二阶奇异耦合微分方程组 Neumann 边值问题的解

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摘要: 利用 Schauder 不动点定理研究二阶非自治半正的耦合微分方程组 Neumann 边值问题正解的存在性. 在扰动项积分值符号同正、同负和异号的情况下, 分别获得了该奇异耦合微分方程组 Neumann 边值问题存在正解的条件.

关键词: Neumann 边值问题; 奇异耦合方程组; 正解; Schauder 不动点定理
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Solutions to Neumann Boundary Value Problems of Second Order Singular Coupled Systems

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Abstract: Using Schauder's fixed point theorem, the authors studied Neumann boundary value problems of second order non-autonomous semipositive coupled systems. We established the existence of positive solutions for Neumann boundary value problems of the singular coupled systems under the conditions that the signs of integral disturbance terms are positive, or negative, or different.

Key words: Neumann boundary value problem; singular coupled system; positive solution; Schauder's fixed point theorem

0 引 言

考虑如下耦合方程组的 Neumann 边值问题:

$$\begin{cases} x'' + m_1^2 x = f_1(t,y) + e_1(t), \quad 0 < t < 1, \\ y'' + m_2^2 y = f_2(t,x) + e_2(t), \\ x'(0) = x'(1) = y'(0) = y'(1) = 0, \end{cases} \tag{1}$$

其中: $e_i \in C[0,1]$; $0 < m_i < \pi/2$; $f_i \in C([0,1] \times (0,+\infty),(0,+\infty))$, 并且在零点处可奇异

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($i=1,2$).

目前, 关于微分方程边值问题的研究已有许多结果[1-13], 其中大部分都是对其正解的存在性和唯一性进行讨论.

文献[1-2]用 Krasnosel' skii 不动点定理研究了形为

$$\begin{cases} x''+m^2x=f(t,x), & 0<t<1, \\ x'(0)=x'(1)=0 \end{cases}$$

的 Neumann 边值问题. 在非线性项 $f(t,x)$ 为超线性或次线性的条件下, 蒋达清等[1]得到了该微分模型系统存在一个正解的结果. 在比文献[1]条件更弱的情况下, Sun 等[2]获得了该微分模型系统存在至少两个正解的条件.

文献[3]利用 Leray-Schauder 型非线性抉择定理研究了 Neumann 边值问题:

$$\begin{cases} x''+m^2x=f(t,x)+e(t), & 0<t<1, \\ x'(0)=x'(1)=0, \end{cases}$$

获得了正解存在的充分条件.

本文利用 Schauder 不动点定理, 建立耦合方程组的 Neumann 边值问题(1)正解的存在性.

1 预备知识

引理 1[1] 设 h: $[0,1]\to[0,+\infty)$ 为一连续函数, 则边值问题

$$\begin{cases} x''+m^2x=h(t), & 0<t<1, \quad m\in(0,\pi/2); \\ x'(0)=x'(1)=0 \end{cases}$$

有唯一的解 $x\in C^2[0,1]$, 其表达式为 $x(t)=\int_0^1 G(t,s)h(s)\mathrm{d}s$, 其中 Green 函数为

$$G(t,s)=\begin{cases} \dfrac{\cos m(1-t)\cos ms}{m\sin ms}, & 0\leqslant s\leqslant t\leqslant 1, \\ \dfrac{\cos m(1-s)\cos ms}{m\sin mt}, & 0\leqslant t\leqslant s\leqslant 1. \end{cases}$$

定义函数 γ_i: $[0,1]\to R$ 如下:

$$\gamma_i(t)=\int_0^1 G_i(t,s)e_i(s)\mathrm{d}s, \quad i=1,2,$$

根据引理 1, $\gamma_i(t)$ 是边值问题

$$\begin{cases} z''+m_i^2=e_i(t), & 0<t<1, \\ z'(0)=z'(1)=0 \end{cases}$$

唯一的解, 且显然有 $G_i(t,s)>0$.

为叙述方便, 令 P^* 表示上确界, P_* 表示下确界. 记 $\gamma_{i*}=\min\limits_t \gamma_i(t)$, $\gamma_i^*=\max\limits_t \gamma_i(t)$.

