continuous* in Ω if there exists a constant C_{log} such that

$$|p(x)-p(y)| \leq \frac{C_{log}}{\log\left(e+\frac{1}{|x-y|}\right)} \quad \forall\, x,y\in\Omega.$$

It was proved in [10], Theorem 3.7, that if one assumes that p is log-Hölder continuous then $C^\infty(\bar{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$ (see also [9, 12, 13, 21, 25]).

We now state the Sobolev embedding Theorem (for the proofs see [12]). Let,

$$p^*(x) := \begin{cases} \frac{p(x)N}{N-p(x)} & \text{if } p(x)<N,\\ +\infty & \text{if } p(x)\geq N, \end{cases}$$

be the Sobolev critical exponent. Then we have the following,

Theorem 2.4. *Let Ω be a Lipschitz domain. Let $p:\Omega\to[1,\infty)$ and p log-Hölder continuous. Then the imbedding $W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{p^*(\cdot)}(\Omega)$ is continuous.*

3. H^2−Regularity for the non-degenerated problem for any dimension

In this section we assume that Ω is a bounded domain in $\mathbb{R}^N$, with $N\geq 2$.

We want to study higher regularity of the weak solution of the regularized equation,

$$\begin{cases} -\operatorname{div}\left((\varepsilon+|\nabla u|^2)^{\frac{p(x)-2}{2}}\nabla u\right)=f & \text{in } \Omega,\\ u=g & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where $0<\varepsilon\leq 1$, and $f\in \mathrm{Lip}(\Omega)$ and $g\in W^{1,p(\cdot)}(\Omega)$.

The existence of a weak solution of (3.2) holds by Theorem 13.3.3 in [12].

Remark 3.1. Given $\varepsilon\geq 0$, $p\in C^{\alpha_0}(\overline{\Omega})$ for some $\alpha_0>0$, and $g\in L^\infty(\Omega)$ we have the following results,

1. (1) Since $f,g\in L^\infty(\Omega)$, by Theorem 4.1 in [18], we have that $u\in L^\infty(\Omega)$.
2. (2) By Theorem 1.1 in [17], $u\in C^{1,\alpha}_{loc}(\Omega)$ for some α depending on p_1,p_2, $\|u\|_{L^\infty(\Omega)},\|f\|_{L^\infty(\Omega)}$. Moreover, given $\Omega_0\subset\subset\Omega$, $\|u\|_{C^{1,\alpha}(\Omega_0)}$ depends on the same constants and $dist(\Omega_0,\partial\Omega)$.
3. (3) Finally, by Theorem 1.2 in [17], if $\partial\Omega\in C^{1,\gamma}$ and $g\in C^{1,\gamma}(\partial\Omega)$ for some $\gamma>0$ then $u\in C^{1,\alpha}(\overline{\Omega})$, where α and $\|u\|_{C^{1,\alpha}(\Omega)}$ depend on $p_1,p_2,N,\|u\|_{L^\infty(\Omega)}$, $\|p\|_{C^{\alpha_0}(\Omega)},\alpha_0,\gamma$.

We will first prove the H^2-local regularity assuming only that $p(x)$ is Lipschitz. Then, we will prove the global regularity under the stronger condition that $\nabla p(x)$ is Hölder.

3.1. H^2-**Local regularity.** While we where finishing this paper, we found the work [4], where the authors give a different proof of the H^2-local regularity of the solutions of (3.2). Anyhow, we leave the proof for the completeness of this paper.

Theorem 3.2. *Let* $p, f \in \mathrm{Lip}(\Omega)$ *with* $p_1 > 1$ *and* u *a weak solution of* (3.2)*, then* $u \in H^2_{loc}(\Omega)$.

Proof. First, let us define for any function F and $h > 0$,

$$\Delta^h F(x) = \frac{F(x+\mathbf{h}) - F(x)}{h},$$

where $\mathbf{h} = he_k$ where e_k is a vector of the canonical base of $\mathbb{R}^N$.

Let $\eta(x) = \xi(x)^2 \Delta^h u(x)$ where ξ is a regular function with compact support. Therefore, if we take $v_\varepsilon = (|\nabla u|^2 + \varepsilon)^{1/2}$ and $h < \mathrm{dist}(\mathrm{supp}(\xi), \partial\Omega)$, we have

$$\int_\Omega \langle v_\varepsilon(x)^{p(x)-2} \nabla u(x), \nabla \eta(x) \rangle \, dx = \int_\Omega f(x) \eta(x) \, dx$$

$$\int_\Omega \langle v_\varepsilon(x+\mathbf{h})^{p(x+\mathbf{h})-2} \nabla u(x+\mathbf{h}), \nabla \eta(x) \rangle \, dx = \int_\Omega f(x+\mathbf{h}) \eta(x) \, dx.$$

Subtracting, using that $\nabla \eta = 2\xi \nabla \xi \Delta^h u + \xi^2 \Delta^h (\nabla u)$ and dividing by h we obtain,

$$\begin{aligned} I =& \int_\Omega \langle \Delta^h (v_\varepsilon(x)^{p(x)-2} \nabla u), \Delta^h(\nabla u) \rangle \xi^2 \, dx \\ =& -2 \int_\Omega \langle \Delta^h (v_\varepsilon(x)^{p(x)-2} \nabla u), \xi \nabla \xi \Delta^h u \rangle \, dx + \int_\Omega \xi^2 \Delta^h f \Delta^h u \, dx \\ =& 2 \int_\Omega \left(\int_0^1 (v_\varepsilon(x+\mathbf{h}t)^{p(x+\mathbf{h}t)-2} \nabla u(x+\mathbf{h}t) \, dt \right) \frac{\partial}{\partial x_k} (\xi \nabla \xi \Delta^h u) dx \\ &+ \int_\Omega \xi^2 \Delta^h f \Delta^h u \, dx \\ =& II + III. \end{aligned}$$

Now, let as fix a ball B_R such that $B_{3R} \subset\subset \Omega$ and take $\xi \in C_0^\infty(\Omega)$ supported in B_{2R} such that $0 \le \xi \le 1$, $\xi = 1$ in B_R, $|\nabla \xi| \le 1/R$ and $|D^2 \xi| \le CR^{-2}$.

By Remark 3.1, there exist a constant $C_1 > 0$ such that $|\nabla u| \le C_1$ in B_{3R}, therefore we get

$$\begin{aligned} II &\le 2 \int_{B_{2R}} \frac{C}{R} |\Delta^h u_{x_k}| \xi \, dx + 2 \int_{B_{2R}} \frac{C}{R^2} |\Delta^h u| \, dx \\ &\le \frac{C}{R} \int_{B_{2R}} |\Delta^h (\nabla u)| \xi \, dx + CR^{N-2}. \end{aligned}$$

On the other hand, since f is Lipschitz we have that,

$$|f(x+\mathbf{h}) - f(x)| \le C_2 h$$

for some constant $C_2 > 0$. This implies that,

$$III \le C_2 R^N.$$

Therefore, summing II and III, and using Young's inequality, we have that for any $\delta > 0$

$$I \leq \delta \int_{B_{2R}} |\Delta^h(\nabla u)|^2 \xi^2 \, dx + C, \tag{3.3}$$

for some constant C depending on R and δ.

