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31阶 Steiner三连系的构造与计数*

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摘 要: 阐明了 v 阶 Steiner三连系构造的基本思路, 给出了完全图 K_v 的边矩阵的定义, 提出了 $2t+1$阶 Steiner三连系构造的一种方法. 介绍了依据已存 15阶 Steiner三连系 $ST(15)$构造 31阶 Steiner三连系的全过程, 并讨论了 $2t+1$阶 Steiner三连系的计数问题.

关键词: Steiner三连系; 构造; 阶; 完全图; 边矩阵
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区组设计理论是组合数学的一个重要分支, 它在试验设计、竞赛安排及数字通讯等许多领域中均有重要作用. 早在 1850年, Kirkman提出了一个有趣的"15名女生"问题, 并于同年做出解答. 1971年, D. R. Ray-Chaudhuri与 R. M. Wilson共同发表论文"Kirkman女生问题的解"以阐明 $6n+3$阶 Kirkman三连系的构造. 百余年来, 就是否对每个 $n=0, 1, 2, 3, \cdots$ 总是存在 $6n+3$阶 Kirkman三连系, 一直是个难题[1-4]. 1971年, 中国数学家陆家羲提出了 BIBD设计可分解的充要条件[5].

1 基本思路

设 $G(V,E)$为一完全图 K_v, 若完全图 K_v 的阶数 $v=2t+1$, t为已存 Steiner三连系的阶数, 则 $2t+1$阶 Steiner三连系的构造等价于 $2t+1$阶完全图 K_v 的 $v(v-1)/6$个完全图 K_3 的分解; 然而, 当完全图 K_v 的阶数 v较高时, 将完全图 K_v 直接分解出 $v(v-1)/6$个完全图 K_3 是相当困难的, 倘若将完全图 K_v 先分解出 t个完全图 $K_3^{(i)}$ 和 $t(t-1)/6$个完全三分图 $\boldsymbol{K}_{2,2,2}^{(i\,j\,k)}$, 再让 $t(t-1)/6$个完全三分图 $\boldsymbol{K}_{2,2,2}^{(i\,j\,k)}$ 各分解出 2×2个完全图 K_3, 则 $t(t-1)/6$个完全三分图 $\boldsymbol{K}_{2,2,2}^{(i\,j\,k)}$ 中的 $t(t-1)/6$个完全三分图 K_3 和 t个完全图 $K_3^{(i)}$ 恰好构成 $v=2t+1$阶 Steiner三连系中的 $v(v-1)/6$个区组. 至于完全图 K_v 的 t个完全图 $K_3^{(i)}$ 及 $t(t-1)/6$个完全三分图 $\boldsymbol{K}_{2,2,2}^{(i\,j\,k)}$ 的分解, 以及将 $t(t-1)/6$个完全三分图 $\boldsymbol{K}_{2,2,2}^{(i\,j\,k)}$ 各分解出 2×2个完全图 K_3, 则需要借助于一个有效工具 —— 完全图 K_v 的边矩阵来实现[6-9].

定义 1 设 $G(V,E)$为一完全图 K_v, 若将完全图 K_v 中的 $v(v-1)/2$个边按自然顺序排成上三角阵, 使得 K_v 的任意边 v_iv_j 分别与项 v_i 和 v_j 相关联, 则所得到的上三角阵就称为完全图 K_v 的边矩阵, 并记为 $\boldsymbol{K}_v'$,

$$\boldsymbol{K}_v' = \left(\begin{array}{cccccccc|c} v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & \cdots & v_n & \\ 1_2 & 1_3 & 1_4 & 1_5 & 1_6 & 1_7 & \cdots & 1_n & v_1 \\ & 2_3 & 2_4 & 2_5 & 2_6 & 2_7 & \cdots & 2_n & v_2 \\ & & 3_4 & 3_5 & 3_6 & 3_7 & \cdots & 3_n & v_3 \\ & & & 4_5 & 4_6 & 4_7 & \cdots & 4_n & v_4 \\ & & & & 5_6 & 5_7 & \cdots & 5_n & v_5 \\ & & & & & 6_7 & \cdots & 6_n & v_6 \\ & & & & & & \ddots & \vdots & \vdots \\ & & & & & & & (n-1)_n & v_{n-1} \end{array}\right).$$

