云南大学学报（自然科学版） 2003, 25 (3): 176~ 180 CN 53- 1045/N ISSN 0258- 7971
Journal of Yunnan University

Solitary and periodic traveling wave solutions in Klein-Gordon-Schrodinger Equations*

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Abstract: Solitary waves and periodic waves for Klein-Gordon-Schrodinger Equations are studied, by using the theory of dynamical systems. Bifurcation parameter sets are shown. Under given parameter conditions, explicit formulas of solitary wave solutions and periodic wave solutions are obtained.

Key words: solitary wave; periodic wave; Klein-Gordon-Schrodinger equations
CLC number: O 192; O 175.12 **Document code**: A **Article ID**: 0258- 7971(2003)03- 0176- 05
AMS(MOS) subject classification: 34A20; 34C35; 35B65; 58F05

In this paper we consider the following coulped $(1+n)$- dimensional Klein-Gordon-Schrodinger equations (KGS equations in short)

$$i\psi_t+\frac{1}{2}\Delta\psi=-\varphi\psi, \qquad (1_a)$$

$$\varphi_{tt}-\Delta\varphi+m^2\varphi=|\psi|^2, \qquad (1_b)$$

where

$$\Delta=\frac{\partial}{\partial x_1^2}+\frac{\partial}{\partial x_2^2}+\cdots+\frac{\partial}{\partial x_n^2}$$

is the Laplacian opearator. This system is used to describe a classical model of the Yukawa interaction of conserved complex nucleon field with neutral real meson field. Here, ψ is a complex scalar nucleon field, φ is a real scalar meson field, and m is the mass of a meson. For the background materials of model equations, we refer to the paper[1,2] and the references therein. The unique global existence theorem for the Cauchy problem to (1) with $n=3$ is already established (see [1, 3, 4]). Xia Jingna etc[5] gave a explict solitary wave solution for KGS equations by using so-called the homogeneous balance principle. Unfortunately, the results of [5] are not complete since the authors did not study the bifurcation behaviour of phase portraits for the corresponding traveling wave equations. In this paper, we consider bifurcation problem of solitary waves and periodic waves for (1), by using the theory of dynamical systems[6-8]. Under fixed parameter conditions, all explicit formulas of solitary wave and periodic solutions can be easily obtained.

To find the travelling wave solusions of KGS equations, we first assume that

$$\begin{cases}\psi=e^{i\eta}u(x,t),\\ x=(x_1,x_2,\cdots,x_n),\\ \eta=\sum_{j=1}^{n}\alpha_j x_j+\beta t.\end{cases} \qquad (2)$$

Substituing (2) into (1), canceling $e^{i\eta}$, we have

$$u_t+\sum_{j=1}^{n}\alpha_j u_{x_j}=0, \qquad (3_a)$$

$$\Delta u+2u\varphi-\left(\sum_{j=1}^{n}\alpha_j^2+2\beta\right)u=0, \qquad (3_b)$$

* **Received date**: 2002- 12- 05
Foundation item: Research is supported by the Natural Science Foundation of China(10231020) and Yunnan province(20010049).

$$\varphi_{tt} - \Delta\varphi + m^2\varphi - u^2 = 0. \quad (3_c)$$

Letting $\xi = \sum_{j=1}^{n} \gamma_j x_j - ct$, $u(x, t) = u(\xi)$, it follows

$$u_{\xi\xi} - \frac{\alpha^2 + 2\beta}{\gamma^2} u + \frac{2}{\gamma^2} u\varphi = 0, \quad (4_a)$$

$$\varphi_{\xi\xi} - \frac{m^2}{\gamma^2 - c^2}\varphi + \frac{1}{\gamma^2 - c^2} u^2 = 0, \quad (4_b)$$

where $\alpha^2 = \sum_{j=1}^{n} \alpha_j^2$, $\gamma^2 = \sum_{j=1}^{n} \gamma_j^2$, $c = \sum_{j=1}^{n} \alpha_j \gamma_j$.

