EXTENDING CUTOFF RESOLVENT ESTIMATES VIA PROPAGATION OF SINGULARITIES

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Abstract. We use a gluing method developed in joint work with András Vasy to show that polynomially bounded cutoff resolvent estimates at the real axis imply, up to a constant factor, the same estimates in a neighborhood of the real axis.

1. Introduction

Let

$$P = -h^2\Delta + V(x), \qquad V \in C_0^\infty(\mathbb{R}^n).$$

Suppose $\operatorname{supp} V \subset \{|x| < R_0\}$, and fix $E > 0$ and $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi(x) = 1$ near $\{|x| \le R_0 + 5\}$. Let $R(\lambda) = (P-\lambda)^{-1}$. We show that polynomial semiclassical estimates for $\chi R(E+i0)\chi$ imply, up to a constant factor, the same estimates for the meromorphic continuation of $\chi R(\lambda)\chi$ to λ near E. Cutoff functions (or at least weights) are needed to define the limit $R(E+i0) = \lim_{\varepsilon\to 0^+} R(E+i\varepsilon)$ and the continuation to λ near E because $E \in [0,\infty)$, which is the essential spectrum of P.

Theorem. *Suppose there exist $h_0 > 0$ and $a\colon (0,h_0] \to (0,\infty)$ satisfying $a(h) \le h^{-N}$ for some $N \in \mathbb{N}$, such that*

$$\|\chi R(E+i0)\chi\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le a(h), \qquad 0 < h \le h_0. \tag{1.1}$$

Then there exist $C, h_1 > 0$ such that the meromorphic continuation of $\chi R(\lambda)\chi$ from $\{\arg\lambda > 0\}$ to $\{\arg\lambda < 0\}$ obeys

$$\|\chi R(\lambda)\chi\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le Ca(h), \qquad |\lambda - E| \le \frac{1}{Ca(h)},\ 0 < h \le h_1. \tag{1.2}$$

The function $a(h)$ for which the estimate (1.1) is satisfied is governed by the trapped set of the Hamiltonian $p = |\xi|^2 + V(x)$:

$$K_E = \{\rho \in p^{-1}(E);\ \exists C_\rho > 0,\ \forall t \in \mathbb{R},\ |\exp(tH_p)\rho| \le C_\rho\}. \tag{1.3}$$

Here $H_p = 2\xi\cdot\nabla_x - \nabla V\cdot\nabla_\xi$ is the Hamiltonian vector field associated to p. We have (1.1) with $a(h) = C/h$ if and only if $K_E = \varnothing$, and in this case the result of the Theorem is well known: indeed, see [NSZ03, Proposition 3.1] for a better statement.

The most interesting case of the Theorem is when $a(h) = C\log(1/h)/h$. In [BBR10], Bony-Burq-Ramond show that this is the best bound possible when $K_E \ne \varnothing$, although in general the optimal $a(h)$ is $\exp(C/h)$ (see Burq [Bur02]). The estimate (1.1) with $a(h) = C\log(1/h)/h$ is true in many situations when the trapping is hyperbolic: see work of Burq [Bur04], Christianson [Chr07], Nonnenmacher-Zworski [NoZw09a, NoZw09b] and Wunsch-Zworski [WuZw11]. In particular it is true if K_E consists of a single hyperbolic orbit. In the case of a single orbit which is degenerately hyperbolic, Christianson-Wunsch [ChWu11] give examples where (1.1) holds for $a(h) = h^{-k}$ where $k > 1$ depends on the degeneracy.

Vodev [Vod99, Vod02] and Christianson [Chr09] prove similar results using different methods (see also those papers for applications of such results to wave decay). In the present paper we use the gluing method via propagation of singularities developed in collaboration with Vasy [DaVa11]; this is perhaps the simplest application of that method. The same proof holds with only minor modifications when V is replaced by a metric perturbation or an obstacle or when χ is noncompactly supported but suitably decaying. As in [DaVa11], it can similarly treat suitably decaying perturbations or asymptotically hyperbolic manifolds in the sense of [Vas10, Vas11]. In the interest of simplicity we do not pursue this here.

