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Chromatic Number, Spectral Radius and Spanning Bipartite Subgraph*

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Abstract: Let G be a simple undirected graph with order n　1. Denote by　(G) and　(G) the spectral radius of the adjacency matrix and the Laplacian matrix of G, respectively. In this paper, by the way of spanning bipartite subgraphs, it is showed that: let G be a simple graph with chromatic number k　1. If k is even,　(G)　$\frac{k-1}{k}$　(G); if k is odd,　(G)　$\frac{k}{k+1}$　(G).

Key words: spectral radius; Laplacian spectral radius; chromatic number; spanning bipartite subgraph
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1　Introduction

Let G be a simple graph with vertex set $\{v_1, v_2,\quad, v_n\}$. The spectral radius of G,　(G), is the largest eigenvalue of its adjacency matrix $A(G)$, and $A(G)= (a_{ij})$ is defined to be the n　n matrix where $a_{ij} = 1$ if v_j is adjacent to v_j, and $a_{ij} = 0$ otherwise.

When G is connected, $A(G)$ is irreducible and by the Perron-Frobenius Theorem, the spectral radius　(G) is simple and there is a unique positive unit eigenvector. We shall refer to such an eigenvector as the Perron vector of G.

The number of edges incident with a vertex v_i in a graph G is called the degree of the vertex v_i. The degree of v_i is denoted by $d_i(G)$. The degree matrix D of a graph G is a diagonal matrix with the degree $d_i(G)$ vertex v_i in the position (i, i).

The Laplacian matrix is the matrix $L(G)= D(G)- A(G)$. It is well known that $L(G)$ is positive semidefinite symmetric. We denote the eigenvalues of $L(G)$ in decreasing order by

$$(G) = \quad_1(G) \qquad _2(G) \qquad\qquad _n(G) = 0,$$

and call　(G), the Laplacian spectral radius of G.

It is easy to show the following results:

Theorem 1　Let G be a simple graph with n vertices and Laplacian spectral radius　(G). Then　(G)　n. Equality holds if and only if the complement of G is disconnected.

For a graph G, its chromatic number k is the minimum number of colors needed to color the vertices of G in such a way that no two adjacent vertices are assigned the same color.

Theorem 2[1]　Let G be a simple graph with n vertices, spectral radius　(G) and Chromatics number k. Then　(G)　$\frac{k-1}{k}n$.

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Naturally, we can ask the following question: are the inequality $(G)\ \frac{k-1}{k}\ (G)$ right?

The answer is given partly in this paper. By the way of spanning bipartite subgraphs, we have showed that: let G be a simple graph with chromatic number $k\quad 1$. If k is even, then $(G)\quad \frac{k-1}{k}\ (G)$; if k is odd, then $(G)\quad \frac{k}{k+1}\ (G)$.

The terminology not defined here can be found in ref. [2].

2 Spanning Bipartite Subgraph

We study the problem of bounding the maximum number of spectral radius in a spanning bipartite subgraph of a graph G.

Let G be a simple connected graph with n vertices and chromatic number k. We may suppose that the vertices of G are labelled in such a way that its adjacency matrix can be written in the form $\boldsymbol{A} = \begin{pmatrix} 0 & \boldsymbol{A}_{12} & & \boldsymbol{A}_{1,k-1} & \boldsymbol{A}_{1k} \\ \boldsymbol{A}_{21} & 0 & & \boldsymbol{A}_{2,k-1} & \boldsymbol{A}_{2k} \\ & & & & \\ \boldsymbol{A}_{k-1,1} & \boldsymbol{A}_{k-1,2} & & 0 & \boldsymbol{A}_{k-1,k} \\ \boldsymbol{A}_{k1} & \boldsymbol{A}_{k2} & & \boldsymbol{A}_{k,k-1} & 0 \end{pmatrix}$, where $\boldsymbol{A}_{ij}$ is an $n_i\quad n_j$ matrix and of course, $\boldsymbol{A}_{ij} = \boldsymbol{A}_{ji}^{\mathrm{T}}$ and $\boldsymbol{A}_{ij}\quad 0, \boldsymbol{A}_{ij}\quad 0 (i\quad j, 1\quad i\quad k, 1\quad j\quad k)$; here n_i is the number of vertices with color i.

