Complexity of Total $\{k\}$-Domination and Related Problems*

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Abstract. In this paper, we study the $\{k\}$-domination, total $\{k\}$-domination, $\{k\}$-domatic number, and total $\{k\}$-domatic number problems, from complexity and algorithmic points of view. Let $k \geq 1$ be a fixed integer and $\epsilon > 0$ be any constant. Under the hardness assumption of $NP \not\subseteq DTIME(n^{O(\log \log n)})$, we obtain the following results.

1. The total $\{k\}$-domination problem is NP-complete even on bipartite graphs.
2. The total $\{k\}$-domination problem has a polynomial time $\ln n$ approximation algorithm, but cannot be approximated within $(\frac{1}{k} - \epsilon) \ln n$ in polynomial time.
3. The total $\{k\}$-domatic number problem has a polynomial time $(\frac{1}{k} + \epsilon) \ln n$ approximation algorithm, but does not have any polynomial time $(\frac{1}{k} - \epsilon) \ln n$ approximation algorithm.

All our results hold also for the non-total variants of the problems.

1 Introduction

Domination is a central concept in graph theory and has been thoroughly studied in numerous scenarios. The corresponding optimization problem, namely the dominating set problem, is also very important in various applications such as ad-hoc networks. In this paper, we study several generalizations and variations of the traditional dominating set problem, namely, the $\{k\}$-domination, the total $\{k\}$-domination, the $\{k\}$-domatic number, and the total $\{k\}$-domatic number problems. The first two problems are special cases of Y-domination introduced in [2], and the last two are their corresponding partition variants. While previous work mainly focuses on graph-theoretic properties of these concepts, we emphasize on algorithmic and complexity aspects, obtaining NP-completeness results and tight approximation thresholds for the problems.

1.1 Preliminaries

In this paper we only consider simple undirected graphs. We refer to [4] for standard notations and terminologies not given here. Let $G = (V, E)$ be a graph with

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$|V| = n$. We say the *order* of G is n. For each $v \in V$, let $N(v) = \{u \mid \{u, v\} \in E\}$ denote the neighborhood of v, and $N[v] = N(v) \cup \{v\}$. Let $d(v) = |N(v)|$ denote the degree of v, and $\delta(G) = \min_{v \in V} d(v)$ be the minimum degree of any vertex in G. A set of vertices $S \subseteq V$ is called a *dominating set* of G if every vertex in V either is contained in S or has a neighbor in S; that is, $N[v] \cap S \neq \emptyset$ for all $v \in V$. S is called a *total dominating set* of G if every vertex in V has a neighbor in S; that is, $N(v) \cap S \neq \emptyset$ for all $v \in V$ [6]. Notice that a graph with isolated vertices does not have any total dominating set. Hereafter we may always assume that the considered graph is connected, thus a total dominating set of a graph always exists. Let $\gamma(G)$ (resp. $\gamma_t(G)$) denote the minimum size of any dominating set (resp. total dominating set) of G. $\gamma(G)$ (resp. $\gamma_t(G)$) is also called the *domination number* (resp. *total domination number*) of G. The problem of computing the domination number of a given graph is NP-complete [11], and has a tight approximation threshold around $\ln n$ modulo the hardness assumption of $NP \not\subseteq DTIME(n^{O(\log \log n)})$ [8]. Similar results also hold for the total domination number. See [12,13] for extensive studies and surveys on related topics.

For a function $f : V \to \mathbb{R}$, the weight of f is $\omega(f) := \sum_{v \in V} f(v)$, and the weight of any subset $S \subseteq V$ is $f(S) := \sum_{v \in S} f(v)$. Let k be a positive integer. A function $f : V \to \{0, 1, \ldots, k\}$ is called a $\{k\}$-*dominating function* [7] (resp. *total* $\{k\}$-*dominating function* [15]) of G if for each $v \in V$, $f(N[v]) \geq k$ (resp. $f(N(v)) \geq k$). The $\{k\}$-*domination number* $\gamma^{\{k\}}(G)$ (resp. *total* $\{k\}$-*domination number* $\gamma_t^{\{k\}}(G)$) of G is the minimum weight of any $\{k\}$-dominating function (resp. total $\{k\}$-dominating function) of G. It is clear that $\gamma^{\{1\}}(G) = \gamma(G)$ and $\gamma_t^{\{1\}}(G) = \gamma_t(G)$. The $\{k\}$-*domination problem* (resp. *total* $\{k\}$-*domination problem* is to find a $\{k\}$-dominating function (resp. total $\{k\}$-dominating function) of G of minimum weight. The $\{k\}$-domination problem is proved to be NP-complete for every fixed k [10], and remains NP-complete even on bipartite graphs when $k \leq 3$ ([14], Theorem 2.15; see also Table 1.4 on page 38). In this paper we will give the first NP-completeness result for the total $\{k\}$-domination problem, and generalize the hardness results for $\{k\}$-domination on bipartite graphs to all values of k.