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2. Proof of Theorem

If the cutoff χ were not present, (1.2) would follow directly from the resolvent identity

$$R(\lambda) = R(E)(\operatorname{Id} + (E-\lambda)R(E))^{-1}. \tag{2.1}$$

In order to use this identity, we introduce a complex absorbing barrier function which suppresses the effects of infinity and removes the need for cutoffs in the estimates. Indeed, take $W \in C^\infty(\mathbb{R}^n;[0,\infty))$ with $W(x) = 0$ near $|x| \le R_0 + 4$ and with $W(x) = 1$ near $|x| \ge R_0 + 5$. Put $P_W = P - iW$ and $R_W(\lambda) = (P_W - \lambda)^{-1}$, and note that the essential spectrum of P_W is $-i[0,\infty)$, so that $R_W(\lambda)$ is meromorphic (and, as we will see, actually holomorphic) for λ near E, without needing to be multiplied by cutoffs. We will show that (1.1) implies

$$\|R_W(E)\| \le Ca(h), \tag{2.2}$$

and that

$$\|R_W(\lambda)\| \le Ca(h), \qquad |\lambda - E| \le \frac{1}{Ca(h)}, \tag{2.3}$$

implies (1.2). The implication (2.2) $\Longrightarrow$ (2.3) follows from (2.1). Here and below all operator norms are $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, and all function norms are $L^2(\mathbb{R}^n)$. The large constant C may change from line to line, and $h \in (0, h_{\max}]$ where $h_{\max}$ may change from line to line.

We prove (1.1) $\Longrightarrow$ (2.2) and (2.3) $\Longrightarrow$ (1.2) using two more model operators, where the potential V is removed. Put $P_0 = -h^2\Delta$ and $P_{W,0} = P_0 - iW$, and define similarly R_0, $R_{W,0}$. These resolvents obey the usual nontrapping bounds (see e.g. [NSZ03, Proposition 3.1]):

$$\|\chi R_0(\lambda)\chi\| + \|R_{W,0}(\lambda)\| \le \frac{C}{h}, \qquad |\lambda - E| \le \frac{h}{C}. \tag{2.4}$$

Proof that (1.1) $\Longrightarrow$ (2.2). Take $\chi_K \in C^\infty(\mathbb{R})$ with $\chi_K(r) = 1$ near $r \le R_0 + 2$ and $\chi_K(r) = 0$ near $r \ge R_0 + 3$, and put $\chi_\infty = 1 - \chi_K$. Let

$$F = \chi_K(|x|-1)R(E+i0)\chi_K(|x|) + \chi_\infty(|x|+1)R_{W,0}(E)\chi_\infty(|x|).$$

Now put

$$\begin{aligned}&(P_W - E)F \\ &\quad = \operatorname{Id} + [P, \chi_K(|x|-1)]R(E+i0)\chi_K(|x|) + [P, \chi_\infty(|x|+1)]R_{W,0}(E)\chi_\infty(|x|) \\ &\quad \overset{\text{def}}{=} \operatorname{Id} + A_K + A_\infty.\end{aligned}$$

These errors are not small (we only have $\|A_K\| \le Ca(h)h$ and $\|\chi A_\infty \chi\| \le C$), but solving them away using F we obtain errors which we can control using propagation of singularities. In fact, using $A_K^2 = A_\infty^2 = 0$ we obtain

$$(P_W - E)F(\mathrm{Id} - A_K - A_\infty + A_K A_\infty) = \mathrm{Id} - A_\infty A_K + A_\infty A_K A_\infty.$$

Using $\chi_K(|x|)A_K = 0$, (1.1) and (2.4) we find that

$$\|\chi F(\mathrm{Id} - A_K - A_\infty + A_K A_\infty)\chi\| \le Ca(h),$$

so the conclusion follows from

$$\|A_\infty A_K\| = \mathcal{O}(h^\infty). \tag{2.5}$$

But this is a consequence of propagation of singularities along nontrapping bicharacteristics. Indeed, recall that the semiclassical wavefront set of a function $u \in L^2(\mathbb{R}^n)$ with $\|u\| \le h^{-N}$, denoted $\mathrm{WF}_h\, u$, is defined as follows: a point $\rho \in T^*\mathbb{R}^n$ is not in $\mathrm{WF}_h\, u$ if there exists $a \in C^\infty(T^*\mathbb{R}^n)$, bounded together with all derivatives and with $a(\rho) \neq 0$, such that $\|\operatorname{Op}(a)u\| = \mathcal{O}(h^\infty)$. Define the backward bicharacteristic at $\rho \in T^*\mathbb{R}^n$ by

$$\gamma_\rho^- = \bigcup_{t \le 0} \exp(H_p t)\rho,$$

with H_p as in (1.3). We will use propagation of singularities in the following form: if $f \in L^2(\mathbb{R}^n)$ has compact support and $\|f\| \le h^{-N}$ for some $N \in \mathbb{N}$, then for all $\rho \in T^*\mathbb{R}^n$,

$$\rho \in \mathrm{WF}_h\,(R_{W,0}(E)f) \Longrightarrow \gamma_\rho^- \cap \mathrm{WF}_h\, f \neq \varnothing. \tag{2.6}$$