定理 1(**Schauder 不动点定理**)[14] 令 E 为一 Banach 空间, $B\subset E$ 为一有界闭凸子集, 若连续紧算子 T: $E\to E$ 使得 $T(B)\to B$ 成立, 则 T 在 B 上有一个不动点.

2 主要结果

1) $\gamma_{1*}\geqslant 0$, $\gamma_{2*}\geqslant 0$.

定理 2 假设存在函数 $b_i>0$, $\hat{b}_i>0$ 和 $0<\alpha_i<1$, 使得

$$0\leqslant \frac{\hat{b}_i(t)}{x^{\alpha_i}}\leqslant f_i(t,x)\leqslant \frac{b_i(t)}{x^{\alpha_i}} \quad (i=1,2) \tag{2}$$

对任意的 $x>0$ 和几乎处处的 $t\in(0,1)$ 都成立, 若 $\gamma_{1*}\geqslant 0$, $\gamma_{2*}\geqslant 0$, 则耦合方程组边值问题(1)存在一个正解.

证明: 根据引理 1, 耦合方程组边值问题(1)的解即为如下定义的全连续紧算子

$A(x,y)=(A_1x,A_2y)$: $C[0,1]\times C[0,1]\to C[0,1]\times C[0,1]$的不动点,

$$(A_1x)(t):=\int_0^1 G_1(t,s)[f_1(s,y(s))+e_1(s)]\mathrm{d}s=\int_0^1 G_1(t,s)f_1(s,y(s))\mathrm{d}s+\gamma_1(t);$$

$$(A_2y)(t):=\int_0^1 G_2(t,s)[f_2(s,x(s))+e_2(s)]\mathrm{d}s=\int_0^1 G_2(t,s)f_2(s,x(s))\mathrm{d}s+\gamma_2(t).$$

定义闭凸集

$$K=\{(x,y)\in C[0,1]\times C[0,1]: r_1\leqslant x(t)\leqslant R_1,\ r_2\leqslant y(t)\leqslant R_2,\ t\in[0,1]\},$$

其中 $R_1>r_1>0$ 和 $R_2>r_2>0$ 是固定的适宜正常数. 根据 Schauder 不动点定理, 如果能证明算子 A 是 $K\to K$ 的映射, 则结论得证. 为方便, 引进记号:

$$\beta_i(t)=\int_0^1 G_i(t,s)b_i(s)\mathrm{d}s,\quad \hat{\beta}_i(t)=\int_0^1 G_i(t,s)\hat{b}_i(s)\mathrm{d}s,\quad i=1,2.$$

给定$(x,y)\in K$, 根据函数 G_i 和 $f_i(i=1,2)$符号的非负性, 有

$$(A_1x)(t)=\int_0^1 G_1(t,s)f_1(s,y(s))\mathrm{d}s+\gamma_1(t)\geqslant\int_0^1 G_1(t,s)\frac{\hat{b}_1(s)}{y^{\alpha_1}(s)}\mathrm{d}s\geqslant\int_0^1 G_1(t,s)\frac{\hat{b}_1(s)}{R_2^{\alpha_1}}\mathrm{d}s\geqslant\hat{\beta}_{1*}\cdot\frac{1}{R_2^{\alpha_1}},$$

$$(A_1x)(t)=\int_0^1 G_1(t,s)f_1(s,y(s))\mathrm{d}s+\gamma_1(t)\leqslant\int_0^1 G_1(t,s)\frac{b_1(s)}{y^{\alpha_1}(s)}\mathrm{d}s+\gamma_1^*\leqslant$$

$$\int_0^1 G_1(t,s)\frac{b_1(s)}{r_2^{\alpha_1}}\mathrm{d}s+\gamma_1^*\leqslant\beta_1^*\cdot\frac{1}{r_2^{\alpha_1}}+\gamma_1^*.$$

