Dynkin game under ambiguity in continuous time

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Abstract

In this paper, we want to investigate some kind of Dynkin's game under ambiguity which is represented by Backward Stochastic Differential Equation (shortly BSDE) with standard generator function $g(t, y, z)$. Under regular assumptions, a pair of saddle point can be obtained and the existence of the value function follows. The constrained case is also treated in this paper.

Keywords: Ambiguity, BSDE, Dynkin game, RBSDE.

1 Introduction

Dynkin's stopping games was first introduced and studied by Dynkin in [3], and was generalized in J.Neveu [9], N.V.EIbakidze[11], Yu.I.Kifer [18] , Y.Ohtsubo [19], [20], [21] etc. with discrete parameter with or without a finite constraint. The continuous time version was also studied in many literature (for examples, H.Morimoto [5], L.Stettner [10] and N.V.Krylov [12] etc.). We want to investigate some kind of Dynkin's game under ambiguity in continuous time in this paper.

A general formulation of Dynkin's game states as follows. Define the lower and upper value function as

$$\underline{V}_t := ess \sup_{\tau \in \mathcal{T}_t} ess \inf_{\sigma \in \mathcal{T}_t} E[R_t(\tau, \sigma)|\mathcal{F}_t], \tag{1.1}$$

and

$$\overline{V}_t := ess \inf_{\sigma \in \mathcal{T}_t} ess \sup_{\tau \in \mathcal{T}_t} E[R_t(\tau, \sigma)|\mathcal{F}_t]. \tag{1.2}$$

where $R_t(\tau, \sigma)$ is a function of two stopping times τ and σ satisfying some suitable assumptions. One often try to find sufficient conditions when $\overline{V}_t = \underline{V}_t$ holds. It is easy to see that $\overline{V}_t \geq \underline{V}_t$, to get the reverse inequality, one often look for a pair of saddle point (τ_t^*, σ_t^*) for which

$$E[R_t(\tau, \sigma_t^*)|\mathcal{F}_t] \leq E[R_t(\tau_t^*, \sigma_t^*)|\mathcal{F}_t] \leq E[R_t(\tau_t^*, \sigma)|\mathcal{F}_t] \tag{1.3}$$

holds for any σ and τ taking values in t and T. If (1.3) is true, then $V(t) := \overline{V}_t = \underline{V}_t$ by the definition of (1.1) and (1.2) and $V(t)$ is called as the value function of the Dynkin's stopping game.

There are many ways to solve this game. Since stopping game is an extension of optimal stopping problem, the martingale approach is a nice choice. In fact, we can find a pair of saddle point and the value function by solving a double optimal stopping problem, for reference see E.B.Dynkin [3], N.V.Krylov [12] etc. Since Reflected Backward Stochastic Differential Equation (shortly for RBSDE) with lower barrier has been proved useful to solve optimal stopping problem, some author find out a way to solve Dynkin's game by solving RBSDE with lower and upper obstacles in J.Cvitanic; I.Karatzas [8], S.Hamadène; J.-P.Lepeltier [17] etc. Moreover, A.Bensoussan; A.Friedman [2] and A.Friedman [1] developed the analytical theory

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of stochastic differential games with stopping times in Markov setting, they studied the value and saddle-points of such a game using appropriate partial differential equations , variational inequalities, and free-boundary problems. Of course, there are still other ways to solve this game such as by pathwise approach (see I.KaratzasSource [7]) and by connection with singular control problem (see I.Karatzas; H.Wang [6]).

Inspired by J.Cvitanic; I.Karatzas [8], in this paper we want to study a similar Dynkin's game in the stochastic environment with ambiguity and we evaluate the reward process by nonlinear g-expectations. More explicitly, our problem can be formulated as follows. We define the lower value function and the upper value function as

$$\underline{V}_t := ess \sup_{\tau\in\mathcal{T}_t} ess \inf_{\sigma\in\mathcal{T}_t} \mathcal{E}_t^g[R(\tau,\sigma)] \tag{1.4}$$

and

$$\overline{V}_t = ess \inf_{\sigma\in\mathcal{T}_t} ess \sup_{\tau\in\mathcal{T}_t} \mathcal{E}_t^g[R(\tau,\sigma)] \tag{1.5}$$

respectively.

Where $R(\tau,\sigma) := L(\tau)1_{(\tau\leq\sigma)}+U(\sigma)1_{(\sigma<\tau)}$ and $\mathcal{T}_t$ are stopping times taking values between t and T, the finite termination of problem. Under some suitable assumptions on the two processes $L(t)$ and $U(t)$, we want to find out a pair of saddle point (τ_t^*,σ_t^*) such that

$$\mathcal{E}_t^g[R(\tau,\sigma_t^*)] \leq \mathcal{E}_t^g[R(\tau_t^*,\sigma_t^*)] \leq \mathcal{E}_t^g[R(\tau_t^*,\sigma)] \tag{1.6}$$

for any $\tau,\sigma\in\mathcal{T}_t$ and then by definition of (1.4) and (1.5), the game has a value function.

