Article ID: 1007-2985(2011)01-0001-03

Various General Total Colorings of Graphs*

CHEN Xiang-en[1], GAO Yu-ping[1], YANG Sui-yi[2]
(1. College of Mathematics and Information Science, Northwest Normal University, Lanzhou 730070, China;
2. School of Mathematics and Statistics, Tianshui Normal University, Tianshui 741001, Gansu China)

Abstract: A proper total coloring of a graph G is an assignment of colors to the edges and vertices of G in such a way that no two adjacent vertices, no two adjacent edges, and no incident vertex and edge receive the same color. In this paper, the authors weaken the constraints of the proper total coloring to obtain several general total colorings and discuss their chromatic numbers.

Key words: total coloring; chromatic number; edge chromatic index
CLC number: O157.5 **Document code:** A

1 Introduction and Definitions

All graphs mentioned in this article are simple, undirected and finite. We denote the vertex set, edge set, maximum degree, minimum degree, chromatic number, and edge chromatic index of a graph G by $V(G)$, $E(G)$, $\Delta(G)$, $\delta(G)$, $\chi(G)$, and $\chi'(G)$, respectively. A famous result of Vizing's says that for any graph G, $\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G)+1$.

A proper total coloring of a graph G is an assignment of colors to the edges and vertices of G in such a way that no two adjacent vertices, no two adjacent edges, and no incident vertex and edge receive the same color. The minimum number of colors required for a proper total coloring of G is called the total chromatic number of G, and is denoted by $\chi_t(G)$. For any graph G, it's clear that $\chi_t(G) \geqslant \Delta(G)+1$. Behzad M[1] and Vizing V G[2] independently made the conjecture that for any graph G, $\chi_t(G) \leqslant \Delta(G)+2$. This is known as the total coloring conjecture (TCC) and is still unproven. See ref. [3-8] for surveys on proper total colorings.

In this paper, we will weaken the conditions for the proper total coloring and give various general total colorings of a graph. And we will obtain their chromatic numbers.

Definition 1 Let f be an assignment of colors to the edges and vertices of a graph G. There are three possible constraints on f in the following: (i) no two adjacent vertices receive the same color; (ii) no two adjacent edges receive the same color; (iii) no vertex receive the same color as any one of its incident edges does.

(a) f is called a VIE-total coloring if f satisfies none of the above three conditions. A VIE-total coloring of G using k colors is called a k-VIE-total coloring of G. The minimum number of colors required for a VIE-total coloring of G is called VIE-total chromatic number of G, and is denoted by $\chi_t^0(G)$.

* **Received date:** 2010-11-03
Foundation item: National Natural Science Foundation of China (10771091)
Biography: CHEN Xiang-en (1965-), male, was born in Tianshui City, Gansu Province, professor, master, research area are coloring theory of graphs and algebra theory of graphs.

(b) f is called an IE-total coloring if f satisfies only (i). An IE-total coloring of G using k colors is called a k-IE-total coloring of G. The minimum number of colors required for an IE-total coloring of G is called IE-total chromatic number of G, and is denoted by $\chi_t^{ie}(G)$.

(c) f is called a VI-total coloring if f satisfies only (ii). A VI-total coloring of G using k colors is called a k-VI-total coloring of G. The minimum number of colors required for a VI-total coloring of G is called VI-total chromatic number of G, and is denoted by $\chi_t^{vi}(G)$.

(d) f is called a VE-total coloring if f satisfies only (iii). A VE-total coloring of G using k colors is called a k-VE-total coloring of G. The minimum number of colors required for a VE-total coloring of G is called VE-total chromatic number of G, and is denoted by $\chi_t^{ve}(G)$.

(e) f is called a V-total coloring if f satisfies both (ii) and (iii). A V-total coloring of G using k colors is called a k-V-total coloring of G. The minimum number of colors required for a V-total coloring of G is called V-total chromatic number of G, and is denoted by $\chi_t^{v}(G)$.

(f) f is called an E-total coloring if f satisfies both (i) and (iii). An E-total coloring of G using k colors is called a k-E-total coloring of G. The minimum number of colors required for an E-total coloring of G is called E-total chromatic number of G, and is denoted by $\chi_t^{e}(G)$.

(g) f is called an I-total coloring if f satisfies both (i) and (ii). An I-total coloring of G using k colors is called a k-I-total coloring of G. The minimum number of colors required for an I-total coloring of G is called I-total chromatic number of G, and is denoted by $\chi_t^{i}(G)$.

We can see that f is a proper total coloring if f satisfies all the above three conditions.