A set $\{f_1, f_2, \ldots, f_d\}$ of $\{k\}$-dominating functions (resp. total $\{k\}$-dominating functions) of G is called a $\{k\}$-*dominating family* (resp. *total* $\{k\}$-*dominating family*) of G if $\sum_{i=1}^{d} f_i(v) \leq k$ for all $v \in V$. The $\{k\}$-*domatic number* (resp. *total* $\{k\}$-*domatic number*) of G, denoted by $d^{\{k\}}(G)$ (resp. $d_t^{\{k\}}(G)$), is the maximum number of functions in a $\{k\}$-dominating family (resp. total $\{k\}$-dominating family) of G. These concepts were defined in [16,17], and has been further explored in [1,5]. The $\{k\}$-*domatic number problem* (resp. *total* $\{k\}$-*domatic number problem*) is to find a $\{k\}$-dominating family (resp. total $\{k\}$-dominating family) of G with maximum number of functions. When $k = 1$, the $\{k\}$-domatic number is conventionally called the *domatic number*, and can be alternatively defined as follows: A *domatic partition* (resp. *total domatic partition*) of G is a partition of V into disjoint sets $V_1, V_2, \ldots, V_t$ of V such that for each $i \in [t]$, V_i is a dominating set (resp. total dominating set) of G; t is called the *size* of this partition. The *domatic number* $d(G)$ (resp. *total domatic number* $d_t(G)$) of G is the maximum size of a domatic partition (resp. total domatic partition) of G. Clearly $d(G) = d^{\{1\}}(G)$ and $d_t(G) = d_t^{\{1\}}(G)$. Feige et al. [9] proved that the domatic number can be approximated within $\ln n$ in polynomial time, but cannot be approximated to $(1-\epsilon)\ln n$ in polynomial time unless $NP \subseteq DTIME(n^{O(\log\log n)})$. In this paper we will generalize this threshold behavior to (both total and non-total) $\{k\}$-domatic number problems.

We list some easy observations regarding these graph parameters.

Observation 1. *Let $k \geq 1$ be an integer. Then,*

- $\gamma_t(G) \leq \gamma_t^{\{k\}}(G) \leq k\gamma_t(G)$;
- $d_t(G) \leq d_t^{\{k\}}(G) \leq \delta(G)$;
- $d_t^{\{k\}}(G) \leq \frac{k|V|}{\gamma_t^{\{k\}}(G)} \leq \frac{k|V|}{\gamma_t(G)}$.

Moreoever, all the inequalities hold similarly for the non-total counterparts of the concepts (for the second one we need to replace $\delta(G)$ with $\delta(G)+1$).

1.2 Summary of Our Contributions

In Section 2, we prove that the total $\{k\}$-domination problem is NP-complete for every fixed k, even on bipartite graphs. To our knowledge, this is the first NP-completeness result for the problem. We also prove that the $\{k\}$-domination problem remains NP-complete on bipartite graphs for every fixed k, extending the result in [14] which only works for $k \leq 3$. We then study the problem from the approximation point of view, and obtain asymptotically tight approximation ratio for every fixed k.

In Section 3, we derive tight approximability results for the total $\{k\}$-domatic number problem. More precisely, we show that this problem admits a polynomial-time $(\frac{1}{k}+\epsilon)\ln n$ approximation algorithm for every fixed $\epsilon > 0$, but does not have any polynomial-time $(\frac{1}{k}-\epsilon)\ln n$ approximation algorithm unless $NP \subseteq DTIME$ $(n^{O(\log\log n)})$. The results easily apply to the $\{k\}$-domatic number problem.

2 $\{k\}$-Domination and Total $\{k\}$-Domination

In this section we study the $\{k\}$-domination and the total $\{k\}$-domination problems from both exact optimization and approximation aspects.