On the other hand observe that $I = I_1 + I_2$ where,

$$I_1 = \frac{1}{h} \int_{B_{2R}} \langle (v_\varepsilon(x+\mathbf{h})^{p(x+\mathbf{h})-2} \nabla u(x+\mathbf{h}) - v_\varepsilon(x)^{p(x+\mathbf{h})-2} \nabla u(x)), \Delta^h(\nabla u) \rangle \xi^2 \, dx,$$

and

$$I_2 = \frac{1}{h} \int_{B_{2R}} \langle \Big(v_\varepsilon(x)^{p(x+\mathbf{h})} - v_\varepsilon(x)^{p(x)} \Big) \frac{\nabla u(x)}{v_\varepsilon(x)^2}, \Delta^h(\nabla u) \rangle \xi^2 \, dx.$$

Using that $p(x)$ is Lipschitz and the fact that $|\nabla u(x)| \leq C_1$ we have that, for some b between $p(x+h)$ and $p(x)$,

$$\frac{1}{h} \left| v_\varepsilon(x)^{p(x+\mathbf{h})} - v_\varepsilon(x)^{p(x)} \right| = \left| v_\varepsilon(x)^b \log(v_\varepsilon(x)) \frac{p(x+\mathbf{h}) - p(x)}{h} \right| \leq C,$$

for some constant $C > 0$ depending on $p_1, p_2, \varepsilon, C_1$ and the Lipschitz constant of $p(x)$.

Therefore, we have that

$$-I_2 \leq C C_1 \varepsilon^{-1} \int_{B_{2R}} |\Delta^h(\nabla u)| \xi^2 \, dx.$$

By (3.3), the last inequality and using again Young's inequality we have that, for any $\delta > 0$

$$I_1 \leq \delta \int_{B_{2R}} |\Delta^h(\nabla u)|^2 \xi^2 \, dx + C, \tag{3.4}$$

for some constant $C > 0$ depending on $p_1, p_2, \varepsilon, C_1$ and the Lipschitz constant of $p(x)$.

To finish the proof, we have to find a lower bound for I_1. By a well known inequality, we have that

$$\langle (v_\varepsilon(x+\mathbf{h})^{p(x+h)-2} \nabla u(x+\mathbf{h}) - v_\varepsilon(x)^{p(x+\mathbf{h})-2} \nabla u(x)), (\nabla u(x+\mathbf{h}) - \nabla u(x)) \rangle \geq C_\varepsilon |\nabla u(x+\mathbf{h}) - \nabla u(x)|^2,$$

where

$$C_\varepsilon = \begin{cases} \varepsilon^{p(x+\mathbf{h})-2/2} & \text{if } p(x+\mathbf{h}) \geq 2, \\ (p(x+\mathbf{h})-1)\varepsilon^{p(x+\mathbf{h})-2/2} & \text{if } p(x+\mathbf{h}) \leq 2. \end{cases}$$

Therefore, using that $p_1 > 1$, we arrive at

$$I_1 \geq \int_{B_{2R}} C h^{-2} |\nabla u(x+\mathbf{h}) - \nabla u(x)|^2 \xi^2 \, dx = C \int_{B_{2R}} |\Delta^h(\nabla u(x))|^2 \xi^2 \, dx.$$

Finally combining the last inequality with (3.4) we have that,

$$\int_{B_R} |\Delta^h(\nabla u(x))|^2 \, dx \leq C(N, p, f, \varepsilon).$$

This proves that $u \in H^2_{loc}(\Omega)$. □

3.2. H^2-**Global Regularity.** Now we want to prove that if $f \in \mathrm{Lip}(\Omega)$ and $g \in C^{1,\beta}(\partial\Omega)$, the regularized equation (3.2) has a weak solution $u \in C^2(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ for an $\alpha \in (0,1)$. We already know, by Remark 3.1, that $u \in C^{1,\alpha}(\overline{\Omega})$. Then, we only need to prove that $u \in C^2(\Omega)$.

Lemma 3.3. *Let Ω be a bounded domain in $\mathbb{R}^N$ with $\partial\Omega \in C^{1,\gamma}$, $p \in C^{1,\beta}(\Omega) \cap C^{\alpha_0}(\overline{\Omega})$, $f \in \mathrm{Lip}(\Omega)$ and $g \in C^{1,\beta}(\partial\Omega)$. Then, the Dirichlet Problem (3.2) has a solution $u \in C^2(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$.*

Proof. Observe that by Theorem 3.2, we know that the solution is in $H^2_{loc}(\Omega)$. Then for any $\Omega' \subset\subset \Omega$ we can derive the equation and look the solution of (3.2) as the solution of the following equation,

$$\begin{cases} L_\varepsilon u = a(x) & \text{in } \Omega', \\ u = u & \text{on } \partial\Omega'. \end{cases} \tag{3.5}$$

Here,

$$L_\varepsilon u = a^\varepsilon_{ij}(x) u_{x_i x_j}$$

with

$$\begin{aligned} a^\varepsilon_{ij}(x) &= \delta_{ij} + (p(x)-2)\frac{u_{x_i}u_{x_j}}{v_\varepsilon^2}, \quad v_\varepsilon = \left(\varepsilon + |\nabla u|^2\right)^{\frac{1}{2}}, \\ a_\varepsilon(x) &= \ln(v_\varepsilon)\langle \nabla u, \nabla p\rangle + f v_\varepsilon^{2-p}. \end{aligned} \tag{3.6}$$

The operator L_ε is uniformly elliptic in Ω, since for any $\xi \in \mathbb{R}^N$

$$\min\{(p_1-1),1\}|\xi|^2 \le a^\varepsilon_{ij}\xi_i\xi_j \le \max\{(p_2-1),1\}|\xi|^2. \tag{3.7}$$

On the other hand, by Remark 3.1, $u \in C^{1,\alpha}(\overline{\Omega})$. Then, $a^\varepsilon_{ij} \in C^\alpha(\overline{\Omega})$, since $\varepsilon > 0$. Using that $f \in \mathrm{Lip}(\Omega)$, we have that $a \in C^\rho(\Omega)$ where $\rho = \min(\alpha,\beta)$. If $\partial\Omega' \in C^2$, as u is the unique solution of (3.5), by Theorem 6.13 in [19], we have that $u \in C^{2,\rho}(\Omega')$. This ends the proof.

□

Remark 3.4. By the H^2 global estimate for linear elliptic equations with $L^\infty(\Omega)$ coefficients in two variables (see Lemma A.1 and (3.7)) we have that,

$$\|u\|_{H^2(\Omega)} \le C\left(\|a_\varepsilon\|_{L^2(\Omega)} + \|g\|_{H^2(\Omega)}\right)$$

where u is the solution of (3.2) and C is a constant independents of ε.