Denote that $q_1 = u$, $p_1 = u_\xi$, $q_2 = \varphi$, $p_2 = \varphi_\xi$. Then, (4) becomes the following 4 - dimensional dynamical system

$$\dot{q}_1 = p_1, \quad (5_a)$$

$$\dot{q}_2 = p_2, \quad (5_b)$$

$$\dot{p}_1 = \frac{\alpha^2 + 2\beta}{\gamma^2} q_1 - \frac{2}{\gamma^2} q_1 q_2, \quad (5_c)$$

$$\dot{p}_2 = \frac{m^2}{\gamma^2 - c^2} q_2 - \frac{1}{\gamma^2 - c^2} q_1^2. \quad (5_d)$$

The phase orbits defined by the vector fields of system (5) determine all travelling wave solutions of (1). Suppose that $u(x, t) = u(\xi)$ is a continuous solution of (1) for $\xi \in (-\infty, \infty)$ and $\lim_{\xi\to\infty} u(\xi) = \alpha$, $\lim_{\xi\to-\infty} u(\xi) = \beta$. It is well known that (i) $u(x, t)$ is called a solitary wave solution if $\alpha = \beta$. (ii) $u(x, t)$ is called a kink or antikink solution if $\alpha \neq \beta$. Usually, a solitary wave solution corresponds to a homoclinic orbit. A kink (or antikink) wave solution correspods to a heteroclinic orbit. Thus, to investigate all bifurcations of solitary waves and periodic waves of (1), we shall find all bounded solutions of (5) depending on the parameter space of this system. The bifurcation theory of dynamical systems (see [6]) plays an important role in our study.

This paper is organized as follows. In section 1, we condider the dynamical behaviour of (4) in a plane. We obtain an explicit formula of solitary wave solutions and periodic wave solutions of (4) under given parameter conditions. In section 2, we point out some possible bifurcation behaviour of (5) and the Hamiltonian case in (5). Our study results contain some results in [5] as special examples.

1 The solitary wave and periodic wave solutions of the completely integrable case

In this section, we consider the dynamical behaviour of (4) on the plane $u - a\varphi = 0$, for some $a > 0$. Substituting $u = a\varphi$ and $\varphi = \frac{u}{a}$ into (4_a) and (4_b), respectively, we obtain two uncoupled integrable systems

$$u_{\xi\xi} - \frac{\alpha^2 + 2\beta}{\gamma^2} u + \frac{2}{a\gamma^2} u^2 = 0, \quad (6_a)$$

$$\varphi_{\xi\xi} - \frac{m^2}{\gamma^2 - c^2}\varphi + \frac{a^2}{\gamma^2 - c^2}\varphi^2 = 0, \quad (6_b)$$

which correspond to two planar dynamical systems

$$\frac{du}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{\alpha^2 + 2\beta}{\gamma^2} u - \frac{2}{a\gamma^2} u^2, \quad (7_a)$$

$$\frac{d\varphi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{m^2}{\gamma^2 - c^2}\varphi - \frac{a^2}{\gamma^2 - c^2}\varphi^2. \quad (7_b)$$

There exist two critical points of (7_a) and (7_b) at $O_1(u, y) = (0, 0)$, $O_2(\varphi, y) = (0, 0)$ and $A_1(u, y) = A_1\left(\frac{a(\alpha^2 + 2\beta)}{2}, 0\right)$, $A_2(\varphi, y) = A_2\left(\frac{m^2}{a^2}, 0\right)$, reapectively.

We assume that $\alpha^2 + 2\beta > 0$ and $\gamma^2 - c^2 > 0$.

In this case, the origin O_1 and O_2 are saddle points of (7), A_1 and A_2 are centers of (7). These two systems have the first integrals

$$H_1(u, y) = \frac{1}{2}y^2 - \frac{\alpha^2 + 2\beta}{2\gamma^2} u^2 + \frac{2}{3a\gamma^2} u^3, \quad (8_a)$$

$$H_2(\varphi, y) = \frac{1}{2}y^2 - \frac{m^2}{2(\gamma^2 - c^2)}\varphi^2 + \frac{a^2}{3(\gamma^2 - c^2)}\varphi^3. \quad (8_b)$$

