An Erdős-Ko-Rado theorem in general linear groups

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Abstract

Let S_n be the symmetric group on n points. Deza and Frankl [M. Deza and P. Frankl, On the maximum number of permutations with given maximal or minimal distance, J. Combin. Theory Ser. A 22 (1977) 352–360] proved that if $\mathcal{F}$ is an intersecting set in S_n then $|\mathcal{F}| \le (n-1)!$. In this paper we consider the q-analogue version of this result. Let $\mathbb{F}_q^n$ be the n-dimensional row vector space over a finite field $\mathbb{F}_q$ and $GL_n(\mathbb{F}_q)$ the general linear group of degree n. A set $\mathcal{F}_q \subseteq GL_n(\mathbb{F}_q)$ is *intersecting* if for any $T, S \in \mathcal{F}_q$ there exists a non-zero vector $\alpha \in \mathbb{F}_q^n$ such that $\alpha T = \alpha S$. Let $\mathcal{F}_q$ be an intersecting set in $GL_n(\mathbb{F}_q)$. We show that $|\mathcal{F}_q| \le q^{(n-1)n/2} \prod_{i=1}^{n-1}(q^i - 1)$.

Keywords: Erdős-Ko-Rado theorem, general linear group

The Erdős-Ko-Rado theorem [5] is a central result in extremal combinatorics. There are many interesting proofs and extensions of this theorem, for a summary see [4].

Let S_n be the symmetric group on n points. A set $\mathcal{F} \subseteq S_n$ is *intersecting* if for any $f, g \in \mathcal{F}$ there exists an $x \in [n]$ such that $f(x) = g(x)$. The following result is an Erdős-Ko-Rado theorem for intersecting families of permutations.

Theorem 1. *Let $\mathcal{F}$ be an intersecting set in S_n. Then*

(i) (Deza and Frankl [3]) $|\mathcal{F}| \le (n-1)!$.

(ii) (Cameron and Ku [1]) *Equality in* (i) *holds if and only if $\mathcal{F}$ is a coset of the stabilizer of a point.*

Wang and Zhang [8] gave a simple proof of Theorem 1. Recently, Godsil and Meagher [6] presented another proof.

In this paper we consider the q-analogue of Theorem 1, and obtain an Erdős-Ko-Rado theorem in general linear groups.

Let $\mathbb{F}_q$ be a finite field and $\mathbb{F}_q^n$ the n-dimensional row vector space over $\mathbb{F}_q$. The set of all $n \times n$ nonsingular matrices over $\mathbb{F}_q$ forms a group under matrix multiplication, called the *general linear*

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group of degree n over $\mathbb{F}_q$, denoted by $GL_n(\mathbb{F}_q)$. There is an action of $GL_n(\mathbb{F}_q)$ on $\mathbb{F}_q^n$ defined as follows:

$$\begin{array}{rcl} \mathbb{F}_q^n \times GL_n(\mathbb{F}_q) & \longrightarrow & \mathbb{F}_q^n \\ ((x_1, x_2, \ldots, x_n), T) & \longmapsto & (x_1, x_2, \ldots, x_n)T. \end{array}$$

Let P be an m-subspace of $\mathbb{F}_q^n$. Denote also by P an $m \times n$ matrix of rank m whose rows span the subspace P and call the matrix P a matrix representation of the subspace P.

Definition 1. *A set $\mathcal{F}_q \subseteq GL_n(\mathbb{F}_q)$ is intersecting if for any $T, S \in \mathcal{F}_q$ there exists a non-zero vector $\alpha \in \mathbb{F}_q^n$ such that $\alpha T = \alpha S$.*

In this paper, we shall prove the following result:

Theorem 2. *Let $\mathcal{F}_q$ be an intersecting set in $GL_n(\mathbb{F}_q)$. Then $|\mathcal{F}_q| \leq q^{(n-1)n/2} \prod_{i=1}^{n-1}(q^i - 1)$.*

For the group $GL_n(\mathbb{F}_q)$ we can define a graph, denoted by Γ, on vertex set $GL_n(\mathbb{F}_q)$ by joining T and S if they are intersecting. Since $GL_n(\mathbb{F}_q)$ is an automorphism group of Γ, this graph is vertex-transitive.

In order to prove Theorem 2, we require a useful lemma obtained by Cameron and Ku and a classical result about finite geometry.

Lemma 3. ([1]) *Let C be a clique and A a coclique in a vertex-transitive graph on v vertices. Then $|C||A| \leq v$. Equality implies that $|C \cap A| = 1$.*

An *n-spread* of $\mathbb{F}_q^l$ is collection of n-subspaces $\{W_1, \ldots, W_t\}$ such that every non-zero vector in $\mathbb{F}_q^l$ belongs to exactly one W_i.