4. Proof of Theorem 1.1

Before proving the theorem, we will need a global bound for the derivatives of the solutions of (3.2).

Lemma 4.1. *Let $f \in L^{q(x)}(\Omega)$ with $q'(x) \le p^*(x)$, $g \in W^{1,p(\cdot)}(\Omega)$, $\varepsilon > 0$ and u_ε be the weak solution of (3.2) then*

$$\|\nabla u_\varepsilon\|_{L^{p(\cdot)}(\Omega)} \le C$$

where C is a constant depending on $\|f\|_{L^{q(\cdot)}(\Omega)}, \|g\|_{W^{1,p(\cdot)}(\Omega)}$ but not on ε.

Proof. Let

$$J(v) := \int_\Omega \frac{1}{p(x)}(|\nabla v|^2 + \varepsilon)^{p(x)/2}\, dx.$$

By the convexity of J and using (3.2) we have that,

$$\begin{aligned} J(u_\varepsilon) &\le J(g) - \int_\Omega (|\nabla u_\varepsilon|^2+\varepsilon)^{(p-2)/2}\nabla u_\varepsilon(\nabla g - \nabla u_\varepsilon)\, dx \\ &\le C\left(1 + \int_\Omega f(u_\varepsilon - g)\, dx\right) \\ &\le C\left(1 + \|f\|_{L^{q(\cdot)}(\Omega)}\|u_\varepsilon - g\|_{L^{q'(\cdot)}(\Omega)}\right) \\ &\le C\left(1 + \|f\|_{L^{q(\cdot)}(\Omega)}\|\nabla u_\varepsilon - \nabla g\|_{L^{p(\cdot)}(\Omega)}\right), \end{aligned}$$

where in the last inequality we are using that $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$ continuously and Poincare's inequality.

Thus we have that there exist a constant independent of ε such that,

$$\int_\Omega |\nabla u_\varepsilon|^{p(x)}\, dx \le C(1 + \|\nabla u_\varepsilon\|_{L^{p(\cdot)}(\Omega)}),$$

and using the properties of the $L^{p(\cdot)}(\Omega)-$ norms this means that

$$\|\nabla u_\varepsilon\|^m_{L^{p(\cdot)}(\Omega)} \le C(1 + \|\nabla u_\varepsilon\|_{L^{p(\cdot)}(\Omega)}),$$

for some $m > 1$. Therefore $\|\nabla u_\varepsilon\|_{L^{p(\cdot)}(\Omega)}$ is bounded independent of ε. □

To prove Theorem 1.1, we will use the results of Section 3. Therefore, we will first need to assume that $p \in C^{1.\beta}(\Omega) \cap C(\overline{\Omega})$.

Theorem 4.2. *Let Ω be a bounded domain in $\mathbb{R}^2$ with C^2 boundary, $p \in C^{1.\beta}(\Omega)\cap C^{\alpha_0}(\overline{\Omega})$ with $p(x) \ge p_1 > 1$, $g \in H^2(\Omega)$ and u be the weak solution of* (1.1)*. If f satisfies (F1) and (F2) then $u \in H^2(\Omega)$.*

Proof. Let $f_\varepsilon \in \mathrm{Lip}(\Omega)$ and $g_\varepsilon \in C^{2,\alpha}(\overline{\Omega})$ such that

$$\begin{aligned} &f_\varepsilon \to f \text{ strongly in } L^{q(\cdot)}(\Omega), \\ &g_\varepsilon \to g \text{ strongly in } H^2(\Omega), \end{aligned}$$

as $\varepsilon \to 0$. Observe that, since $f(x) = 0$ if $p(x) > 2$, we can take $f_\varepsilon \equiv 0$ in $\{x \in \Omega\colon p(x) > 2\}$.

Now, let us consider the solution of (3.2) as the solution of

$$\begin{cases} a_{11}^\varepsilon(x)\frac{\partial^2 u_\varepsilon}{\partial x_1^2} + 2a_{12}^\varepsilon(x)\frac{\partial^2 u_\varepsilon}{\partial x_1 \partial x_2} + a_{22}^\varepsilon(x)\frac{\partial^2 u_\varepsilon}{\partial x_2^2} = a_\varepsilon(x) & \text{in } \Omega, \\ u_\varepsilon = g_\varepsilon & \text{on } \partial\Omega, \end{cases}$$

where $a_{11}^\varepsilon, a_{22}^\varepsilon, a_{12}^\varepsilon, a_\varepsilon$ are defined as in Lemma 3.3, substituting f and g by f_ε and g_ε respectively. By Lemma 3.3 we know that $u_\varepsilon \in C^2(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$.

First we will prove the $\{u_\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $H^2(\Omega)$. By Remark 3.4, we have that

$$\begin{aligned} \|u_\varepsilon\|_{H^2(\Omega)} &\le C(\|a_\varepsilon(x)\|_{L^2(\Omega)} + \|g_\varepsilon\|_{H^2(\Omega)}) \\ &\le C(\|\ln(v_\varepsilon)\nabla u_\varepsilon \nabla p\|_{L^2(\Omega)} + \|f_\varepsilon v^{2-p}\|_{L^2(\Omega)} + \|g_\varepsilon\|_{H^2(\Omega)}). \end{aligned} \tag{4.8}$$

Taking $\Omega_1 = \{x \in \Omega : |\nabla u_\varepsilon(x)| > 1\}$, using that $p(x)$ is Lipschitz and Hölder's inequality, we have

$$\| \ln(v_\varepsilon)\nabla u_\varepsilon \nabla p\|_{L^2(\Omega)} \le C\| \ln^2(v_\varepsilon)\nabla u_\varepsilon\|^{1/2}_{L^{p'(\cdot)}(\Omega_1)}\|\nabla u_\varepsilon\|^{1/2}_{L^{p(\cdot)}(\Omega_1)} + C. \tag{4.9}$$

On the other hand, since $q(x) \ge q_1 > 2$, we have that $q'(x) \le p^*(x)$. Then, as $\|f_\varepsilon\|_{L^{q(\cdot)}(\Omega)}$ and $\|g_\varepsilon\|_{H^2(\Omega)}$ are bounded independent of ε, using Lemma 4.1 we conclude that $\|\nabla u_\varepsilon\|_{L^{p(\cdot)}(\Omega)}$ is uniformly bounded.

Observe that, for all $s > 0$ there exist a constant $C > 0$ such that

$$\ln(v_\varepsilon) \le C v_\varepsilon^{s/2} < C|\nabla u_\varepsilon|^{s/2} \quad \text{in } \Omega_1,$$

thus

$$\begin{aligned}\| \ln^2(v_\varepsilon)|\nabla u_\varepsilon|\|_{L^{p'(\cdot)}(\Omega_1)} &\le C\| |\nabla u_\varepsilon|^{1+s}\|_{L^{p'(\cdot)}(\Omega_1)} \\ &\le C\|\nabla u_\varepsilon\|^{(1+s)}_{L^{p'(\cdot)(1+s)}(\Omega_1)} \\ &\le C\|u_\varepsilon\|^{(1+s)}_{H^2(\Omega_1)}.\end{aligned}$$

In the last line, we are using that $2^* = \infty$, since $N = 2$.