同理有

$$(A_2y)(t)=\int_0^1 G_2(t,s)f_2(s,x(s))\mathrm{d}s+\gamma_2(t)\geqslant\int_0^1 G_2(t,s)\frac{\hat{b}_2(s)}{x^{\alpha_2}(s)}\mathrm{d}s\geqslant\int_0^1 G_2(t,s)\frac{\hat{b}_2(s)}{R_1^{\alpha_2}}\mathrm{d}s\geqslant\hat{\beta}_{2*}\cdot\frac{1}{R_1^{\alpha_2}},$$

$$(A_2y)(t)=\int_0^1 G_2(t,s)f_2(s,x(s))\mathrm{d}s+\gamma_2(t)\leqslant\int_0^1 G_2(t,s)\frac{b_2(s)}{x^{\alpha_2}(s)}\mathrm{d}s+\gamma_2^*\leqslant$$

$$\int_0^1 G_2(t,s)\frac{b_2(s)}{r_1^{\alpha_2}}\mathrm{d}s+\gamma_2^*\leqslant\beta_2^*\cdot\frac{1}{r_1^{\alpha_2}}+\gamma_2^*.$$

如果 r_1,r_2,R_1 和 R_2 的选择使得

$$\hat{\beta}_{1*}\cdot\frac{1}{R_2^{\alpha_1}}\geqslant r_1,\qquad \beta_1^*\cdot\frac{1}{r_2^{\alpha_1}}+\gamma_1^*\leqslant R_1,$$

$$\hat{\beta}_{2*}\cdot\frac{1}{R_1^{\alpha_2}}\geqslant r_2,\qquad \beta_2^*\cdot\frac{1}{r_1^{\alpha_2}}+\gamma_2^*\leqslant R_2$$

成立, 则$(A_1x,A_2y)\in K$.

注意到$\hat{\beta}_{i*}$, $\beta_{i*}>0$, 不妨取 $R=R_1=R_2$, $r=r_1=r_2$, $r=\frac{1}{R}$, 因为 $\alpha_i<1$, 所以能找到足够大的 $R>1$, 使得不等式

$$\hat{\beta}_{1*}\cdot R^{1-\alpha_1}\geqslant 1,\qquad \beta_1^*\cdot R^{\alpha_1}+\gamma_1^*\leqslant R,$$

$$\hat{\beta}_{2*}\cdot R^{1-\alpha_2}\geqslant 1,\qquad \beta_2^*\cdot R^{\alpha_2}+\gamma_2^*\leqslant R$$

成立. 因此可以通过对 r_1,r_2,R_1 和 R_2 选择使得$(A_1x,A_2y)\in K$是合理的.

2) $\gamma_1^*\leqslant 0$, $\gamma_2^*\leqslant 0$.

如果 $\gamma_1^*\leqslant 0$, $\gamma_2^*\leqslant 0$, 则表明弱奇异方程(1)也存在正解.

定理 3　假设存在 $b_i,\hat{b}_i>0$ 和 $0<\alpha_i<1$, 使得条件(2)成立. 如果 $\gamma_1^*\leqslant 0$, $\gamma_2^*\leqslant 0$, 且满足:

$$\begin{aligned}\gamma_{1*}&\geqslant\left[\alpha_1\alpha_2\cdot\frac{\hat{\beta}_{1*}}{(\beta_2^*)^{\alpha_1}}\right]^{1/(1-\alpha_1\alpha_2)}\left(1-\frac{1}{\alpha_1\alpha_2}\right),\\ \gamma_{2*}&\geqslant\left[\alpha_1\alpha_2\cdot\frac{\hat{\beta}_{2*}}{(\beta_1^*)^{\alpha_2}}\right]^{1/(1-\alpha_1\alpha_2)}\left(1-\frac{1}{\alpha_1\alpha_2}\right),\end{aligned}\tag{3}$$

则耦合方程组边值问题(1)存在一个正解.

证明：当 $\gamma_1^* \leqslant 0$, $\gamma_2^* \leqslant 0$ 时，为了证明 $A: K \to K$，只需找到使得不等式

$$\frac{\hat{\beta}_{1*}}{R_2^{\alpha_1}} + \gamma_{1*} \geqslant r_1, \qquad \frac{\beta_1^*}{r_2^{\alpha_1}} \leqslant R_1, \tag{4}$$

$$\frac{\hat{\beta}_{2*}}{R_1^{\alpha_2}} + \gamma_{2*} \geqslant r_2, \qquad \frac{\beta_2^*}{r_1^{\alpha_2}} \leqslant R_2 \tag{5}$$

成立的 $0 < r_1 < R_1$, $0 < r_2 < R_2$ 即可.