This problem looks very like with the problem stated and solved in J.Cvitanic; I.Karatzas [8] , but there is difference between them, although the solutions are same as we will prove. To make our problem meaningful, we will point out the main difference in section 3 and treat a more complicated case with constraints, in which case we evaluate reward process by g_Γ-expectation whose definition will be given in section 2.

Our paper organized as follows. In section two, the necessary framework and some useful propositions of BSDE was reviewed, and the main result and its proof is stated in section 3.

2 BSDE, Reflected BSDE and Constrained BSDE

Given a probability space $(\Omega,\mathcal{F},P)$ and R^d-valued Brownian motion $W(t)$, we consider a sequence $\{(\mathcal{F}_t); t\in[0,T]\}$ of filtrations generated by Brownian motion $W(t)$ and augmented by P-null sets. $\mathcal{P}$ is the σ-field of predictable sets of $\Omega\times[0,T]$. We use $L^2(\mathcal{F}_T)$ to denote the space of all F_T-measurable random variables $\xi:\Omega\to R^d$ for which

$$\parallel \xi \parallel^2 = E[|\xi|^2] < +\infty.$$

and use $H_T^2(R^d)$ to denote the space of predictable process $\varphi:\Omega\times[0,T]\to R^d$ for which

$$\parallel \varphi \parallel^2 = E[\int_0^T |\varphi|^2] < +\infty.$$

S_n^k denotes the space of $(\mathcal{F}_t)$-progressively measurable processes $\varphi:[0,T]\times\Omega\mapsto R^n$ with $E(\sup_{0\leq t\leq T}\parallel\varphi\parallel^k)<\infty, k\in N$.

S_{ci}^2 denotes the space of continuous, increasing, $(\mathcal{F}_t)$-adapted processes $A:[0,T]\times\Omega\mapsto[0,\infty)$ with $A(0)=0, E(A^2(T))<\infty$.

Given a function $g : [0,T] \times R \times R^d \to R$, following assumptions always used in theory of BSDE.

$$|g(\omega, t, x_1, y_1) - g(\omega, t, x_2, y_2)| \le M(|x_1 - x_2| + |y_1 - y_2|), \quad \forall (x_1, y_1), (x_2, y_2) \tag{A1}$$

for some $M > 0$.

$$g(\cdot, x, y) \in H_T^2(R) \quad \forall x \in R,\, y \in R^d. \tag{A2}$$

The BSDE driven by $g(t, x, y)$ is given by

$$-dX(t) = g(t, X(t), Y(t))dt - Y'(t)dW(t) \tag{2.1}$$

where $X(t) \in R$ and $W(t) \in R^d$. Suppose that $\xi \in L^2(\mathcal{F}_T)$ and g satisfies (A1) and (A2), E.Pardoux and S.G.Peng[4] proved the existence of adapted solution $(X(t), Y(t))$ of such BSDE. We call (g, ξ) standard parameters for the BSDE.

For later use, we collect some useful propositions of BSDE below, its proof can be found in many papers such as S.G.Peng[13].

Proposition 2.1. *If the generator function $g(t,x,y) : [0,T] \times R \times R^d \mapsto R$ satisfies assumptions (A1) and (A2). For any stopping time $\tau \le \sigma \le T$, we denote $X(\tau)$ in the following BSDE as $\mathcal{E}^g_{\tau,\sigma}(\zeta)$,*

$$X(\tau) = \zeta + \int_\tau^\sigma g(s, X(s), Y(s))ds - \int_\tau^\sigma Y'(s)dW(s).$$

where ζ is $\mathcal{F}_\sigma$ measurable, then we have

- *(i) (Comparison proposition) If $\mathcal{F}(\sigma)$-measurable variables $\xi \ge \eta$ a.s , then $\mathcal{E}^g_{\tau,\sigma}(\xi) \ge \mathcal{E}^g_{\tau,\sigma}(\eta)$ for any stopping times $0 \le \tau \le \sigma \le T$ a.s.*
- *(ii) If $g(t, y, 0) = 0$, then for any stopping times $\tau \le \sigma \le T$ and $\mathcal{F}_\sigma$-measurable variable ζ, we have $\mathcal{E}^g_{\tau,\sigma}(\zeta) = \mathcal{E}^g_{\tau,T}(\zeta)$ and we write $\mathcal{E}^g_{\tau,\sigma}(\zeta)$ shortly as $\mathcal{E}^g_\tau(\zeta)$ when $\sigma = T$.*
- *(iii) (Coherence) If $g(t, y, 0) = 0$, then for any stopping times $\tau \le \sigma \le T$ and $\mathcal{F}_\sigma$-measurable variable ζ, we have $\mathcal{E}^g_\tau(\mathcal{E}^g_\sigma(\zeta)) = \mathcal{E}^g_\tau(\zeta)$.*

The theory of BSDE has wildly used in many fields such as financial mathematics and stochastic optimal control problems. Some brilliant use of these is that one can connect the optimal stopping problem with BSDE reflected by some lower barrier and connect Dynkin's game problem with BSDE reflected from below and above by lower barrier and upper barrier respectively. Here, for later proof of our problem we need to introduce Reflected BSDE.