Then, by the last inequality, (4.8) and (4.9), we get

$$\|u_\varepsilon\|_{H^2(\Omega)} \le C\left(\|u_\varepsilon\|^{(1+s)/2}_{H^2(\Omega)} + \|f_\varepsilon v_\varepsilon^{2-p}\|_{L^2(\Omega)} + 1\right). \tag{4.10}$$

Taking

$$A_1 = \{x \in \Omega : p(x) = 2\} \quad \text{and} \quad A_2 = \{x \in \Omega : p(x) < 2\}$$

and using that $f_\varepsilon \equiv 0$ in $\{x \in \Omega : p(x) > 2\}$, we have that

$$\|f_\varepsilon v_\varepsilon^{2-p}\|_{L^2(\Omega)} \le \|f_\varepsilon\|_{L^2(A_1)} + \|f_\varepsilon v_\varepsilon^{2-p}\|_{L^2(A_2)}.$$

Since $\|f_\varepsilon\|_{L^2(A_1)}$ is bounded, to prove that $\{u_\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $H^2(\Omega)$, we only have to find a bound of $\|f_\varepsilon v_\varepsilon^{2-p}\|_{L^2(A_2)}$.

Let as define in A_2 the function

$$\widetilde{q}(x) = \begin{cases} \frac{1}{2p(x)-3} + 1 & \text{if} \quad \frac{1}{q(x)} + \frac{3}{2} \le p(x) < 2, \\ \frac{q(x)}{2} + 1 & \text{if} \quad p(x) < \frac{1}{q(x)} + \frac{3}{2}. \end{cases}$$

It is easy to see that $2 < \widetilde{q}(x) \le q(x)$ for any $x \in A_2$.

On the other hand, let us denote $\mu(x) = \frac{2\widetilde{q}(x)}{\widetilde{q}(x)-2}$ and $\gamma(x) = \mu(x)(2-p(x))$ then

$$1 < 1 + \frac{2}{q_2} \le \gamma(x) \le \max\left\{2, 2 + \frac{8}{q_1 - 2}\right\} \quad \forall x \in A_2.$$

Now, using Hölder's inequality with exponent $\widetilde{q}(x)/2$, we have

$$\|f_\varepsilon v_\varepsilon^{2-p}\|_{L^2(A_2)} \le C\|f_\varepsilon\|_{L^{\widetilde{q}(\cdot)}(A_2)}\|v_\varepsilon^{2-p}\|_{L^{\mu(\cdot)}(A_2)}. \tag{4.11}$$

Then, if $\|v_\varepsilon\|_{L^{\gamma(\cdot)}(A_2)} \le 1$ we have $\|v_\varepsilon^{2-p}\|_{L^{\mu(\cdot)}(A_2)} \le 1$ and since $\widetilde{q}(x) \le q(x)$ we get

$$\|f_\varepsilon v_\varepsilon^{2-p}\|_{L^2(A_2)} \le C.$$

If $\|v\|_{L^{\gamma(\cdot)}(A_2)} \ge 1$, we have

$$\|v_\varepsilon^{2-p}\|_{L^{\mu(\cdot)}(A_2)} \le \|v_\varepsilon\|^{2-p_1}_{L^{\gamma(\cdot)}(A_2)} \le C(1 + \|\nabla u_\varepsilon\|^{2-p_1}_{L^{\gamma(\cdot)}(A_2)}), \tag{4.12}$$

where in the last inequality we are using that $\varepsilon \leq 1$.

Since $2^* = \infty$ and $1 < \gamma_1 \leq \gamma(x) \leq \gamma_2 < \infty$, by the Sobolev embedding inequality, we have that

$$\|\nabla u_\varepsilon\|^{2-p_1}_{L^{\gamma(\cdot)}(A_2)} \leq C\|u_\varepsilon\|^{2-p_1}_{H^2(A_2)} \leq C\|u_\varepsilon\|^{2-p_1}_{H^2(\Omega)}.$$

Combining this last inequality with inequalities (4.12), (4.11), (4.10) and the fact that $\widetilde{q}(x) \leq q(x)$, we get

$$\|u_\varepsilon\|_{H^2(\Omega)} \leq C(\|u_\varepsilon\|^{(1+s)/2}_{H^2(\Omega)} + \|u_\varepsilon\|^{2-p_1}_{H^2(\Omega)} + 1).$$

Finally, we get that for any $0 < s < 1$ there exist a constant $C = C(p, g, f, s)$ such that

$$\|u_\varepsilon\|_{H^2(\Omega)} \leq C.$$

Then, there exist a subsequence still denoted $\{u_\varepsilon\}_{\varepsilon\in(0,1]}$ and $u \in H^1(\Omega)$ such that

$$u_\varepsilon \to u \text{ strongly in } H^1(\Omega),$$
$$u_\varepsilon \rightharpoonup u \text{ weakly in } H^2(\Omega),$$

It is clear that u satisfies the boundary condition.

Lastly, by Proposition 3.2 in [2], there exist a constant $M > 0$ independent of ε such that,

$$|(\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p(x)-2}{2}}\nabla u_\varepsilon - (\varepsilon + |\nabla u|^2)^{\frac{p(x)-2}{2}}\nabla u| \leq M|\nabla(u_\varepsilon - u)|^{p(x)-1} \tag{4.13}$$

for all $x \in \Omega$. Then, passing to the limit in the weak formulation of (3.2) and using the above inequality, we have that

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\varphi\, dx = \int_\Omega f\varphi\, dx$$

for any $\varphi \in C_0^\infty(\Omega)$. Therefore $u \in H^2(\Omega)$ and solves (1.1). □

Now, we are able to prove the theorem.

Proof of Theorem 1.1. First, we consider the case $p \in C^1(\overline{\Omega})$. Let $p_\varepsilon \in C^\infty(\overline{\Omega})$ such that $p_\varepsilon \to p$ in $C^1(\Omega)$. Now, we define

$$f_\varepsilon(x) = \begin{cases} f(x) & \text{if } p_\varepsilon(x) \leq 2, \\ 0 & \text{if } p_\varepsilon(x) > 2. \end{cases} \tag{4.14}$$

Observe that $f_\varepsilon \to f$ in $L^{q(\cdot)}(\Omega)$ as $\varepsilon \to 0$.