固定 $R_1 = \dfrac{\beta_1^*}{r_2^{\alpha_1}}$, $R_2 = \dfrac{\beta_2^*}{r_1^{\alpha_2}}$，如果 r_2 满足不等式 $\hat{\beta}_{2*}(\beta_1^*)^{-\alpha_2} r_2^{\alpha_1\alpha_2} + \gamma_{2*} \geqslant r_2$，或者不等式 $\gamma_{2*} \geqslant g(r_2) := r_2 - \dfrac{\hat{\beta}_{2*}}{(\beta_1^*)^{\alpha_2}} r_2^{\alpha_1\alpha_2}$. 则式(5)中的第一个不等式成立.

函数 $g(r_2)$ 在点 $r_{20} := \left[\alpha_1\alpha_2 \cdot \dfrac{\hat{\beta}_{2*}}{(\beta_1^*)^{\alpha_2}}\right]^{1/(1-\alpha_1\alpha_2)}$ 处有最小值. 取 $r_{2*} = r_{20}$，如果

$$\gamma_{2*} \geqslant g(r_{20}) = \left[\alpha_1\alpha_2 \cdot \frac{\hat{\beta}_{2*}}{(\beta_1^*)^{\alpha_2}}\right]^{1/(1-\alpha_1\alpha_2)} \left(1 - \frac{1}{\alpha_1\alpha_2}\right),$$

则式(5)成立.

类似地有

$$\gamma_{1*} \geqslant h(r_1) := r_1 - \frac{\hat{\beta}_{1*}}{(\beta_2^*)^{\alpha_1}} r_1^{\alpha_1\alpha_2},$$

函数 $h(r_1)$ 在点 $r_{10} := \left[\alpha_1\alpha_2 \cdot \dfrac{\hat{\beta}_{1*}}{(\beta_2^*)^{\alpha_1}}\right]^{1/(1-\alpha_1\alpha_2)}$ 处取得最小值. 如果

$$\gamma_{1*} \geqslant \left[\alpha_1\alpha_2 \cdot \frac{\hat{\beta}_{1*}}{(\beta_2^*)^{\alpha_1}}\right]^{1/(1-\alpha_1\alpha_2)} \left(1 - \frac{1}{\alpha_1\alpha_2}\right),$$

则式(4)成立.

设 $r_1 = r_{10}$, $r_2 = r_{20}$，如果 $\gamma_{1*} \geqslant g(r_1)$ 和 $\gamma_{2*} \geqslant g(r_2)$ 成立，则式(4)和(5)中的第一个不等式均成立，而不等式 $\gamma_{1*} \geqslant g(r_1)$ 和 $\gamma_{2*} \geqslant g(r_2)$ 即为条件(3). 由 R_1 和 R_2 的选择直接可得式(4)和(5)中的第二个不等式，因此下面仅需证明 $R_1 = \dfrac{\beta_1^*}{r_{20}^{\alpha_1}} > r_{10}$ 和 $R_2 = \dfrac{\beta_2^*}{r_{10}^{\alpha_2}} > r_{20}$ 成立即可.

因为 $\hat{\beta}_{i*} \leqslant \beta_i^*$ $(i=1,2)$，所以计算可得

$$R_1 = \frac{\beta_1^*}{r_{20}^{\alpha_1}} = \frac{\beta_1^*}{\left\{\left[\alpha_1\alpha_2 \cdot \dfrac{\hat{\beta}_{2*}}{(\beta_1^*)^{\alpha_2}}\right]^{1/(1-\alpha_1\alpha_2)}\right\}^{\alpha_1}} = \frac{\beta_1^*}{\left[\alpha_1\alpha_2 \cdot \dfrac{\hat{\beta}_{2*}}{(\beta_1^*)^{\alpha_2}}\right]^{\alpha_1/(1-\alpha_1\alpha_2)}} =$$