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2 31阶 Steiner三连系的构造

31阶 Steiner三连系为 $v = 2t + 1$阶 Steiner三连系, 是以 $t = 15$为已存 Steiner三连系的阶而得到的 $v = 2t + 1$阶 Steiner三连系. 用于构造 31阶 Steiner三连系的 15阶 Steiner三连系 $ST(15)$ 如下:

$ST(15) = \{\lfloor 1,2,3\rceil, \lfloor 1,4,5\rceil, \lfloor 1,6,7\rceil, \lfloor 1,8,9\rceil, \lfloor 1,10,11\rceil, \lfloor 1,12,13\rceil, \lfloor 1,14,15\rceil, \lfloor 2,9,11\rceil, \lfloor 2,12,14\rceil, \lfloor 2,5,7\rceil, \lfloor 2,13,15\rceil, \lfloor 2,8,10\rceil, \lfloor 2,4,6\rceil, \lfloor 3,12,15\rceil, \lfloor 3,8,11\rceil, \lfloor 3,13,14\rceil, \lfloor 3,5,6\rceil, \lfloor 3,4,7\rceil, \lfloor 3,9,10\rceil, \lfloor 4,10,14\rceil, \lfloor 4,9,13\rceil, \lfloor 4,11,15\rceil, \lfloor 4,8,12\rceil, \lfloor 5,9,12\rceil, \lfloor 5,8,13\rceil, \lfloor 5,11,14\rceil, \lfloor 5,10,15\rceil, \lfloor 6,11,13\rceil, \lfloor 6,8,14\rceil, \lfloor 6,10,12\rceil, \lfloor 7,8,15\rceil, \lfloor 7,10,13\rceil, \lfloor 7,9,14\rceil, \lfloor 7,11,12\rceil\}$.

以 $ST(15)$ 作为已存 15阶 Steiner三连系, 对 $t = 15$的 31阶 Steiner三连系的构造可概括为下列步骤:

Step 1 将 31阶完全图 K_{31}的 $31(31-1)/2$个边排成上三角阵, 得完全图 K_{31} 的边矩阵 $\boldsymbol{K}_{31}'$.

Step 2 将完全图 K_{31} 的边矩阵 $\boldsymbol{K}_{31}'$ 划分为 $t = 15$个完全图 $K_3^{(1)}, K_3^{(2)}, K_3^{(3)}, \cdots, K_3^{(15)}$ 和 $t(t-1)/2 = 105$个完全二分图的边矩阵 $\boldsymbol{K}_{2,2}^{(i,j)}, \boldsymbol{K}_{2,2}^{(i,k)}, \boldsymbol{K}_{2,2}^{(j,k)}$:

$$\boldsymbol{K}_{31}' = \left(\begin{array}{ccccccccc|c} v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & \cdots & v_{31} & \\ 1_2 & 1_3 & 1_4 & 1_5 & 1_6 & 1_7 & 1_8 & \cdots & 1_{31} & v_1 \\ & 2_3 & 2_4 & 2_5 & 2_6 & 2_7 & 2_8 & \cdots & 2_{31} & v_2 \\ & & 3_4 & 3_5 & 3_6 & 3_7 & 3_8 & \cdots & 3_{31} & v_3 \\ & & & 4_5 & 4_6 & 4_7 & 4_8 & \cdots & 4_{31} & v_4 \\ & & & & 5_6 & 5_7 & 5_8 & \cdots & 5_{31} & v_5 \\ & & & & & 6_6 & 6_7 & \cdots & 6_{31} & v_6 \\ & & & & & & 7_8 & \cdots & 7_{31} & v_7 \\ & & & & & & & \ddots & \vdots & \vdots \\ & & & & & & & & 30_{31} & v_{31} \end{array}\right), \quad \boldsymbol{K}_{31}' = \left(\begin{array}{cccccc|c} & v_2 & v_3 & v_4 & \cdots & v_{15} & \\ K_3^{(1)} & \boldsymbol{K}_{2,2}^{(1,2)} & \boldsymbol{K}_{2,2}^{(1,3)} & \boldsymbol{K}_{2,2}^{(1,4)} & \cdots & \boldsymbol{K}_{2,2}^{(1,15)} & v_1 \\ & K_3^{(2)} & \boldsymbol{K}_{2,2}^{(2,3)} & \boldsymbol{K}_{2,2}^{(2,4)} & \cdots & \boldsymbol{K}_{2,2}^{(2,15)} & v_2 \\ & & K_3^{(3)} & \boldsymbol{K}_{2,2}^{(3,4)} & \cdots & \boldsymbol{K}_{2,2}^{(3,15)} & v_3 \\ & & & K_3^{(4)} & \cdots & \boldsymbol{K}_3^{(4,15)} & v_4 \\ & & & & \ddots & \vdots & \vdots \\ & & & & & K_3^{(15)} & v_{15} \end{array}\right);$$