2.1 NP-Completeness Results

Theorem 1. *For every fixed integer $k > 0$, the $\{k\}$-domination problem is NP-complete even on bipartite graphs.*

Proof. Let k be a fixed integer. It is proved in [10] that the $\{k\}$-domination problem in general graphs is NP-complete. Let $G = (V, E)$ be an input graph to the $\{k\}$-domination problem. We construct a bipartite graph $H = (X, Y, E')$

as follows. Let $X = \{x_v \mid v \in V\} \cup \{x\}$, and $Y = \{y_v \mid v \in V\} \cup \{y\}$. Let $E = \{\{x_v, y_u\} \mid u \in N[v]\} \cup \{\{y, x'\} \mid x' \in X\}$.

We will show that $\gamma^{\{k\}}(H) = \gamma^{\{k\}}(G) + k$, thus proving Theorem 1 since this forms a Karp reduction. On one hand, let g be a $\{k\}$-dominating function of G with weight $\gamma^{\{k\}}(G)$. We construct a function $h : X \cup Y \to \{0, 1, \ldots, k\}$ as follows. For every $v \in V$ let $h(x_v) = g(v)$ and $h(y_v) = 0$. Let $h(x) = 0$ and $h(y) = k$. It is easy to verify that h is indeed a $\{k\}$-dominating function of H, and $\omega(h) = \omega(g) + k$. Hence, $\gamma^{\{k\}}(H) \leq \gamma^{\{k\}}(G) + k$.

On the other hand, let h be a $\{k\}$-dominating function of H with weight $\gamma^{\{k\}}(H)$. We modify h in the following way to get a finer form, which will help us extract a corresponding weight function of G. Since x only neighbors y in graph H, we have $h(x) + h(y) \geq k$. We may assume that $h(x) = 0$ and $h(y) = k$, since otherwise we can change h in this way to get a $\{k\}$-dominating function of H with no larger weight. Now every vertex in X is $\{k\}$-dominated by y. As long as there exists $y_v \in Y \setminus \{y\}$ such that $h(y_v) > 0$, we can set $h(y_v) \leftarrow 0$ and $h(x_v) \leftarrow \min\{h(x_v) + h(y_v), k\}$ to get a new $\{k\}$-dominating function of H with no larger weight. (This is because reducing the weight of y_v can only affect the vertices in $X \setminus \{x\}$, and every vertex in $X \setminus \{x\}$ has already been $\{k\}$-dominated by y.) Thus, we may assume that $h(y_v) = 0$ for all $v \in V$. Now we construct a function $g : V \to \{0, 1, \ldots, k\}$ by letting $g(v) = h(x_v)$ for all $v \in V$. It is easy to check that g is a $\{k\}$-dominating function of G with weight $\omega(h) - k$, implying that $\gamma^{\{k\}}(H) \geq \gamma^{\{k\}}(G) + k$. Therefore, we have shown that $\gamma^{\{k\}}(H) = \gamma^{\{k\}}(G) + k$, completing the proof of Theorem 1.

Theorem 1 generalizes the results in [14] for $k \leq 3$. We next consider the total variant of the problem.

Theorem 2. *For every fixed integer $k > 0$, the total $\{k\}$-domination problem is NP-complete even on bipartite graphs.*

Proof. We perform a similar reduction as used in the proof of Theorem 1. Let k be a fixed integer and $G = (V, E)$ be an input graph to the $\{k\}$-domination problem. Construct a bipartite graph $H = (X, Y, E')$ by making $X = \{x_v \mid v \in V\} \cup \{x\}$, $Y = \{y_v \mid v \in V\} \cup \{y\}$, and $E = \{\{x_v, y_u\} \mid u \in N(v)\} \cup \{\{y, x'\} \mid x' \in X\}$. (Note that we do not need edges of the form (x_v, y_v) any more.) It suffices to prove that $\gamma_t^{\{k\}}(H) = \gamma^{\{k\}}(G) + k$.

Let g be a $\{k\}$-dominating function of G with weight $\gamma^{\{k\}}(G)$. We obtain a function $h : X \cup Y \to \{0, 1, \ldots, k\}$ as follows. Let $h(x_v) = g(v)$ and $h(y_v) = 0$ for all $v \in V$. Let $h(x) = 0$ and $h(y) = k$. It is easy to verify that h is a total $\{k\}$-dominating function of H with weight $\omega(g) + k$, which indicates that $\gamma_t^{\{k\}}(H) \leq \gamma^{\{k\}}(G) + k$.