$$\frac{(\beta_1^*)^{1+\alpha_1\alpha_2/(1-\alpha_1\alpha_2)}}{(\alpha_1\alpha_2 \cdot \hat{\beta}_{2*})^{\alpha_1/(1-\alpha_1\alpha_2)}} = \frac{(\beta_1^*)^{1/(1-\alpha_1\alpha_2)}}{[(\alpha_1\alpha_2 \cdot \hat{\beta}_{2*})^{\alpha_1}]^{1/(1-\alpha_1\alpha_2)}} =$$

$$\left[\frac{\beta_1^*}{(\alpha_1\alpha_2 \cdot \hat{\beta}_{2*})^{\alpha_1}}\right]^{1/(1-\alpha_1\alpha_2)} = \left[\frac{1}{(\alpha_1\alpha_2)^{\alpha_1}} \cdot \frac{\beta_1^*}{(\hat{\beta}_{2*})^{\alpha_1}}\right]^{1/(1-\alpha_1\alpha_2)} >$$

$$\left[\alpha_1\alpha_2 \cdot \frac{\hat{\beta}_{1*}}{(\beta_2^*)^{\alpha_1}}\right]^{1/(1-\alpha_1\alpha_2)} = r_{10}$$

成立. 类似地，也能得到 $R_2 > r_{20}$.

3) $\gamma_{1*} \geqslant 0$, $\gamma_2^* \leqslant 0$ ($\gamma_1^* \leqslant 0$, $\gamma_{2*} \geqslant 0$).

定理 4 假设条件(2)成立. 如果 $\gamma_{1*} \geqslant 0$, $\gamma_2^* \leqslant 0$，且

$$\gamma_{2*} \geqslant r_{21} - \hat{\beta}_{2*} \cdot \frac{r_{21}^{\alpha_1\alpha_2}}{(\beta_1^* + \gamma_1^* r_{21}^{\alpha_1})^{\alpha_2}}, \tag{6}$$

其中 $0 < r_{21} < +\infty$ 是方程

$$r_2^{1-\alpha_1\alpha_2}(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{1+\alpha_2} = \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*}$$

唯一的正解，则耦合方程组边值问题(1)存在一个正解.

证明：当 $\gamma_{1*} \geqslant 0$，$\gamma_2^* \leqslant 0$ 时，为了证明 $A: K \to K$，只需找到使不等式

$$\frac{\hat{\beta}_{1*}}{R_2^{\alpha_1}} \geqslant r_1, \qquad \frac{\beta_2^*}{r_1^{\alpha_2}} \leqslant R_2, \tag{7}$$

$$\frac{\hat{\beta}_{2*}}{R_1^{\alpha_2}} + \gamma_{2*} \geqslant r_2, \qquad \frac{\beta_1^*}{r_2^{\alpha_1}} + \gamma_1^* \leqslant R_1 \tag{8}$$

成立的 $r_1 < R_1$，$r_2 < R_2$ 即可.

固定 $R_2 = \frac{\beta_2^*}{r_1^{\alpha_2}}$，如果 r_1 满足不等式 $\frac{\hat{\beta}_{1*}}{(\beta_2^*)^{\alpha_1}} \cdot r_1^{\alpha_1\alpha_2} \geqslant r_1$ 或不等式 $0 < r_1 \leqslant \left[\frac{\hat{\beta}_{1*}}{(\beta_2^*)^{\alpha_1}}\right]^{1/(1-\alpha_1\alpha_2)}$，则式(7)中的第一个不等式成立. 如果选择 $r_1 > 0$ 足够小，则不等式(9)成立，同时 R_2 也足够大. 固定 $R_1 = \frac{\beta_1^*}{r_2^{\alpha_1}} + \gamma_1^*$，如果 r_2 满足不等式

$$\gamma_{2*} \geqslant r_2 - \frac{\hat{\beta}_{2*}}{R_1^{\alpha_2}} = r_2 - \hat{\beta}_{2*} \cdot \frac{1}{(\beta_1^*/r_2^{\alpha_1} + \gamma_1^*)^{\alpha_2}} =$$

$$r_2 - \hat{\beta}_{2*} \cdot \frac{1}{[(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})/r_2^{\alpha_1}]^{\alpha_2}} = r_2 - \hat{\beta}_{2*} \cdot \frac{r_2^{\alpha_1\alpha_2}}{(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{\alpha_2}},$$