We see from (7) and (8) that corresponding to the level curves defined by $H_1 = H_2 = 0$, two homoclinic orbits connecting the origin of (7_a) and (7_b) have the following parametric representations:

$$u(\xi) = \frac{3a(\alpha^2 + 2\beta)}{4}\operatorname{sech}^2\left(\frac{\sqrt{\alpha^2 + 2\beta}}{2\gamma}\xi\right), \quad (9)$$

$$\varphi(\xi) = \frac{m^2}{2a^2}\operatorname{sech}^2\left(\frac{m}{2\sqrt{\gamma^2 - c^2}}\xi\right). \quad (10)$$

The relation $u = a\varphi$ gives the following parameter conditions

$$\frac{\alpha^2+2\beta}{m^2}=\frac{\gamma^2}{\gamma^2-c^2},$$

$$\alpha^2+2\beta=\frac{2m^2}{a^2},$$

i. e.,

$$(2-a^2)\gamma^2=2c^2,\ a^2<2. \qquad (11)$$

Thus, we have

$$u(\xi)=\frac{3m^2}{2a}\mathrm{sech}^2\left(\frac{\sqrt{2}m}{2a\gamma}\xi\right), \qquad (12)$$

$$\varphi(\xi)=\frac{3m^2}{2a^2}\mathrm{sech}^2\left(\frac{\sqrt{2}m}{2a\gamma}\xi\right). \qquad (13)$$

We see from the above discussion that the following conclusion holds.

Theorem 1 Suppose that conditions (11) holds, i. e., for any $a\in(0,\sqrt{2})$

$$\left(\sum_{j=1}^{n}\alpha_j^2+2\beta\right)/(m^2)=$$

$$\left(\sum_{j=1}^{n}\gamma_j^2\right)\Big/\left(\sum_{j=1}^{n}\gamma_j^2-\left(\sum_{j=1}^{n}\alpha_j\gamma_j\right)^2\right),$$

$$(2-a^2)\sum_{j=1}^{n}\gamma_j^2=2\left(\sum_{j=1}^{n}\alpha_j\gamma_j\right)^2.$$

Then, KGS equations have uncountablely infinite many (corresponding to different a) exact solitary solutions

$$\phi(x,t)=\frac{3m^2}{2a}\mathrm{sech}^2\left(\frac{\sqrt{2}m}{2a\gamma}\sum_{j=1}^{n}\gamma_j(x_j-\alpha_j t)\right)\times$$

$$\exp\left(i\left(\sum_{j=1}^{n}\alpha_j x_j+\beta t\right)\right), \qquad (14)$$

$$\varphi(x,t)=\frac{3m^2}{2a^2}\mathrm{sech}^2\left(\frac{\sqrt{2}m}{2a\gamma}\sum_{j=1}^{n}\gamma_j(x_j-\alpha_j t)\right). \qquad (15)$$

Obviously, by using more simple and natural method than one in [5], theorem 1 corrects and generalizes the maian result in [5].

Under the condition of Theorem 1, H_1 and H_2 defined by (8) determine the same Hamiltonian values. When $h\in\left(-\frac{m^6}{3a^4\gamma^2},0\right)$, there exists a family of closed orbits of (7a) and (7b) having the level curves $H_1=H_2=h$. Denote that

$$2h+\frac{\alpha^2+2\beta}{2\gamma^2}u^2-\frac{2}{3a\gamma^2}u^3=$$

$$(r_1-u)(u-r_2)(u-r_3),$$

$$h\in\left(-\frac{m^6}{3a^4\gamma^2},0\right), \qquad (16)$$

where $r_j=r_j(h)(j=1,2,3)$ are functions of h. Using (7) and (16) to culculate, we obtain

$$u(\xi)=r_1-(r_1-r_2)\mathrm{sn}^2\left(\frac{\sqrt{r_1-r_3}}{2}\xi,k\right),$$

$$k=\sqrt{\frac{r_1-r_2}{r_1-r_3}}, \qquad (17)$$

where $sn(x,k)$ is the Jacobian elliptic function with the modulus $k\in(0,1)$. Clearly, $u(\xi)$ is a periodic function with the period $T(k)=\frac{4}{\sqrt{r_1-r_2}}K(k)$, where $K(k)$ is the complete elliptic integral of the first kind. The function $\varphi(\xi)$ has the same formula as (17).