Dfinition 2.1. *(Backward stochastic differential equation (BSDE) with upper and lower reflecting barriers). Let ξ be a given random variable in $L^2(\mathcal{F}_T)$, and $g : [0,T] \times \Omega \times R \times R^d \mapsto R$ a given $\mathcal{P} \otimes \mathcal{B}(R) \otimes \mathcal{B}(R^d)$-measurable functions satisfying (A1) and (A2).*

Consider two continuous processes L, U in S_1^2 satisfying

$$L(t) \le U(t), \qquad \forall\, 0 \le t \le T \quad \text{and} \quad L(T) \le \xi \le U(T) \quad a.s.$$

We say that a triple (X, Y, K) of F-progressively measurable processes $X : [0,T] \times \Omega \mapsto R$, $Y : [0,T] \times \Omega \mapsto R^d$ and $K : [0,T] \times \Omega \mapsto R$ is a solution of the Backward Stochastic Differential Equation (BSDE) with reflecting barriers U, L (upper and lower, respectively), terminal condition ξ and coefficient g, if the following hold:

(i) $K = K^+ - K^-$ *with* $K^{\pm} \in S^2_{ci}$,

(ii) $Y \in H^2_d$,

and

$$X(t) = \xi + \int_t^T g(s, X(s), Y(s))ds + K^+(T) - K^+(t) - (K^-(T) - K^-(t)) - \int_t^T Y'(s)dW(s), \qquad 0 \le t \le T, \tag{2.2}$$

$$L(t) \le X(t) \le U(t), \qquad 0 \le t \le T, \tag{2.3}$$

$$\int_0^T (X(t) - L(t))dK^+(t) = \int_0^T (U(t) - X(t))dK^-(t) = 0, \tag{2.4}$$

almost surely.

The processes L, U play the role of reflecting barriers, these are allowed to be random and time-varying, and the state-process X is not allowed to cross them on its way to the prescribed terminal target condition $X_T = \xi$. J.Cvitanic; I.Karatzas [8] has solved this kind of BSDE with two reflected barriers in S^2_1 via solving Dynkin'game and double optimal stopping problem. S.G.Peng and M.Y.Xu [16] treated different case of such problem with different class of barriers which is sufficient for the use of our paper.

We will also treat constrained case in our paper, so we introduce Constrained Backward Stochastic Differential Equation (CBSDE) at the same time.

The constraints in our paper is like that in S.G.Peng [14], namely for a given nonnegative function $\phi(t, x, y) : [0, T] \times R \times R^d \mapsto R^+$ we ask the solution $(X(t), Y(t))$ of BSDE satisfying

$$\phi(t, X(t), Y(t)) = 0, a.s \quad \text{for any} \quad t \in [0, T]. \tag{C}$$

In constrained case, it often need an increasing process acting as singular control to force the solution stays in the constrained filed. BSDE with an increasing process is called a g-super-solution and the smallest one plays a crucial role.

Dfinition 2.2. *(super-solution) A super-solution of a BSDE associated with the standard parameters* (g, ξ) *is a vector process* $(X(t), Y(t), C(t))$ *satisfying*

$$-dX(t) = g(t, X(t), Y(t))dt + dC(t) - Y'(t)dW(t), \quad X(T) = \xi, \tag{2.5}$$

or being equivalent to

$$X(t) = \xi + \int_t^T g(s, X(s), Y(s))ds - \int_t^T Y'(s)dW(s) + \int_t^T dC(s), \tag{2.5'}$$

where $(C_t, t \in [0, T])$ *is an increasing, adapted, right-continuous process with* $C_0 = 0$.

Dfinition 2.3. *(* g_Γ*-solution or the minimal solution) A* g*-super-solution* $(X(t), Y(t), C(t))$ *is said to be the the minimal solution of a constrained backward differential stochastic equation (shortly CBSDE), given* $y_T = \xi$*, subjected to the constraint* (C) *if for any other* g*-super-solution* $(\tilde{X}(t), \tilde{Y}(t), \tilde{C}(t))$ *satisfying* (C) *with* $\tilde{X}(T) = \xi$*, we have* $X(t) \le \tilde{X}(t)$ *a.e., a.s.. When both* $g(t, x, 0) = 0$ *and* $\phi(t, x, 0) = 0$ *for any* $x \in R, t \in [0, T]$*, the minimal solution is denoted by* $\mathcal{E}^{g,\phi}_t(\xi)$ *and for convenience called as* g_Γ*-solution. Sometimes, we also call* g_Γ*-expectation* $\mathcal{E}^{g,\phi}_t(\xi) \triangleq X(t)$ *the dynamic* g_Γ*-expectation with constraints* (C).