Then, by Theorem 4.2, the solution u_ε of (1.1) (with p_ε and f_ε instead of p and f) is bounded in $H^2(\Omega)$ by a constant independent of ε. Therefore, there exist a subsequence still denoted $\{u_\varepsilon\}_{\varepsilon\in(0,1]}$ and $u \in H^2(\Omega)$ such that

$$\begin{aligned} u_\varepsilon &\to u && \text{in } H^1(\Omega), \\ u_\varepsilon &\rightharpoonup u && \text{weakly in } H^2(\Omega). \end{aligned} \tag{4.15}$$

It remains to prove that u is a solution of (1.1). Let $\varphi \in C_0^\infty(\Omega)$, then

$$\begin{aligned} \int_\Omega f_\varepsilon \varphi \, dx &= \int_\Omega |\nabla u_\varepsilon|^{p_\varepsilon(x)-2} \nabla u_\varepsilon \nabla \varphi \, dx \\ &= \int_\Omega |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \nabla \varphi \, dx \\ &+ \int_\Omega (|\nabla u_\varepsilon|^{p_\varepsilon(x)-2} - |\nabla u_\varepsilon|^{p(x)-2}) \nabla u_\varepsilon \nabla \varphi \, dx. \end{aligned} \tag{4.16}$$

Therefore, using that $H^2(\Omega) \hookrightarrow W^{1,p(\cdot)}(\Omega)$ compactly, we have that

$$\int_\Omega |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon \nabla \varphi \, dx \to \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \varphi \, dx. \tag{4.17}$$

On the other hand, we have

$$|\nabla u_\varepsilon(x)|^{p_\varepsilon(x)-1} - |\nabla u_\varepsilon(x)|^{p(x)-1} = |\nabla u_\varepsilon(x)|^{b_\varepsilon(x)} \log(|\nabla u_\varepsilon(x)|)(p_\varepsilon(x) - p(x)),$$

where $b_\varepsilon(x) = p_\varepsilon(x)\theta + (1-\theta)p(x) - 1$ for some $0 < \theta < 1$. Therefore, using that $2^* = \infty$ and that $p_\varepsilon \to p$ uniformly, we obtain

$$\int_\Omega (|\nabla u_\varepsilon|^{p_\varepsilon(x)-2} - |\nabla u_\varepsilon|^{p(x)-2}) \nabla u_\varepsilon \nabla \varphi \, dx \to 0. \tag{4.18}$$

Then, using that $f_\varepsilon \to f$ in $L^{q(\cdot)}(\Omega)$, (4.16), (4.17) and the (4.18) we conclude that u is a solution of (1.1).

Now, we consider the case $p \in \mathrm{Lip}(\overline{\Omega})$. By Lemmas B.1 and B.2 there exists $p_\varepsilon \in C^1(\overline{\Omega})$ such that $|\Omega \setminus \Omega_0| < \varepsilon$ where

$$\Omega_0 = \{x \in \Omega \colon p_\varepsilon(x) = p(x) \text{ and } \nabla p_\varepsilon(x) = \nabla p(x)\}.$$

We define f_ε as in (4.14). Then, the solution u_ε of (1.1) with p_ε and f_ε instead of p and f is bounded in $H^2(\Omega)$ by a constant independent of ε. Therefore there exist a subsequence still denoted $\{u_\varepsilon\}_{\varepsilon \in (0,1]}$ and $u \in H^2(\Omega)$ satisfying (4.15).

Lastly, we prove that u is a solution of (1.1). Let $\varphi \in C_0^\infty(\Omega)$. By Hölder inequality, since $2^* = \infty$ and by (3) of Lemma B.2 we have

$$\begin{aligned} &\int_{\Omega\setminus\Omega_0} (|\nabla u_\varepsilon|^{p_\varepsilon(x)-2} - |\nabla u_\varepsilon|^{p(x)-2}) \nabla u_\varepsilon \nabla \varphi \, dx \\ &\quad \le C(\|\nabla u_\varepsilon\|_{L^{p_\varepsilon}(\Omega)} \|1\|_{L^{p_\varepsilon}(\Omega\setminus\Omega_0)} + \|\nabla u_\varepsilon\|_{L^p(\Omega)} \|1\|_{L^p(\Omega\setminus\Omega_0)}) \\ &\quad \le C\|u_\varepsilon\|_{H^2(\Omega)} (\|1\|_{L^{p_\varepsilon}(\Omega\setminus\Omega_0)} + \|1\|_{L^p(\Omega\setminus\Omega_0)}). \end{aligned}$$

Then, since $\|u_\varepsilon\|_{H^2(\Omega)}$ is bounded independent of ε and $|\Omega \setminus \Omega_0| < \varepsilon$ we obtain that

$$\int_{\Omega\setminus\Omega_0} (|\nabla u_\varepsilon|^{p_\varepsilon(x)-2} - |\nabla u_\varepsilon|^{p(x)-2}) \nabla u_\varepsilon \nabla \varphi \, dx \to 0.$$

Therefore, since (4.16), (4.17) again hold, using that $f_\varepsilon \to f$ in $L^{q(\cdot)}(\Omega)$, and the above equation, we conclude that u is a solution of (1.1). $\square$

5. The convex case

Lastly, we want to prove that the solution is in $H^2(\Omega)$ if we only assume that $\partial\Omega$ is convex. We want to remark here that this result generalize the one in Theorem 2.2 in [22] in two ways. In that paper the authors consider the case $p = constant$ and $g = 0$. Instead, we are allowed to cover the case where g is any function in $H^2(\Omega)$ and $p(x) \in \mathrm{Lip}(\overline{\Omega})$.

Remark 5.1. Let Ω be a convex set and $p : \Omega \to [1, \infty)$ be log −continuous in $\overline{\Omega}$. Then, there exists a sequence $\{\Omega_m\}_{m\in\mathbb{N}}$ of convex subset of Ω with C^2 boundary such that $\Omega_m \subset \Omega_{m+1}$ for any $m \in \mathbb{N}$ and $|\Omega \setminus \Omega_m| \to 0$.

(1) Then, there exists a constant C depending on $p(x), |\Omega|$ such that

$$\|v\|_{L^{p(\cdot)}(\Omega_m)} \le C\|\nabla v\|_{L^{p(\cdot)}(\Omega_m)} \quad \forall v \in W_0^{1,p(\cdot)}(\Omega_m),$$

for any $m \in \mathbb{N}$. This follows by Theorem 3.3 in [21], using that $\Omega_m \subset \Omega_{m+1}$ for any $m \in \mathbb{N}$.

(2) The Lipschitz constants of Ω_m ($m \in \mathbb{N}$) are uniformly bounded (see Remark 2.3 in [22]). Therefore, the extension operators

$$E_{1,m} : W^{1,p(\cdot)}(\Omega_m) \to W^{1,p(\cdot)}(\Omega) \text{ and } E_{2,m} : H^2(\Omega_m) \to H^2(\Omega)$$

define as Theorem 4.2 in [11] satisfy that $\|E_{1,m}\|$ and $\|E_{2,m}\|$ are uniformly bounded.

(3) By (2) and Corollary 8.3.2 in [12], there exists a constant C independent of m such that

$$\|v\|_{L^{p^*(\cdot)}(\Omega_m)} \le C\|v\|_{W^{1,p(\cdot)}(\Omega_m)} \quad \forall v \in W^{1,p(\cdot)}(\Omega_m),$$

for any $m \in \mathbb{N}$.