Theorem 2 Under the same condition of Theorem 1, for $h\in\left(-\frac{m^6}{3a^4\gamma^2},0\right)$, $a\in(0,\sqrt{2})$ there exist uncountablely infinite many families of periodic wave solutions of KGS equations as follows

$$\phi(x,t)=\left(r_1-(r_1-r_2)\mathrm{sn}^2\left(\frac{\sqrt{r_1-r_3}}{2}\cdot\sum_{j=1}^{n}\gamma_j(x_j-\alpha_j t),k\right)\right)\times$$

$$\exp\left(i\left(\sum_{j=1}^{n}\alpha_j x_j+\beta t\right)\right), \qquad (18)$$

$$\varphi(x,t)=r_1-(r_1-r_2)\cdot\mathrm{sn}^2\left(\frac{\sqrt{r_1-r_3}}{2}\sum_{j=1}^{n}\gamma_j(x_j-\alpha_j t),k\right). \qquad (19)$$

As a particular example, we consider the (1+1)-dimensional case of (1). For $n=1$, we have $c=\alpha\gamma$, $\gamma^2-c^2=\gamma^2(1-\alpha^2)$. The condition of Theorem 1 becomes

$$\alpha^2+2\beta=\frac{m^2}{1-\alpha^2},\ a^2=2(1-\alpha^2). \qquad (20)$$

Hence, we have the following exact travelling wave solution of (1)

$$u(x,t)=\frac{3\sqrt{2}m^2}{4\sqrt{1-\alpha^2}}\mathrm{sech}\left(\frac{m}{2\sqrt{1-\alpha^2}}(x-\alpha t)\right)\cdot$$

$$\exp(i(\alpha x+\beta t)), \qquad (21)$$

$$\varphi(x,t)=\frac{3m^2}{4(1-\alpha^2)}\mathrm{sech}\left(\frac{m}{2\sqrt{1-\alpha^2}}(x-\alpha t)\right). \qquad (22)$$

2 Possible bifurcation behaviour of (5) and Hamiltonian case

In this section, we consider some possible bifurcations for the system (5). Clearly, in the 4- dimensional phase space (q_1, q_2, p_1, p_2), there are 3 equilibriums of (5) at

$O(0, 0, 0, 0)$, $A_+\left(\sqrt{\frac{\alpha^2+2\beta}{2}}m, \frac{\alpha^2+2\beta}{2}, 0, 0\right)$ and $A_-\left(-\sqrt{\frac{\alpha^2+2\beta}{2}}m, \frac{\alpha^2+2\beta}{2}, 0, 0\right)$, if $\alpha^2+2\beta>0$. At the equilibrium $(q_1^0, q_2^0, p_1^0, p_2^0)$, the Jacobian matrax of the linearized system of (5) has the form

$$J(q_1^0, q_2^0, p_1^0, p_2^0)=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\alpha^2+2\beta}{\gamma^2}-\frac{2}{\gamma^2}q_2^0 & \frac{2}{\gamma^2}q_1^0 & 0 & 0 \\ -\frac{2}{\gamma^2-c^2}q_1^0 & \frac{m^2}{\gamma^2-c^2} & 0 & 0 \end{pmatrix}.$$

Thus, the characteristic equations of the linearized systems of (5) at the origin O and $A_\pm$ respectively has the forms as follows

$$\left(\lambda^2-\frac{m^2}{\gamma^2-c^2}\right)\left(\lambda^2-\frac{\alpha^2+2\beta}{\gamma^2}\right)=0, \quad (23)$$

$$\lambda^4-\left(\frac{m^2}{\gamma^2-c^2}\right)\lambda^2-\frac{2m^2(\alpha^2+2\beta)}{\gamma^2(\gamma^2-c^2)}=0. \quad (24)$$

We know from (23) that if the condition $\alpha^2+2\beta<0$, $\gamma^2-c^2<0$ holds, (23) has two purely imaginarypairs of eigenvalues: $\pm\frac{m}{\sqrt{c^2-\gamma^2}}\mathrm{i}$ and $\pm\frac{\sqrt{|\alpha^2+2\beta|}}{\gamma}\mathrm{i}$. Hence, we can investigate the problem of codimension two bifurcation for (5) at the origin O.