$K_3^{(1)} = \lfloor 1,2,3\rceil, K_3^{(2)} = \lfloor 1,4,5\rceil, K_3^{(3)} = \lfloor 1,6,7\rceil, K_3^{(4)} = \lfloor 1,8,9\rceil, K_3^{(5)} = \lfloor 1,10,11\rceil, \cdots, K_3^{(15)} = \lfloor 1,14,15\rceil;$

$$\boldsymbol{K}_{2,2}^{(1,2)} = \begin{pmatrix} 2_4 & 2_5 \\ 3_4 & 3_5 \end{pmatrix}, \boldsymbol{K}_{2,2}^{(1,3)} = \begin{pmatrix} 2_6 & 2_7 \\ 3_6 & 3_7 \end{pmatrix}, \boldsymbol{K}_{2,2}^{(1,4)} = \begin{pmatrix} 2_8 & 2_9 \\ 3_8 & 3_9 \end{pmatrix}, \boldsymbol{K}_{2,2}^{(1,5)} = \begin{pmatrix} 2_{10} & 2_{11} \\ 3_{10} & 3_{11} \end{pmatrix},$$

$$\boldsymbol{K}_{2,2}^{(1,6)} = \begin{pmatrix} 2_{12} & 2_{13} \\ 3_{12} & 3_{13} \end{pmatrix}, \boldsymbol{K}_{2,2}^{(1,7)} = \begin{pmatrix} 2_{14} & 2_{15} \\ 3_{14} & 3_{15} \end{pmatrix}, \cdots, \boldsymbol{K}_{2,2}^{(1,31)} = \begin{pmatrix} 1_{30} & 1_{31} \\ 1_{30} & 1_{31} \end{pmatrix}, \boldsymbol{K}_{2,2}^{(2,3)} = \begin{pmatrix} 4_6 & 4_7 \\ 5_6 & 5_7 \end{pmatrix},$$

$$\boldsymbol{K}_{2,2}^{(2,4)} = \begin{pmatrix} 4_8 & 4_9 \\ 5_8 & 5_9 \end{pmatrix}, \boldsymbol{K}_{2,2}^{(2,5)} = \begin{pmatrix} 4_{10} & 4_{11} \\ 5_{10} & 5_{11} \end{pmatrix}, \cdots, \boldsymbol{K}_{2,2}^{(2,31)} = \begin{pmatrix} 4_{30} & 4_{31} \\ 5_{30} & 5_{31} \end{pmatrix}, \boldsymbol{K}_{2,2}^{(3,4)} = \begin{pmatrix} 6_8 & 6_9 \\ 7_8 & 7_9 \end{pmatrix},$$

$$\boldsymbol{K}_{2,2}^{(3,5)} = \begin{pmatrix} 6_{10} & 6_{11} \\ 7_{10} & 7_{11} \end{pmatrix}, \boldsymbol{K}_{2,2}^{(3,6)} = \begin{pmatrix} 6_{12} & 6_{13} \\ 7_{12} & 7_{13} \end{pmatrix}, \cdots, \boldsymbol{K}_{2,2}^{(3,31)} = \begin{pmatrix} 6_{30} & 6_{31} \\ 7_{30} & 7_{31} \end{pmatrix}, \boldsymbol{K}_{2,2}^{(4,5)} = \begin{pmatrix} 8_{10} & 8_{11} \\ 9_{10} & 9_{11} \end{pmatrix},$$

$$\boldsymbol{K}_{2,2}^{(4,6)} = \begin{pmatrix} 8_{12} & 8_{13} \\ 9_{12} & 9_{13} \end{pmatrix}, \boldsymbol{K}_{2,2}^{(4,7)} = \begin{pmatrix} 8_{14} & 8_{15} \\ 9_{14} & 9_{15} \end{pmatrix}, \cdots, \boldsymbol{K}_{2,2}^{(4,31)} = \begin{pmatrix} 8_{30} & 8_{31} \\ 9_{30} & 9_{31} \end{pmatrix}, \boldsymbol{K}_{2,2}^{(5,6)} = \begin{pmatrix} 10_{12} & 10_{13} \\ 11_{12} & 11_{13} \end{pmatrix},$$