We want to remark that all the constants of the above inequalities are independent of p_1 (see Section 6 for the applications).

Proof of Theorem 1.2. We begin taking $\{\Omega_m\}_{m\in\mathbb{N}}$ as in Remark 5.1 and u_m the solution of

$$\begin{cases} -\Delta_{p(x)} u_m = f & \text{in } \Omega_m, \\ u_m = g & \text{on } \partial\Omega_m. \end{cases}$$

By Theorem 1.1, $u_m \in H^2(\Omega_m)$ for any $m \in \mathbb{N}$. Moreover, u_m solves

$$\begin{cases} L^m u_m = a^m_{ij}(x) u_{m,x_i x_j} = a^m(x) & \text{in } \Omega_m, \\ u_m = g & \text{on } \partial\Omega_m, \end{cases}$$

with

$$a^m_{ij}(x) = \delta_{ij} + (p(x) - 2)\frac{u_{m,x_i}(x) u_{m,x_j}(x)}{|\nabla u_m(x)|^2},$$
$$a^m(x) = \ln(|\nabla u_m(x)|)\langle \nabla u_m(x), \nabla p(x)\rangle + f(x)|\nabla u_m(x)|^{2-p(x)}.$$

Then $v_m = u_m - g$ solves

$$\begin{cases} L^m v_m = -L^m g + a^m(x) & \text{in } \Omega_m, \\ v_m = 0 & \text{on } \partial\Omega_m. \end{cases}$$

Thus, using that $v_m \in H^2(\Omega_m) \cap H_0^1(\Omega_m)$ and since the coefficients $a_{ij}^m(x)$ are bounded independent of m, we can argue as in Theorem 2.2 in [22] and obtain,

(5.19)
$$\begin{aligned}\|v_m\|_{H^2(\Omega_m)} &\leq C\| - L^m g + f|\nabla u_m|^{2-p(\cdot)} + \ln(|\nabla u_m|)|\nabla u_m|\|_{L^2(\Omega_m)} \\ &\leq C\left(\||\nabla u_m|^{2-p(\cdot)}\|_{L^2(\Omega_m)} + \|\ln(|\nabla u_m|)|\nabla u_m|\|_{L^2(\Omega_m)} + 1\right)\end{aligned}$$

where the constant C is independent of m.

As in Lemma 4.1 we can prove, using Remark 5.1 (1) and (3), that the norms $\|\nabla u_m\|_{L^{p(\cdot)}(\Omega_m)}$ are uniformly bounded. Therefore, proceeding as in Theorem 4.2 we obtain

$$\begin{aligned}&\|\ln(|\nabla u_m|)|\nabla u_m|\|_{L^2(\Omega_m)} + \|f|\nabla u_m|^{2-p}\|_{L^2(\Omega_m)} \\ &\leq C\left(\|\nabla u_m\|^{(1+s)/2}_{L^{p'(\cdot)(1+s)}(\Omega_{1,m})} + \|\nabla u_m\|^{2-p_1}_{L^{\gamma(\cdot)}(A_{2,m})} + 1\right),\end{aligned} \tag{5.20}$$

with C independent of m, where

$$\Omega_{1,m} = \{x \in \Omega_m : |\nabla u_m(x)| > 1\} \text{ and } A_{2,m} = \{x \in \Omega_m : p(x) < 2\}.$$

Now, using Remark 5.1 (3) and (2), we have that for any $r > 1$ that

$$\begin{aligned}\|v_m\|_{W^{1,r}(\Omega_m)} &\leq \|E_{2,m} v_m\|_{W^{1,r}(\Omega)} \\ &\leq C\|E_{2,m} v_m\|_{H^2(\Omega)} \\ &\leq C\|v_m\|_{H^2(\Omega_m)}\end{aligned} \tag{5.21}$$

where C is independent of m.

Therefore, using (5.19), (5.20) and (5.21), we get

$$\begin{aligned}\|v_m\|_{H^2(\Omega_m)} \leq& C(\|v_m\|^{(1+s)/2}_{H^2(\Omega_m)} + \|v_m\|^{2-p_1}_{H^2(\Omega_m)} + \|g\|^{(1+s)/2}_{H^2(\Omega_m)} + \|g\|^{2-p_1}_{H^2(\Omega_m)} + 1) \\ \leq& C(\|v_m\|^{(1+s)/2}_{H^2(\Omega_m)} + \|v_m\|^{2-p_1}_{H^2(\Omega_m)} + 1),\end{aligned}$$

where the constant C is independent of m. This proves that $\{\|v_m\|_{H^2(\Omega_m)}\}_{m\in\mathbb{N}}$ is bounded.

Now we have, as in the proof of Theorem 2.2 in [22], that there exist a subsequence still denote $\{v_m\}_{m\in\mathbb{N}}$ and a function $v \in H^2(\Omega) \cap H_0^1(\Omega)$ such that,

$$v_m \to v \quad \text{strongly in } H^1(\Omega')$$

for any $\Omega' \subset\subset \Omega$. Then $u = v + g \in H^2(\Omega)$ and

$$u_m \to u \quad \text{strongly in } H^1(\Omega')$$

for any $\Omega' \subset\subset \Omega$. Thus, using (4.13), we have

$$|\nabla u_m|^{p(x)-2}\nabla u_m \to |\nabla u|^{p(x)-2}\nabla u \quad \text{strongly in } L^{p'(\cdot)}(\Omega') \tag{5.22}$$

for any $\Omega' \subset\subset \Omega$.

On the other hand, for any $\varphi \in C_0^\infty(\Omega)$ there exist m_0 such that for all $m \geq m_0$

$$\int_{\Omega_m} |\nabla u_m|^{p(x)-2}\nabla u_m \nabla\varphi\, dx = \int_{\Omega_m} f\varphi\, dx.$$

Therefore, using (5.22) we have that u is a weak solution of (1.1). □

Proof of Corollary 1.3. By the previous theorem we have that $u \in H^2(\Omega)$, then we can derive the equation (1.1) and obtain

$$\begin{cases} -a_{ij}(x)u_{x_ix_j} = a(x) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where

$$a_{ij}(x) = \delta_{ij} + (p(x)-2)\frac{u_{x_i}(x)u_{x_j}(x)}{|\nabla u(x)|^2},$$
$$a(x) = \ln(|\nabla u(x)|)\langle \nabla u(x), \nabla p(x)\rangle + f(x)|\nabla u(x)|^{2-p(x)}.$$

Using that $f \in L^{q(\cdot)}(\Omega)$ with $q(x) \geq q_1 > 2$ and following the lines in the proof of Theorem 4.2, we have that $a(x) \in L^s(\Omega)$ with $s > 2$. Therefore, by Remark A.3, we have that $u \in C^{1,\alpha}(\overline{\Omega})$. □

6. Comments

In the image processing problem it is of interest the case where p_1 is close to 1. By this reason, we are also interested in the dependence of the H^2-norm on p_1.