Lemma 1 Let G be a simple graph. If $S_1 \subseteq S_2 \subseteq \{V, I, E\}$, then S_1-total coloring of G must be S_2-total coloring of G (proper total coloring is referred to as $\emptyset$-total coloring, IE-total coloring is referred to as {I, E}-total coloring, and so on). Thus we have: (i) $\chi_t^{vie}(G) \leqslant \min\{\chi_t^{ie}(G), \chi_t^{vi}(G), \chi_t^{ve}(G)\}$; (ii) $\chi_t^{ie}(G) \leqslant \min\{\chi_t^{i}(G), \chi_t^{e}(G)\}$; (iii) $\chi_t^{vi}(G) \leqslant \min\{\chi_t^{v}(G), \chi_t^{i}(G)\}$; (iv) $\chi_t^{ve}(G) \leqslant \min\{\chi_t^{v}(G), \chi_t^{e}(G)\}$; (v) $\chi_t(G) \geqslant \max\{\chi_t^{v}(G), \chi_t^{e}(G), \chi_t^{i}(G)\}$.

2 Exact Results

For empty graph with order $n \geqslant 1$, we can obtain the various general total chromatic numbers easily which are all equal to 1. Thus in this section we consider the graph which has at least one edge.

Theorem 1 $\chi_t^0(G) = 1$.

Proof It's clear that $\chi_t^0(G) \geqslant 1$. Color all the edges and vertices of G with only one color, we have a 1-VIE-total coloring of G. Thus, $\chi_t^0(G) = 1$.

Theorem 2 $\chi_t^{ie}(G) = \chi(G)$.

Proof It's clear that $\chi_t^{ie}(G) \geqslant \chi(G)$. We first color the vertices of G with $\chi(G)$ colors $\{1, 2, \cdots, \chi(G)\}$ such that no two adjacent vertices receive the same color. Then color all the edges with a color $\alpha \in \{1, 2, \cdots, \chi(G)\}$. This coloring is a $\chi(G)$-IE-total coloring of G. Thus, $\chi_t^{ie}(G) = \chi(G)$.

Theorem 3 $\chi_t^{vi}(G) = \chi'(G)$.

Proof It's clear that $\chi_t^{vi}(G) \geqslant \chi'(G)$. We first color the edges of G with $\chi'(G)$ colors $\{1, 2, \cdots, \chi'(G)\}$ such that no two adjacent edges receive the same color. Then color all the vertices with a color $\alpha \in \{1, 2, \cdots, \chi'(G)\}$. This coloring is a $\chi'(G)$-VI-total coloring of G. Thus, $\chi_t^{vi}(G) = \chi'(G)$.

Theorem 4 $\chi_t^{ve}(G) = 2$.

Proof Because no vertex receive the same color as any one of its incident edges does, we have $\chi_t^{ve}(G) \geqslant 2$. Assign color 1 to all the edges of G and 2 to all the vertices of G. This coloring is a 2-VE-total coloring of G. Thus, $\chi_t^{ve}(G) = 2$.

Theorem 5 $\chi_t^{v}(G) = \Delta(G) + 1$.

Proof By lemma 1, we have $\chi_t^v(G) \geqslant \chi_t^{vi}(G) = \chi'(G)$. If $\chi'(G) = \Delta(G)$, in order to make the adjacent edges have different colors and no vertex receives the same color as any one of its incident edges does, it must be $\chi_t^v(G) \geqslant \Delta(G) + 1$. We first color all the edges of G with Δ colors $\{1, 2, \cdots, \Delta\}$ such that no two adjacent edges receive the same color. Then color all the vertices of G with a new color which doesn't belong to $\{1, 2, \cdots, \Delta\}$. The result is a $(\Delta + 1)$-V-total coloring of G. If $\chi'(G) = \Delta(G) + 1$, color all the edges of G with $\Delta + 1$ colors $\{1, 2, \cdots, \Delta + 1\}$ such that no two adjacent edges receive the same color. For any vertex $v \in V(G)$, $d(v) \leqslant \Delta(G)$, there must be a color α which is not represented at v under the above proper edge coloring. Color v with α. We can obtain a $(\Delta + 1)$-V-total coloring of G. Therefore, $\chi_t^v(G) = \Delta(G) + 1$.

Theorem 6 $\chi_t^e(G) = \begin{cases} 3 & \text{if } G \text{ is a bipartite graph;} \\ \chi(G) & \text{if } G \text{ is not a bipartite graph.} \end{cases}$

Proof By lemma 1, we have $\chi_t^e(G) \geqslant \chi_t^{ie}(G) = \chi(G)$. If G is a bipartite graph, then $\chi(G) = 2$. For any edge $e = uv \in E(G)$, the color of e must be different from that of u and v, so $\chi_t^e(G) \geqslant 3$. We can color all the vertices of G with 2 colors $\{1, 2\}$ such that no two adjacent vertices receive the same color. Then color all the edges of G with a new color $\alpha (\neq 1, 2)$. Then the resulting is a 3-E-total coloring of G. So $\chi_t^e(G) = 3$ if G is a bipartite graph. If G is not a bipartite graph, $\chi(G) \geqslant 3$. Give G a vertex coloring using $\chi(G)$ colors such that no two adjacent vertices receive the same color. For any edge $e = xy \in E(G)$, there must be a color α which is different from the colors of x and y, color e with α. We can obtain a $\chi(G)$-E-total coloring of G. Therefore, $\chi_t^e(G) = \chi(G)$ if G is not a bipartite graph.