See for example [DaVa11, Lemma 5.1] for a proof. If further $\gamma_\rho^- \subset \{|x| > R_0\}$ (so that the backward bicharacteristic flowout of ρ is disjoint from $T^* \operatorname{supp} V$), then similarly

$$\rho \in \mathrm{WF}_h\,(R(E + i0)f) \Longrightarrow \gamma_\rho^- \cap \mathrm{WF}_h\, f \neq \varnothing. \tag{2.7}$$

See for example [Dat09, Lemma 2] for a proof. We now use this to show that $\mathrm{WF}_h\, A_\infty A_K f$ is empty for any $f \in L^2(\mathbb{R}^n)$ with $\|f\| = 1$, from which (2.5) follows. To see this, write

$$A_\infty A_K f = [P, \chi_\infty(|x| + 1)]R_{W,0}(E)[P, \chi_K(|x| - 1)]R(E + i0)\chi_K(|x|)f.$$

Then any $\rho \in \mathrm{WF}_h\, A_\infty A_k f$ has $\rho \in T^* \operatorname{supp} d\chi_\infty(|\cdot| + 1) \subset \{R_0 + 1 < |x| < R_0 + 2\}$, and by (2.6) we know that γ_ρ^- must contain a point $\rho' \in T^* \operatorname{supp} d\chi_K(|\cdot| - 1) \subset \{R_0 + 3 < |x| < R_0 + 4\}$. Note that $\gamma_{\rho'}^- \subset \{|x| > R_0 + 3\}$. Because of this we may apply (2.7) to conclude that $\gamma_{\rho'}^-$ (and hence also γ_ρ^-) must contain a point $\rho'' \in T^* \operatorname{supp} \chi_K(|\cdot|) \subset \{|x| < R_0 + 3\}$, which is impossible due to $\gamma_{\rho'}^- \subset \{|x| > R_0 + 3\}$. This shows that $\mathrm{WF}_h\, A_\infty A_k f = \varnothing$, from which (2.5) follows. □

Proof that (2.3) $\Longrightarrow$ (1.2). We use the same χ_K and χ_∞ as in the previous proof, but we redefine F, A_0, and A_1 as follows:

$$F = \chi_K(|x| - 1)R_W(\lambda)\chi_K(|x|) + \chi_\infty(|x| + 1)R_0(\lambda)\chi_\infty(|x|).$$

Now put

$$\begin{aligned}(P - \lambda)F &= \mathrm{Id} + [P, \chi_K(|x| - 1)]R_W(\lambda)\chi_K(|x|) + [P, \chi_\infty(|x| + 1)]R_0(\lambda)\chi_\infty(|x|) \\ &\overset{\mathrm{def}}{=} \mathrm{Id} + A_K + A_\infty.\end{aligned}$$

As before,

$$(P - \lambda)F(\mathrm{Id} - A_K - A_\infty + A_K A_\infty) = \mathrm{Id} - A_\infty A_K + A_\infty A_K A_\infty.$$

Now

$$\|\chi F(\mathrm{Id} - A_K - A_\infty + A_K A_\infty)\chi\| \le C a(h),$$

and once again

$$\|A_\infty A_K\| = \mathcal{O}(h^\infty),$$

follows from propagation of singularities, this time in the following form (the proofs can again be found in, for example, [Dat09, Lemma 2] and [DaVa11, Lemma 5.1]): if $f \in L^2(\mathbb{R}^n)$ has compact support and $\|f\| \le h^{-N}$, then for all $\rho \in T^*\mathbb{R}^n$,

$$\rho \in \mathrm{WF}_h\,(R_0(\lambda) f) \Longrightarrow \gamma_\rho^- \cap \mathrm{WF}_h\, f \neq \varnothing.$$

If further $\gamma_\rho^- \subset \{|x| > R_0\}$, then similarly

$$\rho \in \mathrm{WF}_h\,(R_W(\lambda) f) \Longrightarrow \gamma_\rho^- \cap \mathrm{WF}_h\, f \neq \varnothing.$$

□

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