For any $\xi \in L^2(\mathcal{F}_T)$, we denote $\mathcal{H}^\phi(\xi)$ as the set of g-super-solutions $(X(t), Y(t), C(t))$ subjecting to (C) with $X(T) = \xi$. When $\mathcal{H}^\phi(\xi)$ is not empty, S.G.Peng [14] proved that g_Γ-solution exists.

Similarly with S.G.Peng; M.Y.Xu [15], let

$$\Gamma_t := \{(t, x, y) \in [0, T] \times R \times R^d | \phi(t, x, y) = 0\},$$

the g_Γ-solution is defined as the smallest g-super-solution with constraints (C).

We give a continuous property of g_Γ-solution for later use.

Proposition 2.2. *Suppose the generator function $g(t, x, y)$ and the constraint function $\phi(t, x, y)$ both satisfy conditions (A1) and (A2), $\{\xi_n \in L_T^2(P),\ n = 1, 2, \cdots\}$ is an bounded increasing sequence and converges almost surely to $\xi \in L_T^2(P)$, if $\mathcal{E}_t^{g,\phi}(\zeta)$ exists for $\zeta = \xi, \xi_n, n = 1, 2, \cdots$, then*

$$\lim_{n\to\infty} \mathcal{E}_t^{g,\phi}(\xi_n) = \mathcal{E}_t^{g,\phi}(\xi) \quad a.s \quad \forall\, t \in [0, T].$$

Proof Without noting, all the proofs go on under almost surely.

According to S.G.Peng [14], the solutions $x^m(t)(\xi)$ of

$$x^m(t)(\xi) = \xi + \int_t^T g(x^m(s)(\xi), y^m(s), s)ds + A^m(T) - A^m(t) - \int_t^T y^m(s)dW(s).$$

is an increasing sequence and converges to $\mathcal{E}_t^{g,\phi}(\xi)$, where

$$A^m(t) := m \int_0^t \phi(x^m(s), y^m(s), s)ds.$$

It is easy to see $\{\mathcal{E}_t^{g,\phi}(\xi_n), n = 1, 2, \cdots\}$ is an increasing sequence. We denote its limit at t as a_t, then $a_t \le \mathcal{E}_t^{g,\phi}(\xi)$. Since ξ_n converges almost surely increasingly to $\xi \in L_T^2(R)$, by dominated convergence theorem, it also converges strongly in $L_T^2(R)$, then by the continuous dependence property of g-supersolution, the limit of $\{x^m(t)(\xi_n)\}_{n=1}^\infty$ is $x^m(t)(\xi)$ for any fixed m.

We want to show that $a_t = \mathcal{E}_t^{g,\phi}(\xi)$. If on the contrary on has $a_t < \mathcal{E}_t^{g,\phi}(\xi)$, then there is some $\delta > 0$ such that $\mathcal{E}_t^{g,\phi}(\xi) - \mathcal{E}_t^{g,\phi}(\xi_n) > \delta$ for any n. On the other hand, for any $\epsilon > 0$, $0 \le \mathcal{E}_t^{g,\phi}(\xi) - x^m(t)(\xi) \le \epsilon$ holds for some larger m_0. Fixing m_0, ϵ, there is some n_0 which depends on m_0 and ϵ such that $0 \le x^{m_0}(t)(\xi) - x^{m_0}(t)(\xi_{n_0}) \le \epsilon$, so $\mathcal{E}_t^{g,\phi}(\xi) - x^{m_0}(t)(\xi_{n_0}) \le 2\epsilon$, but we have $\mathcal{E}_t^{g,\phi}(\xi) - x^{m_0}(t)(\xi_{n_0}) \ge \mathcal{E}_t^{g,\phi}(\xi) - \mathcal{E}_t^{g,\phi}(\xi_{n_0}) > \delta$, this is impossible for $\epsilon < \frac{\delta}{2}$. □

3 Dynkin's game under ambiguity

In this section we firs review some exited result about Reflected BSDE and Dykin' game.

In [8], if (X, Y, Z) is the solution of Reflected BSDE stated in above section, then it is said that $X(t)$ equals the value function of the Dynkin's game of (1.1) and (1.2) with

$$R_t(\tau, \sigma) = \int_t^{\tau\wedge\sigma} g(s, (X(s), Y(s))ds + L(\tau)1_{(\tau<T, \tau\le\sigma)} + U(\sigma)1_{(\sigma<\tau)} + \xi 1_{\tau\wedge\sigma=T}. \qquad (I)$$

More generally in S.Hamadène, J.-P.Lepeltier [17], the author considered the mixed zero-sum stochastic differential game with payoff

$$J(u,\tau;v,\sigma)=E^{(u,v)}[\int_0^{\tau\wedge\sigma} g(s,X(s),u(s),v(s))ds+L(\tau)1_{\tau\le\sigma,\sigma<T}+U(\sigma)1_{\sigma<\tau}+\xi 1_{\tau\wedge\sigma=T}]. \quad (II)$$

Under the assumption $g(t,x,0)=0$, we can explore $\mathcal{E}_t^g[L(\tau)1_{(\tau\le\sigma)}+U(\sigma)1_{(\sigma<\tau)}]$ as

$$E[\int_t^{\tau\wedge\sigma} g(s,(X^{\tau,\sigma}(s),Y^{\tau,\sigma}(s))ds+L(\tau)1_{(\tau<T,\tau\le\sigma)}+U(\sigma)1_{(\sigma<\tau)}+\xi 1_{\tau\wedge\sigma=T}|\mathcal{F}_t] \quad (III)$$

with $\xi=L(T)$.