Now assume h is a total $\{k\}$-dominating function of H with weight $\gamma_t^{\{k\}}(H)$. We must have $h(y) = k$ since y is the only neighbor of x. We show that $h(x) = 0$. Consider an arbitrary vertex y_v, $v \in V$. By the definition of total $\{k\}$-domination, $h(N(y_v)) = h(\{x_u \mid u \in N(v)\}) \geq k$. This in particular implies $h(N(y)) \geq k$, since y neighbors all vertices in X. Hence reducing the weight of

x to 0 does not destroy the property of total $\{k\}$-domination, contradicting the optimality of h. Thus, we have $h(x) = 0$.

As long as there exists $y_v \in Y \setminus \{y\}$ such that $h(y_v) > 0$, we can set $h(y_v) \leftarrow 0$ and $h(x_v) \leftarrow \min\{h(x_v) + h(y_v), k\}$ to get a new total $\{k\}$-dominating function of H with no larger weight, since every vertex in X has already been total $\{k\}$-dominated by y. Thus, we may assume that $h(y_v) = 0$ for all $v \in V$. Now we construct a function $g : V \to \{0, 1, \ldots, k\}$ by letting $g(v) = h(x_v)$ for all $v \in V$. It is easy to check that g is a total $\{k\}$-dominating function of G with weight $\omega(h) - k$, which gives $\gamma_t^{\{k\}}(H) \geq \gamma^{\{k\}}(G) + k$. Combined with the previous result, we have $\gamma_t^{\{k\}}(H) = \gamma^{\{k\}}(G) + k$, completing the proof of Theorem 2.

2.2 Approximation Results

This subsection is devoted to the approximability of the two problems.

Theorem 3. *The $\{k\}$-domination problem and the total $\{k\}$-domination problem can both be approximated within $\ln n$ in polynomial time, where n is the order of the input graph.*

Proof. We reduce the $\{k\}$-domination problem to the set k-multicover problem, which is a generalization of the traditional set cover problem in which every element of the universe needs to be covered at least k times by the chosen sets. By [18] (see also [3]) the set k-multicover problem has a polynomial time $\ln n$-approximation algorithm, n being the number of elements in the universe. Let $G = (V, E)$ be an input graph to the $\{k\}$-domination problem. We construct a set k-multicover instance $(U, \mathcal{S})$ as follows. Let the ground set be $U = \{u_v \mid v \in V\}$. For every $v \in V$, include in $\mathcal{S}$ a set $S_v = \{u_{v'} \mid v' \in N[v]\}$. It is easy to see that every $\{k\}$-dominating function of G with weight w naturally induces a set k-multicover of U of cardinality w, and vice versa. Thus, this gives an approximation preserving reduction from the $\{k\}$-domination problem to the set k-multicover problem, proving that the former is $\ln n$-approximable in polynomial time. The proof for the total $\{k\}$-domination problem is very similar, with the only exception that, in the construction of S_v, we let $S_v = \{u_{v'} \mid v' \in N(v)\}$ (instead of using $N[v]$).

Theorem 4. *For any fixed integer $k > 0$ and any constant $\epsilon > 0$, there is no polynomial time $(\frac{1}{k} - \epsilon) \ln n$ approximation algorithm for the $\{k\}$-domination problem, unless $NP \subseteq DTIME(n^{O(\log \log n)})$. The inapproximability result also holds for the total $\{k\}$-domination problem.*

Proof. By Observation 1, we have $\gamma(G) \leq \gamma^{\{k\}}(G) \leq k\gamma(G)$. Moreover, any $\{k\}$-dominating function of G of weight w trivially induces a dominating set of G of size at most w (just take those vertices with positive weights). Thus, if wehave a polynomial time $(\frac{1}{k} - \epsilon) \ln n$ approximation algorithm for $\{k\}$-domination, we can also obtain in polynomial time a dominating set of G whose size is at most $(\frac{1}{k} - \epsilon) \ln n \cdot k\gamma(G) = (1 - k\epsilon) \ln n \cdot \gamma(G)$. For fixed $k, \epsilon > 0$, this implies $NP \subseteq DTIME(n^{O(\log \log n)})$ by [8]. The proof for total $\{k\}$-domination is totally similar.