$$\boldsymbol{K}_{2,2}^{(5,7)} = \begin{pmatrix} 10_{14} & 10_{15} \\ 11_{14} & 11_{15} \end{pmatrix}, \boldsymbol{K}_{2,2}^{(5,8)} = \begin{pmatrix} 10_{16} & 10_{17} \\ 11_{16} & 11_{17} \end{pmatrix}, \cdots, \boldsymbol{K}_{2,2}^{(5,31)} = \begin{pmatrix} 10_{30} & 10_{31} \\ 11_{30} & 11_{31} \end{pmatrix}, \boldsymbol{K}_{2,2}^{(6,7)} = \begin{pmatrix} 12_{14} & 12_{15} \\ 13_{14} & 13_{15} \end{pmatrix},$$

$$\boldsymbol{K}_{2,2}^{(6,8)} = \begin{pmatrix} 12_{16} & 12_{17} \\ 13_{16} & 13_{17} \end{pmatrix}, \boldsymbol{K}_{2,2}^{(6,9)} = \begin{pmatrix} 12_{18} & 12_{19} \\ 13_{18} & 13_{19} \end{pmatrix}, \cdots, \boldsymbol{K}_{2,2}^{(6,31)} = \begin{pmatrix} 12_{30} & 12_{31} \\ 13_{30} & 13_{31} \end{pmatrix}, \boldsymbol{K}_{2,2}^{(7,8)} = \begin{pmatrix} 14_{16} & 14_{17} \\ 15_{16} & 15_{17} \end{pmatrix},$$

$$\boldsymbol{K}_{2,2}^{(7,9)} = \begin{pmatrix} 14_{18} & 14_{19} \\ 15_{18} & 15_{19} \end{pmatrix}, \cdots, \boldsymbol{K}_{2,2}^{(7,10)} = \begin{pmatrix} 14_{20} & 14_{21} \\ 15_{20} & 15_{21} \end{pmatrix}, \cdots, \boldsymbol{K}_{2,2}^{(7,31)} = \begin{pmatrix} 14_{30} & 14_{31} \\ 15_{30} & 15_{31} \end{pmatrix}, \boldsymbol{K}_{2,2}^{(8,9)} = \begin{pmatrix} 16_{18} & 16_{19} \\ 17_{18} & 17_{19} \end{pmatrix},$$

$$\boldsymbol{K}_{2,2}^{(8,10)} = \begin{pmatrix} 16_{20} & 16_{21} \\ 17_{20} & 17_{21} \end{pmatrix}, \boldsymbol{K}_{2,2}^{(8,11)} = \begin{pmatrix} 16_{22} & 16_{23} \\ 17_{22} & 17_{23} \end{pmatrix}, \cdots, \boldsymbol{K}_{2,2}^{(8,31)} = \begin{pmatrix} 16_{30} & 16_{31} \\ 17_{30} & 17_{31} \end{pmatrix}, \boldsymbol{K}_{2,2}^{(9,10)} = \begin{pmatrix} 18_{20} & 18_{21} \\ 19_{20} & 19_{21} \end{pmatrix},$$

$$\boldsymbol{K}_{2,2}^{(9,11)} = \begin{pmatrix} 18_{22} & 18_{23} \\ 19_{22} & 19_{23} \end{pmatrix}, \boldsymbol{K}_{2,2}^{(9,12)} = \begin{pmatrix} 18_{24} & 18_{25} \\ 19_{24} & 19_{25} \end{pmatrix}, \cdots, \boldsymbol{K}_{2,2}^{(9,31)} = \begin{pmatrix} 18_{30} & 18_{31} \\ 19_{30} & 19_{31} \end{pmatrix}, \boldsymbol{K}_{2,2}^{(10,11)} = \begin{pmatrix} 20_{22} & 20_{23} \\ 21_{22} & 21_{23} \end{pmatrix},$$

$$\boldsymbol{K}_{2,2}^{(10,12)} = \begin{pmatrix} 20_{24} & 20_{25} \\ 21_{24} & 21_{25} \end{pmatrix}, \boldsymbol{K}_{2,2}^{(10,13)} = \begin{pmatrix} 20_{26} & 20_{27} \\ 21_{26} & 21_{27} \end{pmatrix}, \cdots, \boldsymbol{K}_{2,2}^{(10,31)} = \begin{pmatrix} 20_{30} & 20_{31} \\ 21_{30} & 21_{31} \end{pmatrix}, \boldsymbol{K}_{2,2}^{(11,12)} = \begin{pmatrix} 22_{24} & 22_{25} \\ 23_{24} & 23_{25} \end{pmatrix},$$