From the expression of (I), (II) and (III) above, we can easily find that the difference between integrands in the integral. In (I), $(X(s),Y(s))$ is fixed at first, in (II), $X(s)$ only depends on controls (u,v), but in our problem (III) $((X^{\tau,\sigma}(s),Y^{\tau,\sigma})$ depends on stopping times (τ,σ).

But with the help of Reflected BSDE and comparison proposition of BSDEs or g-martingale theory, we can find a pair of saddle point of such Dynkin' game under nonlinear expectation. Below is our main result in unconstrained case.

Theorem 3.1. *Let $L(t),U(t)\in S_1^2$ and $L(t)\le U(t), 0\le t\le T$. The generator function $g(t,x,y)$ satisfies assumptions $(A1),A(2)$ and $g(t,x,0)=0,\forall t,x$. Suppose $(X(t),Y(t),K(t))$ is the solution of Reflected BSDE as formulated in definition (2.1) with terminal value $L(T)$, then the Dynkin's game stated in $(1.4),(1.5)$ has a pair of saddle point (τ_t^*,σ_t^*) and hence the value function exists. Furthermore, the pair of saddle point can be represented by*

$$\tau_t^*=\inf\{s\ge t: L(s)=X(s)\}\wedge T \quad (3.1)$$

and

$$\sigma_t^*=\inf\{s\ge t: U(s)=X(s)\}\wedge T. \quad (3.2)$$

and

$$\underline{V}(t)=\overline{V}(t)=X(t).$$

Proof We want to prove (1.6).

Fix σ_t^* first and let τ be arbitrary stopping time taking values in $[t,T]$, then we have

$$\begin{aligned}\mathcal{E}_t^g[R_t(\tau,\sigma_t^*)] &= \mathcal{E}_t^g[L(\tau)1_{(\tau\le\sigma_t^*)}+U(\sigma_t^*)1_{(\sigma_t^*<\tau)}]\\ &\le \mathcal{E}_t^g[X(\tau)1_{(\tau\le\sigma_t^*)}+U(\sigma_t^*)1_{(\sigma_t^*<\tau)}] \quad (\text{for } L\le X \text{ and } X(\sigma_t^*)=U(\sigma_t^*))\\ &= \mathcal{E}_t^g[X(\tau\wedge\sigma_t^*)].\end{aligned}$$

At this step, we need to prove that

$$\mathcal{E}_t^g[X(\tau\wedge\sigma_t^*)]\le\mathcal{E}_t^g[X(\tau_t^*\wedge\sigma_t^*)] \quad (3.3)$$

for any τ values in $[t,T]$. By (2.4) and $(3.1),(3.2)$, we have that when $\tau\le\tau_t^*$, $A^+(\tau)=A^+(t)$; when $\sigma\le\sigma_t^*$, $A^-(\sigma)=A^-(t)$.

So on the set $(\tau\le\tau_t^*)$, by the equation of (2.2), we have

$$X(\tau\wedge\sigma_t^*)=X(\tau_t^*\wedge\sigma_t^*)+\int_{\tau\wedge\sigma_t^*}^{\tau_t^*\wedge\sigma_t^*} g(s,X(s),Y(s))ds-\int_{\tau\wedge\sigma_t^*}^{T} Y'(s)dW(s),$$

this means

$$X(\tau\wedge\sigma_t^*)=\mathcal{E}_{\tau\wedge\sigma_t^*}^g[X(\tau_t^*\wedge\sigma_t^*)]. \quad (3.4)$$

On the other hand, when $\tau > \tau_t^*$, similarly we have

$$X(\tau_t^*\wedge\sigma_t^*) = X(\tau\wedge\sigma_t^*)+\int_{\tau_t^*\wedge\sigma_t^*}^{\tau\wedge\sigma_t^*} g(s,X(s),Y(s))ds+A^+(\tau\wedge\sigma_t^*)-A^+(\tau\wedge\sigma_t^*)-\int_{\tau_t^*\wedge\sigma_t^*}^{\tau\wedge\sigma_t^*} Y'(s)dW(s),$$

and this means

$$\mathcal{E}^g_{\tau_t^*\wedge\sigma_t^*}[X(\tau\wedge\sigma_t^*)] \leq X(\tau_t^*\wedge\sigma_t^*). \tag{3.5}$$

Taking conditional g-expectation on both hands of (3.4) and (3.5), by the coherence property of g-solutions, (iii) of proposition 2.1, one has (3.3).