We see from (24) that

$$\lambda^2=\frac{1}{2}\left(\frac{m^2}{\gamma^2-c^2}\pm\sqrt{\left(\frac{m^2}{\gamma^2-c^2}\right)^2+\frac{4m^2(\alpha^2+2\beta)}{\gamma^2(\gamma^2-c^2)}}\right). \quad (25)$$

Thus, when $\alpha^2+2\beta>0$, $\gamma^2-c^2<0$ and $\Delta=\left(\frac{m^2}{\gamma^2-c^2}\right)^2+\frac{4m^2(\alpha^2+2\beta)}{\gamma^2(\gamma^2-c^2)}>0$, (24) has two purely imaginary pairs of eigenvalues. We can investigate the problem of codimension two bifurcation for (5) at the equilibriums $A_\pm$.

On the other hand, we suppose that $\gamma^2=\gamma^2-c^2$, i.e., $c=0$. Under this condition, the system (5) is a Hamiltonian system of two degree of freedom which can be written as

$$\frac{d}{d\xi}\begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}=J\nabla H=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} -\frac{\alpha^2+2\beta}{\gamma^2}q_1+\frac{2}{\gamma^2}q_1q_2 \\ -\frac{m^2}{\gamma^2}q_2+\frac{1}{\gamma^2}q_1^2 \\ p_1 \\ p_2 \end{pmatrix}$$

where

$$H(q_1, q_2, p_1, p_2)=-\frac{\alpha^2+2\beta}{2\gamma^2}q_1^2-\frac{m^2}{2\gamma^2}q_2^2+\frac{1}{\gamma^2}q_1^2q_2+\frac{1}{2}p_1^2+\frac{1}{2}p_2^2. \quad (26)$$

The dynamical behaviour of the phase orbits defined by (26) will give the properties of travelling wave solutions of (1). We will consider these problems in a new paper.

References:

[1] FUKUDA I, TSUTSUMI M. On coupled Klein-Gordon-Schrödinger equations(II)[J]. J Math Anal Appl, 1978, 66: 358—378.

[2] OHYA M. Stability of stationary states for coupled Klein-Gordon-Schrödinger equations [J]. Nonlinear Analysis, 1996, 27(4): 455—461.

[3] BAILLON J B, CHADAM J M. The cauchy problem for the coupled Klein-Gordon-Schrödinger equations[A]. In Contemporary Development in Continuum Mechanics and Partial Differential Equations [C]. Amsterdam: North-Holland, 1978, 37—44.

[4] HAYASHI N, Von Wahl W. On the global strong solutions of coupled Klein-Gordon-Schrödinger equations [J]. J Math Soc Japan, 1987, 39: 489—497.

[5] XIA Jing-na, HAN Shu-xia, WANG Ming-liang. The exact solitary wave solutions for the coupled Klein-Gordon-Schrödinger equations[J]. Appl Math Mech. 2002, 23(1): 58—64.

[6] GUCKENHEIMER J, HOLMES P. Dynamical systems and bifurcations of vector fields [M]. New York: Springer-Verlay, 1983.

[7] LI Ji-bin, LIU Zheng-rong. Smooth and non-smooth travelling waves in a nonlinearly dispersive equation [J]. Appl Math Modelling, 2000, 25: 41—56.

[8] LI Ji-bin, LIU Zheng-rong. Travelling wave solutions for a class dispersive equations[J]. Chin Ann of Math, 2002, 23B(3): 397—418.

Klein- Gordon- Schrodinger 方程的孤立波和周期行波解*

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摘要: 用动力系统方法研究 Klein- Grodon- Schrodinger 方程的孤立波和周期行波解. 给出了解存在的明显参数条件和孤立波与周期行波解的表达式, 并进一步考虑了行波方程可能的分支问题和 Hamilton 情况.

关键词: 孤立波; 周期行波; Klein- Gordon- Schrodinger 方程

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本文献给母校云南大学 80 周年校庆.