If $N = 2$, $g \in H^2(\Omega)$ and u_ε is the solution of (3.2), we have by Lemma A.1, (3.6) and (3.7), that there exists a constant C independent of p_1 and ε such that

$$\|u_\varepsilon\|_{H^2(\Omega)} \leq \frac{C}{(p_1-1)^\kappa}\left(\|a_\varepsilon\|_{L^2(\Omega)} + \|g\|_{H^2(\Omega)}\right),$$

where $\kappa = 1$ if Ω is convex and $\kappa = 2$ if $\partial\Omega \in C^2$. Therefore, using that the Poincare's inequality and the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega)$ hold in the case $p_1 = 1$ and following the lines of Theorem 1.1 and Theorem 1.2 we have that

$$\|u\|_{H^2(\Omega)} \leq \frac{C}{(p_1-1)^\kappa},$$

where the constant C is independent of p_1.

Appendix A. Regularity results for elliptic linear equations with coefficients in L^∞

Let Ω be an bounded open subset of $\mathbb{R}^2$ and

$$\mathcal{M}u = a_{ij}(x)u_{x_ix_j},$$

such that $a_{ij} = a_{ji}$ and for any $\xi \in \mathbb{R}^N$

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \tag{A.1}$$

and

$$M_1 \leq a_{11}(x) + a_{22}(x) \leq M_2 \quad \text{in } \Omega \tag{A.2}$$

where λ, Λ, M_1 and M_2 are positive constant.

In the next lemma, we will give a H^2-bound for solutions of

$$\begin{cases} \mathcal{M}u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{A.3}$$

In fact, the following result is proved in Theorem 37,III in [23], but it is not explicit the dependence of the bounds on the ellipticity and the $L^\infty-$norm of $(a_{ij}(x))$. Then, following the proof of the mentioned theorem we can prove

Lemma A.1. *Let Ω be a bounded domain in $\mathbb{R}^2$, $f \in L^2(\Omega)$ and $g \in H^2(\Omega)$. Then, if u is a solution of* (A.3) *and $u \in H^2(\Omega)$ we have that*

$$\|u\|_{H^2(\Omega)} \le \frac{C}{\lambda^\kappa}\left(\|f\|_{L^2(\Omega)} + \|g\|_{H^2(\Omega)}\right),$$

where $\kappa = 1$ if Ω is convex and $\kappa = 2$ if $\partial\Omega \in C^2$ and C is a constant independent of λ.

Proof. In this proof, we denote $u_{ij} = u_{x_i x_j}$ for all $i, j = 1, 2$ and C is a constant independent of λ.

First, we consider the case $g \equiv 0$. Using (A.1), we have that

$$\begin{aligned}(a_{11}(x) + a_{22}(x))(u_{12}^2 - u_{11}u_{22}) = &\sum_{i,j,k=1}^{2} a_{ij}u_{ki}u_{kj} - \Delta u \sum_{ij=1}^{2} a_{ij}u_{ij} \\ \ge &\lambda \sum_{ik=1}^{2} u_{ki}^2 - \Delta u f(x).\end{aligned}$$

Then, using Young's inequality, we get

$$\frac{\lambda}{2(a_{11}(x) + a_{22}(x))}\sum_{ik=1}^{2} u_{ki}^2 \le \frac{4}{\lambda(a_{11}(x) + a_{22}(x))} f(x)^2 + u_{12}^2 - u_{11}u_{22},$$

and by (A.2), we have that

$$\sum_{ik=1}^{2} u_{ki}^2 \le \frac{C}{\lambda^2} f(x)^2 + \frac{C}{\lambda}(u_{12}^2 - u_{11}u_{22}), \tag{A.4}$$

Now, using (37.4) and (37.6) in [23], we have that for any $u \in H^2(\Omega)$

$$\int_\Omega (u_{12}^2 - u_{11}u_{22})\, dx = -\int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\right)^2 \frac{H}{2} ds \tag{A.5}$$

where H is the curvature of $\partial\Omega$. If Ω is convex, then $H \ge 0$ and therefore, using (A.4) and (A.5) we have that

$$\|D^2 u\|_{L^2(\Omega)} \le \frac{C}{\lambda}\|f\|_{L^2(\Omega)}. \tag{A.6}$$

In the general case, we can use the following inequality

$$\int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\right)^2 ds \le C\left((1 + \delta^{-1})\int_\Omega |\nabla u|^2\, dx + \delta \int_\Omega \sum_{ik=1}^{2} u_{ki}^2\, dx\right) \tag{A.7}$$

for any $\delta > 0$. See equation (37.6) of [23].

Then, by (A.4), (A.5), using that H is bounded and (A.7) (choosing δ properly) we arrive to

$$\int_\Omega \sum_{ik=1}^{2} u_{ki}^2\, dx \le \frac{C}{\lambda^2}\left(\int_\Omega f(x)^2\, dx + \int_\Omega |\nabla u|^2\, dx\right). \tag{A.8}$$

On the other hand, using that $Lu = f$ in Ω, (A.1) and the Poincare's inequality, we have

$$\|\nabla u\|_{L^2(\Omega)} \leq \frac{C}{\lambda}\|f\|_{L^2(\Omega)}. \tag{A.9}$$

Therefore, by (A.8) and (A.9), we get

$$\|D^2 u\|_{L^2(\Omega)} \leq \frac{C}{\lambda^2}\|f\|_{L^2(\Omega)}.$$

Thus, by the last inequality, (A.9) and (A.6) the lemma is proved in the case $g = 0$.

When g is any function in $H^2(\Omega)$ the lemma follows taking $v = u - g$. □

The following theorem is proved in Corollary 8.1.6 in [20].

Theorem A.2. *Let Ω be a convex polygonal domain in $\mathbb{R}^2$, $\mathcal{M}$ satisfying* (A.1) *and $u \in H^2(\Omega) \cap H_0^1(\Omega)$ be a solution of* (A.3) *with $g = 0$ and $f \in L^p(\Omega)$ with $p > 2$. Then $\nabla u \in C^\mu(\overline{\Omega})$ for some $0 < \mu < 1$.*

Remark A.3. Observe that the above Theorem holds also if we consider any $g \in W^{2,p}(\Omega)$, since we can take $v = u - g$ in (A.3) and use that $W^{2,p}(\Omega) \hookrightarrow C^{1,1-2/p}(\overline{\Omega})$.