Now, we fix τ_t^* , then for any σ taking values in $[t,T]$, we want to show that

$$\mathcal{E}^g_t[X(\tau_t^*\wedge\sigma_t^*)] \leq \mathcal{E}^g_t[X(\tau_t^*\wedge\sigma)]. \tag{3.6}$$

Similarly, when $(\sigma \leq \sigma_t^*)$, we have

$$X(\tau_t^*\wedge\sigma) = X(\tau_t^*\wedge\sigma_t^*)+\int_{\tau_t^*\wedge\sigma}^{\tau_t^*\wedge\sigma_t^*} g(s,X(s),Y(s))ds-\int_{\tau_t^*\wedge\sigma}^{\tau_t^*\wedge\sigma_t^*} Y'(s)dW(s),$$

and

$$X(\tau_t^*\wedge\sigma) = \mathcal{E}^g_{\tau_t^*\wedge\sigma}[X(\tau_t^*\wedge\sigma_t^*)]. \tag{3.7}$$

becomes true.

When $\sigma > \sigma_t^*$, (2.2) becomes

$$X(\tau_t^*\wedge\sigma_t^*) = X(\tau_t^*\wedge\sigma)+\int_{\tau_t^*\wedge\sigma_t^*}^{\tau_t^*\wedge\sigma} g(s,X(s),Y(s))ds-(A^-(\tau_t^*\wedge\sigma)-A^-(\tau_t^*\wedge\sigma_t^*))-\int_{\tau_t^*\wedge\sigma_t^*}^{\tau_t^*\wedge\sigma} Y'(s)dW(s)$$

and this means

$$\mathcal{E}^g_{\tau_t^*\wedge\sigma_t^*}[X(\tau_t^*\wedge\sigma)] \geq X(\tau_t^*\wedge\sigma_t^*). \tag{3.8}$$

Taking g-expectation in (3.7) and (3.8), we have(3.6). □

Remark 3.1. *Comparing the method used to prove Theorem 4.1 in J.Cvitanic; I.Karatzas [8] and the method to prove 3.1 in our paper, although they are very similar, but the advantages of our problem helps us to use g-martingale theory directly and this is very convenient for us to handle constrained case later.*

Next, we then go to the constrained case, that is we evaluate the reward process by the constrained g-expectation which is also named as g_Γ-expectation in S.G.Peng; M.Y.Xu [15]. Define similarly lower and upper value function as non-constrained case,

$$\underline{V}_t := ess\sup_{\tau\in\mathcal{T}_t} ess\inf_{\sigma\in\mathcal{T}_t} \mathcal{E}^{g,\phi}_t[R(\tau,\sigma)] \tag{3.9}$$

$$\overline{V}_t = ess\inf_{\sigma\in\mathcal{T}_t} ess\sup_{\tau\in\mathcal{T}_t} \mathcal{E}^{g,\phi}_t[R(\tau,\sigma)] \tag{3.10}$$

with $R(\tau,\sigma) = L(\tau)1_{(\tau\leq\sigma)} + U(\sigma)1_{(\sigma<\tau)}$.

For any integer m, let $g_m = g + m\phi$. For any fixed m, there is an unique solution of Reflected g_m-solution with double barriers (from below and above by L and U respectively) and we denote it as $(X^m,Y^m,K^m), m = 1,2,\cdots$. Since X^m can be obtained by penalization method, the comparison proposition ensures that $\{X^m\}$ is an increasing sequence of process.

Define the corresponding pairs of stopping times as

$$\tau_t^*(m) = \inf\{s \geq t : L(s) = X^m(s)\} \wedge T \tag{3.11}$$

and

$$\sigma_t^*(m) = \inf\{s \ge t : U(s) = X^m(s)\} \wedge T. \tag{3.12}$$

It is easy to see that $\{\tau_t^*(m)\}$ is increasing and $\{\sigma_t^*(m)\}$ is decreasing and

$$\tau_t^* = \lim_{m\to\infty} \tau_t^*(m),\ \sigma_t^* = \lim_{m\to\infty} \sigma_t^*(m) \tag{3.13}$$

are stopping times.

With these in hand, we can then state and prove our result in constrained case below.

Theorem 3.2. *Let g and ϕ satisfy assumptions (A1) and (A2), $L(t)$ and $U(t)$ are nonnegative continuous processes and there is some constant $B > 0$ such that $L(t) \le B, U(t) \le B$ for any $t \in [0, T]$. We consider the Dynkin's game with lower and upper value function defined in (3.9) and (3.10). If $L(t)$ is increasing, then the pair of stopping time defined in (3.13) is a saddle point.*

Proof For any $n \le m$, by comparison theorem of BSDE and results obtained in unconstrained case of Theorem 3.1, we have

$$\mathcal{E}_t^{g_n}[X(\tau \wedge \sigma_t^*(m))] \le \mathcal{E}_t^{g_m}[X(\tau \wedge \sigma_t^*(m))] \le \mathcal{E}_t^{g_m}[X(\tau_t^*(m) \wedge \sigma_t^*(m))] = X^m(t) \tag{3.14}$$

for any τ and $\tau_t^*(m), \sigma_t^*(m)$ defined in (3.11) and (3.12).