或者不等式

$$\gamma_{2*} \geqslant f(r_2) := r_2 - \hat{\beta}_{2*} \cdot \frac{r_2^{\alpha_1\alpha_2}}{(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{\alpha_2}}.$$

则式(8)中的第一个不等式成立. 根据

$$f'(r_2) = 1 - \hat{\beta}_{2*} \cdot \frac{1}{(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{2\alpha_2}} \cdot$$

$$[\alpha_1\alpha_2 r_2^{\alpha_1\alpha_2-1}(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{\alpha_2} - r_2^{\alpha_1\alpha_2}\alpha_2(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{\alpha_2-1}\alpha_1\gamma_1^* r_2^{\alpha_1-1}] =$$

$$1 - \frac{\hat{\beta}_{2*}\alpha_1\alpha_2 r_2^{\alpha_1\alpha_2-1}}{(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{\alpha_2}}\left[1 - \frac{r_2^{\alpha_1}\gamma_1^*}{\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1}}\right] =$$

$$1 - \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*} r_2^{\alpha_1\alpha_2-1}(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{-1-\alpha_2},$$

有 $f'(0) = -\infty$，$f'(+\infty) = 1$，因此存在 r_{21}，使得 $f'(r_{21}) = 0$，且因为

$$f''(r_2) = -[\alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*}(\alpha_1\alpha_2 - 1)r_2^{\alpha_1\alpha_2-2}(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{-1-\alpha_2} +$$

$$\alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*} r_2^{\alpha_1\alpha_2-1}(-1-\alpha_2)(\beta_1^* + \gamma_1^* \cdot r_2^{\alpha_1})^{-2-\alpha_2}\gamma_1^*\alpha_1 r_2^{\alpha_1-1}] > 0,$$

所以，函数 $f(r_2)$ 在点 r_{21} 处取得最小值，即 $f(r_{21}) = \min\limits_{r_2\in(0,+\infty)} f(r_2)$. 再注意到 $f'(r_{21}) = 0$，则有

$$1 - \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*} r_{21}^{\alpha_1\alpha_2-1}(\beta_1^* + \gamma_1^* \cdot r_{21}^{\alpha_1})^{-1-\alpha_2} = 0,$$

或

$$r_{21}^{1-\alpha_1\alpha_2}(\beta_1^* + \gamma_1^* \cdot r_{21}^{\alpha_1})^{1+\alpha_2} = \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*}$$

成立.

取 $r_2 = r_{21}$，如果 $\gamma_{2*} \geqslant f(r_{21})$，则不等式(8)中的第一个不等式成立，而这正是条件(6). 直接利用 R_2 的选择即可得到式(8)中的第二个不等式. 因此下面只需证明 $r_{21} < R_2$ 和 $r_{10} < R_1$ 即可. 事实上，只要选择 r_1 足够小的同时 R_2 足够大即可.

注 1　在定理 4 中，由于条件(6)的右端总是取负值，因此等价于证明 $f(r_{21}) < 0$. 由定理 4 的证明

可知显然.

类似地，有：

定理5 假设条件(2)满足. 如果 $\gamma_1^* \leqslant 0$, $\gamma_{2*} \geqslant 0$, 且满足

$$\gamma_{1*} \geqslant r_{11} - \hat{\beta}_{1*} \cdot \frac{r_{11}^{\alpha_1\alpha_2}}{(\beta_2^* + \gamma_2^* r_{11}^{\alpha_2})^{\alpha_1}},$$

其中 $0 < r_{11} < +\infty$ 是方程 $r_1^{1-\alpha_1\alpha_2}(\beta_2^* + \gamma_2^* \cdot r_1^{\alpha_2})^{1+\alpha_1} = \alpha_1\alpha_2\beta_2^*\hat{\beta}_{1*}$ 的唯一解，则耦合方程组边值问题(1)存在一个正解.

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