Appendix B. Lipschitz Functions

Using the linear extension operator define in [14], we have the following lemma

Lemma B.1. *Let Ω be a bounded open domain with Lipschitz boundary and $f \in \mathrm{Lip}(\overline{\Omega})$. Then, there exists a function $\overline{f} : \mathbb{R}^N \to \mathbb{R}$ such that $\overline{f}$ is a Lipschitz function, $\sup_{\mathbb{R}^N} \overline{f} = \inf_{\overline{\Omega}} f$ and $\inf_{\mathbb{R}^N} \overline{f} = \max_{\overline{\Omega}} f$.*

Lemma B.2. *Let $f : \mathbb{R}^N \to \mathbb{R}$ be Lipschitz function. Then for each $\varepsilon > 0$, there exists a C^1 function $f_\varepsilon : \mathbb{R}^N \to \mathbb{R}$ such that*

1. *$|\{x \in \mathbb{R}^N : f_\varepsilon(x) \neq f(x)$ or $Df_\varepsilon(x) \neq Df(x)\}| \leq \varepsilon$.*
2. *There exist a constant C depending only on N such that,*
$$\|Df_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \leq CLip(f).$$
3. *If $1 < f_1 \leq f(x) \leq f_2$ in $\mathbb{R}^N$, we have*
$$1 < f_\varepsilon(x) \leq f_2 + C\varepsilon^{\frac{1}{N}} \text{ in } \mathbb{R}^N$$
with C a constant depending only on N.

Proof. Items (1) and (2) follow by Theorem 1, pag. 251 in [16].

To prove (3), let as define

$$\Omega_0 = \{x \in \mathbb{R}^N : f_\varepsilon(x) = f(x) \text{ and } Df_\varepsilon(x) = Df(x)\}$$

and let as suppose that there exist $x \in \mathbb{R}^N \setminus \Omega_0$ such that $f_\varepsilon(x) = f_2 + \delta$ with $\delta > 0$. If $x_0 \in \Omega_0$, by (2), we have

$$CLip(f)|x - x_0| \geq f_\varepsilon(x) - f_\varepsilon(x_0) = f_2 + \delta - f(x_0) \geq \delta.$$

Then $B_\rho(x) \subset \mathbb{R}^N \setminus \Omega_0$ where $\rho = \delta(CLip(f))^{-1}$ and using (1) we get $\delta \leq C\varepsilon^{1/N}$, for some constant C independent of ε.

Analogously we can prove the other inequality. □

References

1. Emilio Acerbi and Giuseppe Mingione , *Regularity results for a class of functionals with non-standard growth*,Arch. Ration. Mech. Anal. **156** (2001), no. 2, 121–140.
2. Jacques Baranger and Khalid Najib, *Analyse numérique des écoulements quasi-newtoniens dont la viscosité obéit à la loi puissance ou la loi de carreau*, Numer. Math. **58** (1990), no. 1, 35–49.
3. Erik M. Bollt, Rick Chartrand, Selim Esedoḡlu, Pete Schultz, and Kevin R. Vixie, *Graduated adaptive image denoising: local compromise between total variation and isotropic diffusion*, Adv. Comput. Math. **31** (2009), no. 1-3, 61–85.
4. S. Challal and A. Lyaghfouri, *Second order regularity for the p(x)-laplace operator*, Math. Nachr. **284** (2011), no. 10, 12701279.
5. Yunmei Chen, Stacey Levine, and Murali Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math. **66** (2006), no. 4, 1383–1406 (electronic).
6. Ph. Ciarlet, *The finite element method for elliptic problems*, vol. 68, North-Holland, Amsterdam, 1978.
7. A. Coscia and G. Mingione *Holder continuity of the gradient of p(x) harmonic mappings*, Comptes Rendus de l'Acadmie des Sciences, Ser. I, Mathematique 328 (1999) 363-368.
8. L. M. Del Pezzo, A. Lombardi, and S. Martínez, *IP-DGFEM method for the $p(x)$-Laplacian*, Preprint. http://arxiv.org/abs/1009.2063.
9. L Diening, *Theoretical and numerical results for electrorheological fluids,*, Ph.D. thesis, University of Freiburg, Germany (2002).
10. L. Diening, *Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$*, Math. Inequal. Appl. **7** (2004), no. 2, 245–253.
11. ———, *Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$*, Math. Nachr. **268** (2004), 31–43.
12. L. Diening, P Harjulehto, P. Hästö, and M. Ruzicka, *Lebesgue and sobolev spaces with variable exponents*, Lecture Notes in Mathematics, vol. 2017, Springer-Verlag, New York, 2011.
13. L. Diening, P. Hästö, and A. Nekvinda, *Open problems in variable exponent Lebesgue and Sobolev spaces*, Function Spaces, Differential Operators and Nonlinear Analysis, Milovy, Math. Inst. Acad. Sci. Czech Republic, Praha, 2005.
14. David E. Edmunds and Jiří Rákosník, *Sobolev embeddings with variable exponent*, Studia Math. **143** (2000), no. 3, 267–293.
15. L. Esposito, F. Leonetti and G. Mingione *Sharp regularity for functionals with (p,q) growth* Journal of Differential Equations 204 (2004) 5-55.
16. L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.
17. Xianling Fan, *Global $C^{1,\alpha}$ regularity for variable exponent elliptic equations in divergence form*, J. Differential Equations **235** (2007), no. 2, 397–417.
18. Xianling Fan and Dun Zhao, *A class of De Giorgi type and Hölder continuity*, Nonlinear Anal. **36** (1999), no. 3, Ser. A: Theory Methods, 295–318.
19. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 224, Springer-Verlag, Berlin, 1983.
20. P. Grisvard, *Elliptic problems in nonsmooth domains*, Monographs and Studies in Mathematics, vol. 24, Pitman (Advanced Publishing Program), Boston, MA, 1985.
21. Kováčik and Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J **41** (1991), 592–618.
22. W. B. Liu and John W. Barrett, *A remark on the regularity of the solutions of the p-Laplacian and its application to their finite element approximation*, J. Math. Anal. Appl. **178** (1993), no. 2, 470–487.
23. Carlo Miranda, *Partial differential equations of elliptic type*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 2, Springer-Verlag, New York, 1970, Second revised edition. Translated from the Italian by Zane C. Motteler.

24. Michael Růžička, *Electrorheological fluids: modeling and mathematical theory*, Lecture Notes in Mathematics, vol. 1748, Springer-Verlag, Berlin, 2000.
25. S. Samko, *Denseness of* $C_0^\infty(\mathbf{R}^N)$ *in the generalized Sobolev spaces* $W^{M,P(X)}(\mathbf{R}^N)$, Direct and inverse problems of mathematical physics (Newark, DE, 1997), Int. Soc. Anal. Appl. Comput., vol. 5, Kluwer Acad. Publ., Dordrecht, 2000, pp. 333–342.

Leandro M. Del Pezzo
CONICET and Departamento de Matemática, FCEyN, UBA,
Pabellón I, Ciudad Universitaria (1428), Buenos Aires, Argentina.
E-mail address: ldpezzo@dm.uba.ar
Web page: http://cms.dm.uba.ar/Members/ldpezzo

Sandra Martínez
IMAS-CONICET and Departamento de Matemática, FCEyN, UBA,
Pabellón I, Ciudad Universitaria (1428),Buenos Aires, Argentina.
E-mail address: smartin@dm.uba.ar