Let $\boldsymbol{x}$ be a Perron vector belonging to (G). Let us break $\boldsymbol{x}$ into $\boldsymbol{x}_1, \boldsymbol{x}_2, \quad, \boldsymbol{x}_k$ of lengths $n_1, n_2, \quad, n_k$, respectively. Set $b_{ij} = \boldsymbol{x}_i^{\mathrm{T}} \boldsymbol{A}_{ij} \boldsymbol{x}_j$. Then $b_{ij} = b_{ji}$. Further, it is easy to show that $b_{ij} > 0 (i\quad j, 1\quad i\quad k, 1\quad j\quad k)$ and $(G) = \sum_{\substack{1\ i\ k \\ 1\ j\ k}} b_{ij}$.

Set $\boldsymbol{B} = \begin{pmatrix} 0 & b_{12} & & b_{1,k-1} & b_{1k} \\ b_{21} & 0 & & b_{2,k-1} & b_{2k} \\ & & & & \\ b_{k-1,1} & b_{k-1,2} & & 0 & b_{k-1,k} \\ b_{k1} & b_{k2} & & b_{k,k-1} & 0 \end{pmatrix}$. The matrix $\boldsymbol{B}$ can be regarded as a adjacent matrix of a weighted complete graph K_k. Let be a set by taking any s rows and s columns correspondingly in $\boldsymbol{B}$. Let $\boldsymbol{B}$ be a matrix obtained from $\boldsymbol{B}$ by replacing the elements by 0, and the elements are covered twice or not covered by . Then the graph, whose adjacent matrix is $\boldsymbol{B}$, is a weighted bipartite subgraph of K_k. And denote the graph with the largest sum of all elements by F. Now we consider two cases.

Case 1 If k is even, then let $k = 2t$. Taking $s = t$ and the number of ways to choose s rows from $\boldsymbol{B}$ is $\binom{2t}{t} = \frac{(2t)!}{t!\ t!}$.

Since the element b_{ij} in $\boldsymbol{B}$ is covered twice if and only if the ith and jth rows are in , when b_{ij} is covered twice by , the number of ways is $\binom{2t-2}{t-2} = \frac{(2t-2)!}{t!\ (t-2)!}$. When b_{ij} is not covered by , the number of ways is $\binom{2t-2}{t} = \frac{(2t-2)!}{t!\ (t-2)!}$. Therefore, when b_{ij} is covered twice or not covered by , the average number of ways is $\frac{t-1}{2t-1}$. Hence the sum of all elements in F is at least $(\frac{t}{2t-1})\ (G)$, and $(F)\quad \frac{t\ (G)}{2t-1} = \frac{k\ (G)}{2(k-1)}$.

Case 2 If k is odd, then let $k = 2t+1$. Taking $s = t$ and the number of ways to choose s rows from $\boldsymbol{B}$ is $\binom{2t+1}{t} = \frac{(2t+1)!}{t!\ (t+1)!}$.

Since the element b_{ij} in $\boldsymbol{B}$ is covered twice if and only if the ith and jth rows are in , when b_{ij} is covered twice by , the number of ways is $\begin{pmatrix} 2t-1 \\ t-2 \end{pmatrix} = \frac{(2t-1)!}{t!\ (t-2)!}$. When b_{ij} is not covered by , the number of ways is $\begin{pmatrix} 2t-1 \\ t \end{pmatrix} = \frac{(2t-1)!}{t!\ (t-1)!}$. Therefore, when b_{ij} is covered twice or not covered by , the average number of ways is $\frac{t}{2t+1}$. Hence the sum of all elements in F is at least $(\frac{t+1}{2t+1})\ (G)$, and $(F)\quad \frac{(t+1)\ (G)}{2t+1} = \frac{(k+1)\ (G)}{2k}$.