On the other hand, one has

$$X^m(t) = \mathcal{E}_t^{g_m}[X(\tau_t^*(m) \wedge \sigma_t^*(m))] \le \mathcal{E}_t^{g_m}[X(\tau_t^*(m) \wedge \sigma)] \le \mathcal{E}_t^{g,\phi}[X(\tau_t^*(m) \wedge \sigma)] \tag{3.15}$$

for any σ taking values in $[0, T]$.

Firs, we take limit in (3.14) and (3.15) as $m \to \infty$, set $X(t) := \lim_{m\to\infty} X^m(t)$, we have

$$\mathcal{E}_t^{g_n}[X(\tau \wedge \sigma_t^*] \le X(t) \tag{3.16}$$

and

$$X(t) \le \mathcal{E}_t^{g,\phi}[X(\tau_t^* \wedge \sigma)]. \tag{3.17}$$

Since g-solution is continuous dependence on its terminal value and, by Proposition 2.2, g_Γ-solution is continuous from below with its terminal variable, see also Remark 3.2 in this paper.

By (3.16) and (3.17), we conclude that The Dynkin's game has a value function X(t).

To prove (τ_t^*, σ_t^*) is a saddle point, we want to prove

$$\mathcal{E}_t^{g,\phi}[X(\tau_t^* \wedge \sigma_t^*)] = X(t) \tag{3.18}$$

.

First note that

$$X^m(t) = \mathcal{E}_t^{g_m}[X(\tau_t^*(m) \wedge \sigma_t^*(m))] \le \mathcal{E}_t^{g_m}[X(\tau_t^*(m) \wedge \sigma_t^*)] \le \mathcal{E}_t^{g,\phi}[X(\tau_t^*(m) \wedge \sigma_t^*)].$$

Taking limit in above equation as $m \to \infty$, because $L(t)$ is continuous and increasing and g_Γ-solution is continuous from below, we have

$$X(t) \le \mathcal{E}_t^{g,\phi}[X(\tau_t^* \wedge \sigma_t^*)]. \tag{3.19}$$

For the other side inequality, note that $\sigma_t^* \le \sigma_t^*(m)$ for any m, we have

$$X^m(t) = \mathcal{E}_t^{g_m}[X(\tau_t^*(m) \wedge \sigma_t^*(m))] = \mathcal{E}_t^{g_m}[X(\tau_t^*(m) \wedge \sigma_t^*)] \ge \mathcal{E}_t^{g_m}[X(\tau_t^* \wedge \sigma_t^*)].$$

Taking limit as $m \to \infty$ in above equation, on has

$$X(t) \geq \mathcal{E}_t^{g,\phi}[X(\tau_t^* \wedge \sigma_t^*)]. \tag{3.20}$$

Combine (3.19) and (3.20) together, we obtain (3.18) and thus complete our proof. □

Remark 3.2. *Note that under the assumptions of Theorem 3.2, $\mathcal{E}_t^{g,\phi}[X(\tau)]$ is meaningful for any stopping time τ taking values in $[t,T]$. It is easy to see that $X^m(t) \leq B$ for any $t \in [0,T]$ and assumptions (A1) and (A2) together with $g(t,x,0) = 0$ and $\phi(t,x,0) = 0$ ensures that g_Γ-solution is well defined on $L_T^\infty(P)$, the space of all essentially bounded $\mathcal{F}_T$-measurable variables. In the paper S.G.Peng and M.Y.Xu [15], the author defined a new subspace of $L_T^2(P)$:*

$$L_{+,\infty}^2(\mathcal{F}_T) \triangleq \{\xi \in L^2(\mathcal{F}_T), \xi^+ \in L^\infty(\mathcal{F}_T)\}.$$

For any $\xi \in L_{+,\infty}^2(\mathcal{F}_T)$ with terminal condition $y_T = \xi$, g_Γ-solution exists if

$$g(t,y,0) \leq L_0 + M|y| \quad \text{and} \quad (y,0) \in \Gamma_t \tag{3.21}$$

holds for a large constant L_0 and for any $y \geq L_0$ and if there exists a deterministic process $a(t)$ such that $L(t) \leq a(t)$ on $[0,T]$. Under assumptions on g and ϕ as above mentioned, (3.21) is satisfied for any $L_0 \geq 0$ and M in (A1) and we can chose $a(t) = B$. It is obvious $L^\infty(\mathcal{F}_T) \subset L_{+,\infty}^2(\mathcal{F}_T)$ and g_Γ-solution is defined well on the whole space $L^\infty(\mathcal{F}_T)$