Theorem 4. ([2]) *An n-spread of $\mathbb{F}_q^l$ exists if and only if n is a divisor of l.*

Lemma 5. *Let $\alpha(\Gamma)$ be the size of the largest coclique of Γ. Then $\alpha(\Gamma) = q^n - 1$.*

Proof. By Theorem 4, there exists an n-spread $\{W_0, W_1, \ldots, W_{q^n}\}$ of $\mathbb{F}_q^{2n}$. Since $W_0 \cap W_{q^n} = \{0\}$ and $W_0 + W_{q^n} = \mathbb{F}_q^{2n}$, by [7, Theorem 1.3], there exists a $G \in GL_{2n}(\mathbb{F}_q)$ such that $W_0 G = (I^{(n)}\ 0^{(n)}), W_{q^n}G = (0^{(n)}\ I^{(n)})$, and $\{W_0G, W_1G, \ldots, W_{q^n}G\}$ is an n-spread of $\mathbb{F}_q^{2n}$, where $I^{(n)}$ is the identity matrix of order n and $0^{(n)}$ is the zero matrix of order n. Without loss of generality, we may assume that $W_0 = (I^{(n)}\ 0^{(n)})$ and $W_{q^n} = (0^{(n)}\ I^{(n)})$. Then each $W_i\,(1 \leq i \leq q^n - 1)$ has the matrix representation of the form $(I^{(n)}\ T_i)$, where $T_i \in GL_n(\mathbb{F}_q)$. For all $1 \leq i \neq j \leq q^n - 1$, since $W_i + W_j$ is of dimension $2n$, $T_i - T_j \in GL_n(\mathbb{F}_q)$. By the fact that $T_i - T_j \in GL_n(\mathbb{F}_q)$ if and only if $\alpha T_i \neq \alpha T_j$ for all $\alpha \in \mathbb{F}_q^n \backslash \{0\}$, $\{T_1, \ldots, T_{q^n-1}\}$ is a coclique of Γ; and so $\alpha(\Gamma) \geq q^n - 1$.

Suppose $\alpha(\Gamma) > q^n - 1$ and $\mathcal{I} = \{T_1, T_2, \ldots, T_{\alpha(\Gamma)}\}$ is a coclique of Γ. Then $T_i - T_j \in GL_n(\mathbb{F}_q)$ for all $1 \leq i \neq j \leq \alpha(\Gamma)$. Take $W_0 = (I^{(n)}\ 0^{(n)})$, $W_{\alpha(\Gamma)+1} = (0^{(n)}\ I^{(n)})$ and $W_i = (I^{(n)}\ T_i)$ $(1 \leq i \leq \alpha(\Gamma))$. Then $W_k \cap W_l = \{0\}$ for all $0 \leq k \neq l \leq \alpha(\Gamma) + 1$. The number of non-zero vectors in $\bigcup_{k=0}^{\alpha(\Gamma)+1} W_k \subseteq \mathbb{F}_q^{2n}$ is $(\alpha(\Gamma) + 2)(q^n - 1) > (q^n + 1)(q^n - 1) = q^{2n} - 1$, a contradiction. □

Combining Lemma 3 and Lemma 5, we complete the proof of Theorem 2.

Let G_v be the stabilizer of a given non-zero vector v in $GL_n(\mathbb{F}_q)$. Then G_v is an intersecting set meeting the bound in Theorem 2. It seems to be interesting to characterize the intersecting sets meeting the bound in Theorem 2.

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References

[1] P.J. Cameron and C.Y. Ku, Intersecting families of permutations, European J. Combin. 24 (2003) 881–890.
[2] P. Dembowski, Finite Geometries, Springer-Verlag, Berlin/Heidelberg, 1968.
[3] M. Deza and P. Frankl, On the maximum number of permutations with given maximal or minimal distance, J. Combin. Theory Ser. A 22 (1977) 352–362.
[4] M. Deza and P. Frankl, Erdős-Ko-Rado theorem–22 years later, SIAM J. Algebraic Discrete Methods 4 (1983) 419–431.
[5] P. Erdős, C. Ko and R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford Ser. 2 (1961) 313–318.
[6] C. Godsil and K. Meagher, A new proof of the Erdős-Ko-Rado theorem for intersecting families of permutations, European J. Combin. 30 (2009) 404–414.
[7] Z. Wan, Geometry of Classical Groups over Finite Fields, 2nd edition, Science Press, Beijing/New York, 2002.
[8] J. Wang and Sophia J. Zhang, An Erdős-Ko-Rado-type theorem in Coxeter groups, European J. Combin. 29 (2008) 1112–1115.