Theorem 7 $\chi_t^i(G) = \max\{\chi(G), \chi'(G)\}$.

Proof By lemma 1, $\chi_t^i(G) \geqslant \max\{\chi_t^{ie}(G), \chi_t^{vi}(G)\} = \max\{\chi(G), \chi'(G)\}$. If $\chi'(G) \geqslant \chi(G)$, we first color all the edges of G with $\chi'(G)$ colors $\{1, 2, \cdots, \chi'(G)\}$ such that no two adjacent edges receive the same color. Then color all the vertices with $\chi(G)$ colors $\{1, 2, \cdots, \chi(G)\} \subseteq \{1, 2, \cdots, \chi'(G)\}$ such that no two adjacent vertices receive the same color. We can obtain a $\chi'(G)$-I-total coloring of G. If $\chi(G) \geqslant \chi'(G)$, we first color all the vertices of G with $\chi(G)$ colors $\{1, 2, \cdots, \chi(G)\} \subseteq \{1, 2, \cdots, \chi(G)\}$ such that no two adjacent vertices receive the same color. Then color all the edges with $\chi'(G)$ colors $\{1, 2, \cdots, \chi'(G)\}$ such that no two adjacent edges receive the same color. We can obtain a $\chi(G)$-I-total coloring of G. Therefore, $\chi_t^i(G) = \max\{\chi(G), \chi'(G)\}$.

Thus the chromatic numbers of various general total colorings of graphs are determinable or rely on the (vertex) chromatic number or edge-chromatic index or both. So we may focus our attention on various general total colorings of graphs which are vertex distinguishing or adjacent vertex distinguishing.

References:

[1] BEHZAD M. Graphs and Their Chromatic Numbers [D]. Michigan State University, 1965.

[2] VIZING V G. Some Unsolved Problems in Graph Theory [J]. Russian Math. Surveys, 1968, 23: 125–141.

[3] HILTON A J W. Recent Results on the Total Chromatic Number [J]. Discrete Math., 1993, 111: 323–331.

[4] CHEW K H. Total Chromatic Number of Graphs of High Maximum Degree [J]. J. Combin. Math. Combin. Comput, 1995, 18: 245–254.

[5] SÁNCHEZ-ARROYO A. A New Upper Bound for Total Colourings of Graphs [J]. Discrete Mathematics, 1995, 138: 375–377.

[6] CAMPOS C N, DE MELLO C P. A Result on the Total Colouring of Powers of Cycles [J]. Discrete Applied Mathematics, 2007, 155: 585–597.

[7] SUN Xiang-yong, WU Jian-liang, WU Yu-wen, et al. Total Colorings of Planar Graphs Without Adjacent Triangles [J]. Discrete Mathematics, 2009, 309: 202–206.

[8] NICOLAS R, ZHU Xu-ding. Total Coloring of Planar Graphs of Maximum Degree Eight [J]. Information Processing Letters, 2010, 110: 321–324.

(next to page 10)

[7] ROCKMORE D N, TAN K. A Note on the Order of the Finite Subgroups of $GL(n, Z)$ [J]. Arch. Math., 1995, 64: 283 - 288.

[8] YOU Xing-zhong. A Note on the Orders of Finite Subgroups in $GL(n, Q)$ [J]. 四川大学学报: 自然科学版, 2008, 45(3): 475- 477.

[9] 游兴中, 陈为敏, 彭大千. $GL(2, Q)$的 2 阶元 共轭类 [J]. 吉首大学学报: 自然科学版, 2010, 31(1): 1- 3.

[10] 柯 召, 孙 琦. 数论讲义(上册) [M]. 北京: 高等教育出版社, 2007.

Conjugacy Classes of Elements of Order 3 in $GL(2, Q)$

LIANG Deng-feng[1], PENG Da-qian[2], YOU Xing-zhong[2]

(1. Dept. of Mathematics, Beijing Technology and Business University, Beijing 100048, China;
2. College of Mathematics and Computing Science, Changsha University of Science and Technology, Changsha 410114, China)

Abstract: By an elementary proof, conjugacy classes of elements of order 3 in the general linear group $GL(2, Q)$ are determined in this paper.

Key words: general linear group; order of element; conjugacy

(责任编辑 向阳洁)

(上接第 3 页)

图的各种一般全染色

陈祥恩[1], 高毓平[1], 杨随义[2]

(1. 西北师范大学数学与信息科学学院, 甘肃 兰州 730070; 2. 天水师范学院数学与统计学院, 甘肃 天水 741001)

摘 要: 图 G 的正常全染色是指若干颜色给 G 的顶点和边的分配, 使任意 2 个相邻顶点、2 条相邻边和任一顶点与它的关联边得到的颜色不同. 将正常全染色的限制条件减弱, 得到了各种一般全染色, 并讨论了它们的色数.

关键词: 全染色; 色数; 边色数

中图分类号: O157.5 文献标识码: A

(责任编辑 向阳洁)