$$\boldsymbol{K}_{2,2}^{(11,13)} = \begin{pmatrix} 22_{26} & 22_{27} \\ 23_{26} & 23_{27} \end{pmatrix}, \boldsymbol{K}_{2,2}^{(11,14)} = \begin{pmatrix} 22_{28} & 22_{29} \\ 23_{28} & 23_{29} \end{pmatrix}, \cdots, \boldsymbol{K}_{2,2}^{(11,31)} = \begin{pmatrix} 22_{30} & 22_{31} \\ 23_{30} & 23_{31} \end{pmatrix}, \boldsymbol{K}_{2,2}^{(12,13)} = \begin{pmatrix} 24_{26} & 24_{27} \\ 25_{26} & 25_{27} \end{pmatrix},$$

$$K_{2,2}^{(12,14)}=\begin{pmatrix}24_{28} & 24_{29}\\ 25_{28} & 25_{29}\end{pmatrix},\ K_{2,2}^{(12,15)}=\begin{pmatrix}24_{30} & 24_{31}\\ 25_{30} & 25_{31}\end{pmatrix},\ K_{2,2}^{(13,14)}=\begin{pmatrix}26_{28} & 26_{29}\\ 27_{28} & 27_{29}\end{pmatrix},\ K_{2,2}^{(14,15)}=\begin{pmatrix}28_{30} & 28_{31}\\ 29_{30} & 29_{31}\end{pmatrix}.$$

Step 3　据 15阶 Steiner三连系 $ST(15)$ 中的 $v(v-1)/6=35$ 个区组将 $t(t-1)/2=105$ 个完全二分图的边矩阵 $K_{2,2}^{(ij)}$ 中的边 v_iv_j 与 $K_{2,2}^{(i,k)}, K_{2,2}^{(j,k)}$ 中的边 v_iv_k 和边 v_jv_k 构成完全图 $K_3^{(i)}=\{i,j,k\}$, 即让边矩阵 $K_{2,2}^{(ij)}, K_{2,2}^{(ik)}, K_{2,2}^{(jk)}$ 重叠成 $v(v-1)/6=35$ 个各由 $2\times2=4$ 个完全 K_3 构成的完全三分图立方边矩阵 $K_{2,2,2}^{(ij,k)}=K^{(ij)}\cup K_{2,2}^{'(i,k)}\cup K_{2,2}^{'(jk)}$. $K_{2,2}^{'(ik)}, K_{2,2}^{'(j,k)}$ 表示边矩阵 $K_{2,2}^{(ik)}, K_{2,2}^{(jk)}$ 各自边的位置是经过调整的. 按照 $K_{2,2,2}^{(i,j,k)}=K^{(i,j)}\cup K_{2,2}^{'(ik)}\cup K_{2,2}^{'(jk)}$ 得到的 35个完全三分图立方边矩阵 $K_{2,2,2}^{(1,2,3)}$, $K_{2,2,2}^{(1,4,5)}, K_{2,2,2}^{(1,6,7)}, K_{2,2,2}^{(1,8,9)}, K_{2,2,2}^{(1,10,11)}, K_{2,2,2}^{(1,12,13)}, K_{2,2,2}^{(1,14,15)}, K_{2,2,2}^{(2,4,6)}, \cdots, K_{2,2,2}^{(7,11,12)}$ 表述为:

$$K_{2,2,2}^{(1,2,3)}=\begin{pmatrix}\{2,4,6\} & \{2,5,7\}\\ \{3,4,7\} & \{3,5,6\}\end{pmatrix},\ K_{2,2,2}^{(1,4,5)}=\begin{pmatrix}\{2,8,10\} & \{2,9,11\}\\ \{3,8,11\} & \{3,9,10\}\end{pmatrix},\ K_{2,2,2}^{(1,6,7)}=\begin{pmatrix}\{2,12,14\} & \{2,13,15\}\\ \{3,12,15\} & \{3,13,14\}\end{pmatrix},$$