Remark 3.3. *Roughly speaking, the continuous property from below of g_Γ-expectation is a simple consequence of the fact that we can change the order of limits in $\lim_{n\to\infty}\lim_{m\to\infty} a_{m,n}$ and $\lim_{m\to\infty}\lim_{n\to\infty} a_{m,n}$ when $a_{m,n}$ are both increasing with n and m. The complete continuous property is more complicated since it concerns mini-max problem. But except g_Γ-expectation is continuous from below, it is still semi-lower-continuous, and we can still conclude some useful continuous property by convex assumptions on coefficients of CBSDE with help of some wonderful results in convex analysis. Of course, we can then make different assumptions on L and U to get through the proof via continuous property in the Theorem 3.2.*

Remark 3.4. *It is an open problem that whether $X(t)$, the limit of $\{X^m(t)\}$ which is a sequence of Reflected solution of BSDEs, is still some solution of some kind of Reflected BSDEs?*

Remark 3.5. *Dynkin's game problem is very similar with Optimal stopping problem under ambiguity. It is well known that the Snell envelope of Optimal stopping problem for g-expectation is same with the solution of Reflected BSDE with one lower obstacle. Roughly speaking, this result is based on the following three deep facts:*

- *(i) The solution of Reflected BSDE with one lower barrier is the same with the g_Γ-solution taking the barrier as a constraint.*
- *(ii) The Snell envelope of the barrier under g-expectation is the smallest g-super-martingale above the barrier.*
- *(iii) Under suitable assumptions, g-super-solution is equivalent to g-super-martingale.*

But in game case, there are no corresponding theories, so the results in this paper are not simple extensions of Optimal stopping case.

References

[1] A.Friedman:Stochastic games and variational inequalities, Arch. Rational Mech. Anal. 51, (1973) 321-346.

[2] A.Bensoussan; A.Friedman: Nonlinear variational inequalities and differential games with stopping times. J. Functional Analysis 16, (1974) 305-352.

[3] E.B.Dynkin : Game variant of a problem on optimal stopping, Soviet Math. Dokl. 10 (1969), 270-274.

[4] E.Pardoux; S.G.Peng: Adapted solution of a backward stochastic differential equation. Systems and control letters. 14 (1990) 55-62.

[5] H.Morimoto : Dynkin games and martingale methods, Stochastics 13 (1984), 213-228.

[6] I.Karatzas; H.Wang: Connections between bounded-variation control and Dynkin games, Optimal control and partial differential equation, (2001) 363-373.

[7] I.KaratzasSource: A Pathwise Approach to Dynkin Games , Statistics, Probability and Game Theory, 30 (1996) 115-125.

[8] J.Cvitanic; I.Karatzas: Backward Stochastic Differential Equations with reflection and Dynkin games, The Annals of Probability 4(24) (1996) 2024-2056

[9] J.Neveu : Discrete-Parameter Martingales, North-Holland, Amsterdom (1975).

[10] L.Stettner : On closedness of general zero-sum stopping game, Bull. Polish Acad. Sci. Math. 32 (1984), 351-361.

[11] N.V.Elbakidze : Construction of the cost and optimal policies in a game problem of stopping a Markov process, Theor. Prob. Appl. 21 (1976), 163-168.

[12] N.V.Krylov : Control of Markov processes and W-spaces, Math. USSR-Izv. 5 (1971), 233-266.

[13] S.G.Peng: Backward SDE and related g-expectation, Pitman Res. Notes Math. Ser. Longman, Harlow, 364 (1997).

[14] S.G.Peng: Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob-Meyer's type. Probab.Theory Related Fields. 113 (1999) 473-499 .

[15] S.G.Peng; M.Y.Xu: Reflected BSDE with a constraint and its applications in an imcomplete market. Bernoulli. 16(3), (2010) 614-640 .

[16] S.G.Peng; M.Y.Xu: The smallest g-supermartingale and reflected BSDE with single and double L^2 obstacles, Ann. I. H. Poincaré-PR 41 (2005) 605-630.

[17] S.Hamadène; J.-P.Lepeltier: Reflected BSDEs and mixed game problem, Stochastic Processes and their Applications 85 (2000) 177-188.

[18] Yu.I.Kifer : Optimal stopped games, Theor. Prob. Appl. 16 (1971), 185-189.

[19] Y.Ohtsubo : Optimal stopping in sequential games vlth or without a constraint of always terminating, to appear in Math. Oper. Res. (1986).

[20] Y.Ohtsubo : The Dynkin stopping with a finite constraint and bisequential decision problems, Mem. Fac. Sci., Kochi Univ., Ser. A 7 (1966), 59-69.

[21] Y.Ohtsubo : Neveu's martingale conditions and closedness in Dynkin stopping problem with a finite constraint, Stochastic Process. Appl. 22 (1986), 333-342.