It follows that:

Theorem 3 Let G be a simple connected graph with chromatic number k. Let H be a spanning bipartite subgraph of G and the spectral radius (H) of H is the largest in all bipartite subgraphs of G. If k is even then $(H)\quad \frac{k\ (G)}{2(k=1)}$; If k is odd then $(H)\quad \frac{(k+1)\ (G)}{2k}$.

3 Main Results

Lemma 1 Let G be a simple connected graph with order $n\quad 1$ and chromatic number k. Then for every bipartite subgraph H of G, $(G)\quad 2\ (H)$.

Proof Let H be a bipartite subgraph of G, then $V(H)=V_1\quad V_2, V_1\quad V_2=$. Every pair vertices in V_i are not adjacent in H. Without loss of generality, we may assume that $V_1=\{1,2,3,\quad ,t\}, V_2=\{t+1,t+2,t+3,\quad ,n\}$, $1\quad t\quad n$. If H is an empty graph (no edges), then $(H)=0$, which implies that $2\ (H)=0\quad (G)$. Hence we assume that H is not empty in the following statement. Let $\boldsymbol{x}$ be a unit column eigenvector of G corresponding to (G). Let $\boldsymbol{y}=(y_1,y_2,\quad ,y_n)^{\mathrm{T}}$ be a nonnegative unit eigenvector of H corresponding to (H). Set $\boldsymbol{z}=(y_1,y_2,\quad ,y_t,-y_{t+1},\quad ,-y_n)^{\mathrm{T}}$.

Then

$$(G)=\ \boldsymbol{x}^{\mathrm{T}}(D(G)-A(G))\boldsymbol{x}\quad \boldsymbol{z}^{\mathrm{T}}(D(G)-A(G))\boldsymbol{z}=\sum_{(i,j)\quad (E_1\quad E_2)}(y_i-y_j)^2+$$
$$\sum_{(i,j)\quad E(H)}(y_i+y_j)^2=\sum_{(i,j)\quad (E_1\quad E_2)}(y_i-y_j)^2+\sum_{(i,j)\quad E(H)}(y_i-y_j)^2+$$
$$4\sum_{(i,j)\quad E(H)}y_iy_j=\sum_{(i,j)\quad (E_1\quad E_2)}(y_i-y_j)^2+4\sum_{(i,j)\quad E(H)}y_iy_j$$
$$4\sum_{(i,j)\quad E(H)}y_iy_j=\ 2\ (H),$$

where $E_i=E(G[V_i])$ and $G[V_i]$ is an induced subgraph of G.

Theorem 4 Let G be a simple graph with chromatic number $k\quad 1$. ( ) If k is even, $(G)\quad \frac{k-1}{k}\ (G)$; ( ) If k is odd, $(G)\quad \frac{k}{k+1}\ (G)$.

Proof This follows from theorem 3 and lemma 1.

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带(G,) - 单调映象的广义多值变分包含的逼近解

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摘 要: 在 Hibert 空间中引入和研究了一类新的带(G,) - 单调映象的广义多值变分包含. 利用(G,) - 单调映象的预解算子技巧, 建立并讨论了这种变分包含解的存在性定理, 也提出了一个新的逼近算法, 证明了由此算法产生的迭代序列的收敛性. 文中给出的结果改进和推广了最近文献的一些相应结果.

关键词: 广义多值变分包含; (G,) - 单调映象; 预解算子技术; 逼近算法; 收敛性

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色数与谱半径和生成偶子图

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摘 要: 设 G 为 n 1 阶简单无向图, (G) 和 (G) 分别表示图 G 的邻接谱谱半径和 Laplacian 谱谱半径. 利用生成偶子图证明了: 当 k 为偶数时, (G) $\frac{k-1}{k}$ (G); 当 k 为奇数时, (G) $\frac{k}{k+1}$ (G). 其中 k(1) 为简单图 G 的色数.

关键词: 谱半径; Laplacian 谱半径; 色数; 生成偶子图

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