$$K_{2,2,2}^{(1,8,9)}=\begin{pmatrix}\{2,16,18\} & \{2,17,19\}\\ \{3,16,19\} & \{3,17,18\}\end{pmatrix},\ K_{2,2,2}^{(1,10,11)}=\begin{pmatrix}\{2,20,22\} & \{2,21,23\}\\ \{3,20,23\} & \{3,21,22\}\end{pmatrix},\ K_{2,2,2}^{(1,12,13)}=\begin{pmatrix}\{2,24,26\} & \{2,25,27\}\\ \{3,24,27\} & \{3,25,26\}\end{pmatrix},$$

$$K_{2,2,2}^{(1,14,15)}=\begin{pmatrix}\{2,28,30\} & \{2,29,31\}\\ \{3,28,31\} & \{3,29,30\}\end{pmatrix},\ K_{2,2,2}^{(2,4,6)}=\begin{pmatrix}\{4,8,12\} & \{4,9,13\}\\ \{5,8,13\} & \{5,9,12\}\end{pmatrix},\ K_{2,2,2}^{(2,5,7)}=\begin{pmatrix}\{4,10,14\} & \{4,11,15\}\\ \{5,10,15\} & \{5,11,14\}\end{pmatrix},$$

$$K_{2,2,2}^{(2,8,10)}=\begin{pmatrix}\{4,16,20\} & \{4,17,21\}\\ \{5,16,21\} & \{5,17,20\}\end{pmatrix},\ K_{2,2,2}^{(2,9,11)}=\begin{pmatrix}\{4,18,22\} & \{4,19,23\}\\ \{5,18,23\} & \{5,19,22\}\end{pmatrix},\ K_{2,2,2}^{(2,12,14)}=\begin{pmatrix}\{4,24,28\} & \{4,25,29\}\\ \{5,24,29\} & \{5,25,28\}\end{pmatrix},$$

$$K_{2,2,2}^{(2,13,15)}=\begin{pmatrix}\{4,26,30\} & \{4,27,31\}\\ \{5,26,31\} & \{5,27,30\}\end{pmatrix},\ K_{2,2,2}^{(3,4,7)}=\begin{pmatrix}\{6,8,14\} & \{6,9,15\}\\ \{7,8,15\} & \{7,9,14\}\end{pmatrix},\ K_{2,2,2}^{(3,5,6)}=\begin{pmatrix}\{6,10,12\} & \{6,11,13\}\\ \{7,10,13\} & \{7,11,12\}\end{pmatrix},$$

$$K_{2,2,2}^{(3,8,11)}=\begin{pmatrix}\{6,8,22\} & \{6,7,23\}\\ \{7,16,23\} & \{7,17,22\}\end{pmatrix},\ K_{2,2,2}^{(3,9,10)}=\begin{pmatrix}\{6,8,20\} & \{6,19,21\}\\ \{7,18,21\} & \{7,19,20\}\end{pmatrix},\ K_{2,2,2}^{(3,12,15)}=\begin{pmatrix}\{6,24,30\} & \{6,25,31\}\\ \{7,24,31\} & \{7,25,30\}\end{pmatrix},$$

$$K_{2,2,2}^{(3,13,14)}=\begin{pmatrix}\{6,26,28\} & \{6,27,29\}\\ \{7,26,29\} & \{7,27,28\}\end{pmatrix},\ K_{2,2,2}^{(4,8,12)}=\begin{pmatrix}\{8,16,24\} & \{8,17,25\}\\ \{9,16,25\} & \{9,17,24\}\end{pmatrix},\ K_{2,2,2}^{(4,9,13)}=\begin{pmatrix}\{8,18,26\} & \{8,19,27\}\\ \{9,18,27\} & \{9,19,26\}\end{pmatrix},$$

$$K_{2,2,2}^{(4,10,14)}=\begin{pmatrix}\{8,20,28\} & \{8,21,29\}\\ \{9,20,29\} & \{9,21,28\}\end{pmatrix},\ K_{2,2,2}^{(4,11,15)}=\begin{pmatrix}\{8,22,30\} & \{8,23,31\}\\ \{9,22,31\} & \{9,23,30\}\end{pmatrix},\ K_{2,2,2}^{(5,8,13)}=\begin{pmatrix}\{10,16,26\} & \{10,17,27\}\\ \{11,16,27\} & \{11,17,26\}\end{pmatrix},$$

$$K_{2,2,2}^{(5,9,12)}=\begin{pmatrix}\{10,18,24\} & \{10,19,25\}\\ \{11,18,25\} & \{11,19,24\}\end{pmatrix},\ K_{2,2,2}^{(5,11,14)}=\begin{pmatrix}\{10,22,28\} & \{10,23,29\}\\ \{11,22,29\} & \{11,23,28\}\end{pmatrix},\ K_{2,2,2}^{(5,10,15)}=\begin{pmatrix}\{10,20,30\} & \{10,21,31\}\\ \{11,20,31\} & \{11,21,30\}\end{pmatrix},$$

$$K_{2,2,2}^{(6,8,14)}=\begin{pmatrix}\{12,16,28\} & \{12,17,29\}\\ \{13,16,29\} & \{13,17,28\}\end{pmatrix},\ K_{2,2,2}^{(6,9,15)}=\begin{pmatrix}\{12,18,30\} & \{12,19,31\}\\ \{3,18,31\} & \{13,19,30\}\end{pmatrix},\ K_{2,2,2}^{(6,10,12)}=\begin{pmatrix}\{12,20,24\} & \{12,21,25\}\\ \{13,20,25\} & \{13,21,24\}\end{pmatrix},$$

$$K_{2,2,2}^{(6,11,13)}=\begin{pmatrix}\{12,22,26\} & \{12,23,27\}\\ \{13,22,27\} & \{13,23,26\}\end{pmatrix},\ K_{2,2,2}^{(7,8,15)}=\begin{pmatrix}\{14,16,30\} & \{14,17,31\}\\ \{15,16,31\} & \{15,17,30\}\end{pmatrix},\ K_{2,2,2}^{(7,9,14)}=\begin{pmatrix}\{14,18,28\} & \{14,19,29\}\\ \{15,18,29\} & \{15,19,28\}\end{pmatrix},$$

$$K_{2,2,2}^{(7,10,13)}=\begin{pmatrix}\{14,20,26\} & \{14,21,27\}\\ \{15,20,27\} & \{15,21,26\}\end{pmatrix},\ K_{2,2,2}^{(7,11,12)}=\begin{pmatrix}\{14,22,24\} & \{14,23,25\}\\ \{15,22,25\} & \{15,23,24\}\end{pmatrix}.$$

Step 4　让 15个完全图 $K_3^{(1)}, K_3^{(2)}, K_3^{(3)}, \cdots, K_3^{(15)}$ 和 $t(t-1)/6=35$ 个完全三分图立方边矩阵 $K_{2,2,2}^{(i,j,k)}$ 中的 $4t(t-1)/6=140$ 个完全图 K_3 恰好构成 31阶 Steiner三连系 $ST^{(1)}(31)$ 中的 $v(v-1)/6=155$ 个区组:

$ST^{(1)}(31)=\{\{1,2,3\}, \{1,4,5\}, \{1,6,7\}, \{1,8,9\}, \{1,10,11\}, \{1,12,13\}, \{1,14,15\}, \{1,16,17\}, \{1,18,19\}, \{1,20,21\}, \{1,22,23\}, \{1,24,25\}, \{1,26,27\}, \{1,28,29\}, \{1,30,31\}, \{2,4,6\}, \{2,5,7\}, \{2,8,10\}, \{2,9,11\}, \{2,12,14\}, \{2,13,15\}, \{2,16,18\}, \{2,17,19\}, \{2,20,22\}, \{2,21,23\}, \{2,24,26\}, \{2,25,27\}, \{2,26,28\}, \{2,27,29\}, \{2,28,30\}, \{2,29,31\}, \{3,4,7\}, \{3,5,6\}, \{3,8,11\}, \{3,9,10\}, \{3,12,15\}, \{3,13,14\}, \{3,16,19\}, \{3,17,18\}, \{3,20,23\}, \{3,21,22\}, \{3,24,27\}, \{3,25,26\}, \{3,28,31\}, \{3,29,30\}, \{4,8,12\}, \{4,9,13\}, \{4,10,14\}, \{4,11,15\}, \{4,16,20\}, \{4,17,21\}, \{4,18,22\}, \{4,19,23\}, \{4,24,28\}, \{4,25,29\}, \{4,26,30\}, \{4,27,31\}, \{5,8,13\}, \{5,19,12\}, \{5,10,15\}, \{5,11,14\}, \{5,16,21\}, \{5,17,20\}, \{5,18,23\}, \{5,19,22\}, \{5,24,29\}, \{5,25,28\}, \{5,26,31\}, \{5,27,30\}, \{6,8,14\}, \{6,9,15\}, \{6,10,12\}, \{6,11,13\}, \{6,16,22\}, \{6,17,23\}, \{6,17,23\}, \{6,18,20\}, \{6,19,21\}, \{6,24,30\}, \{6,25,31\}, \{6,26,28\}, \{6,27,29\}, \{7,8,15\}, \{7,9,14\}, \{7,10,13\}, \{7,11,12\}, \{7,16,23\}, \{7,16,23\}, \{7,17,22\}, \{7,18,21\}, \{7,19,20\}, \{7,24,31\}, \{7,25,30\}, \{7,26,29\}, \{7,27,28\}, \{8,16,24\}, \{8,17,25\}, \{8,18,26\}, \{8,19,27\}, \{8,20,28\}, \{8,21,29\}, \{8,22,30\}, \{8,23,31\}, \{9,16,25\}, \{9,17,24\}, \{9,18,27\}, \{9,19,26\}, \{9,20,29\}, \{9,21,28\}, \{9,22,31\}, \{9,23,30\}, \{10,16,26\}, \{10,17,27\}, \{10,18,24\}, \{10,19,25\}, \{10,22,28\}, \{10,23,29\}, \{10,20,30\}, \{10,21,31\}, \{11,16,27\}, \{11,17,26\}, \{11,18,25\}, \{11,19,24\}, \{11,22,29\}, \{11,23,28\}, \{11,20,31\}, \{11,21,30\}, \{12,16,28\}, \{12,17,29\}, \{12,18,30\}, \{12,19,31\}, \{12,20,24\}, \{12,21,25\}, \{12,22,26\}, \{12,23,27\}, \{13,16,29\}, \{13,17,28\}, \{13,18,31\}, \{13,19,30\}, \{13,20,25\}, \{13,21,24\}, \{13,22,27\}, \{13,23,26\}, \{14,16,30\}, \{14,17,31\}, \{14,18,28\}, \{14,19,29\}, \{14,20,26\}, \{14,21,27\}, \{14,22,24\},$

{14, 23, 25}, {15, 16, 31}, {15, 17, 30}, {15, 18, 29}, {15, 19, 28}, {15, 20, 27}, {15, 21, 26}, {15, 22, 25}, {15, 23, 24}}.

3 31阶 Steiner三连系的计数

31阶 Steiner三连系 $ST^{(1)}(31)$ 是依据边矩阵 $\boldsymbol{K}_{31}'$ 的完全三分图的边矩阵 $\boldsymbol{K}_{2,2,2}^{(i\,j\,k)}$ 的划分方案得出的, 边矩阵 $\boldsymbol{K}_{2,2,2}^{(i\,j\,k)}$ 的划分方案共有 $N^{(1)} = 31$ 个, 而将 $t(t-1)/2 = 105$ 个完全二分图的边矩阵 $\boldsymbol{K}_{2,2}^{(ij)}, \boldsymbol{K}_{2,2}^{(ik)}, \boldsymbol{K}_{2,2}^{(jk)}$ 并成 $t(t-1)/6 = 35$ 个完全二分图的立方边矩阵 $\boldsymbol{K}_{2,2,2}^{(i\,j\,k)}$ 的方案数 $N^{(2)} = 2$ 据乘法法则, 31阶 Steiner三连系的个数 $N = N^{(1)} \times N^{(2)} = 31 \times 2 = 62$

4 结语

该研究为图论研究提供了一个工具, 即借助完全图 K_v 的边矩阵 $\boldsymbol{K}_v'$, 利用边矩阵 $\boldsymbol{K}_v'$ 可构造任意 $2t+1$ 阶 Steiner三连系, 并提出一种 $2t+1$ 阶 Steiner三连系构造的方法, 解决了 $2t+1$ 阶 Steiner三连系的计数问题.

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Construction and Enumeration of Steiner Triple System of Order 31

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Abstract: The basic concept of constructing Steiner triple system of order v is described. The definition of edge matrix of a complete graph K_v is given. A method of constructing Steiner triple system of order $2t+1$ is proposed. The entire procedures of constructing Steiner triple system of order 31 according to the steiner triple system of order 15 is explicated. The enumeration problem of Steiner triple system is discussed.

Key words: Steiner triple system; construction; order; complete graph; edge matrix

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