3 Total $\{k\}$-Domatic Number

In this section we derive a tight approximation threshold for the total $\{k\}$-domatic number problem, modulo the assumption that NP does not have slightly super-polynomial algorithms.

Theorem 5. *Let $k \geq 1$ be a fixed integer and $\epsilon > 0$ be an arbitrarily small constant. Given any graph G of order n with n sufficiently large, we can find in polynomial time a total $\{k\}$-dominating family of G of size at least $(k-\epsilon)\delta(G)/\ln n$.*

Proof. Suppose $G = (V, E)$ is the input graph. Let $\delta = \delta(G)$ and $l = (k - \epsilon)\delta/\ln n$. We assume that $\delta = \omega(1)$, since otherwise the statement is trivial. Denote $[m] = \{1, 2, \ldots, m\}$ for any integer m. For every vertex $v \in V$, associate with it a k-tuple $(v_1, \ldots, v_k) \in [l]^k$ uniformly at random; that is, each coordinate v_i is chosen to be j with probability $1/l$, for all $i \in [k]$ and $j \in [l]$. We can alternatively consider this process as coloring each vertex with k colors, each of which is chosen uniformly at random from a set of l colors $\{1, 2, \ldots, l\}$. Every color j naturally induces a function $f_j : V \rightarrow \{0, 1, \ldots, k\}$: just let $f_j(v)$ be the number of j's in the k-tuple associated with vertex v. We also have that $\sum_{j \in [l]} f_j(v) = k$ for all $v \in V$.

For $v \in V$ and $j \in [l]$, let $\mathcal{E}(v, j)$ denote the event that at most $k-1$ tuple-coordinates, among all the $k \cdot d(v)$ coordinates of v's neighbors, are j. If there is no $v \in V$ such that $\mathcal{E}(v, j)$ holds, then every vertex v "sees" color j among its neighbors for at least k times, and thus f_j is a total $\{k\}$-dominating function of G. We will bound the probability that $\mathcal{E}(v, j)$ happens in order to establish a large (expected) number of total $\{k\}$-dominating functions among $\{f_j \mid j \in [l]\}$.

For all $v \in V$ and $j \in [l]$, we have

$$
\begin{aligned}
Pr[\mathcal{E}(v,j)] &= \sum_{i=0}^{k-1} \binom{k \cdot d(v)}{i} \left(\frac{1}{l}\right)^{i} \left(1 - \frac{1}{l}\right)^{k\cdot d(v)-i} \\
&\leq \left(1 - \frac{1}{l}\right)^{k\cdot d(v)-(k-1)} \left(\frac{1}{l}\right)^{k-1} \sum_{i=0}^{k-1} \binom{k \cdot d(v)}{i} \\
&\leq \exp\left(-\frac{k \cdot d(v) - (k-1)}{l}\right) \left(\frac{1}{l}\right)^{k-1} (k \cdot d(v))^{k-1} \\
&\qquad \text{(using } 1 + x \leq e^x \text{ and } \sum_{i=0}^{t} \binom{n}{i} \leq n^t\text{)} \\
&= \exp\left(-\frac{k \cdot d(v) - (k-1)}{(k-\epsilon)\delta/\ln n}\right) \left(\frac{k \cdot d(v)}{(k-\epsilon)\delta/\ln n}\right)^{k-1}. \\
&= n^{-\frac{k\cdot d(v)-(k-1)}{(k-\epsilon)\delta}} \left(\frac{k \cdot d(v) \ln n}{(k-\epsilon)\delta}\right)^{k-1} \\
&\leq n^{-\frac{k\cdot \delta-(k-1)}{(k-\epsilon)\delta}} \left(\frac{k \cdot \delta \ln n}{(k-\epsilon)\delta}\right)^{k-1}
\end{aligned}
$$

$$\text{(using } d(v) \geq \delta \text{ and } n \text{ is sufficiently large)}$$
$$\leq n^{-(1+\epsilon')} \left((1+\epsilon') \ln n\right)^{k-1},$$

where $\epsilon' > 0$ is some constant depending on k and ϵ. (In the last step we also used that $\delta = \omega(1)$.)

By the linearity of expectations, the expected number of pairs (v, j) such that $\mathcal{E}(v, j)$ holds is at most $nl \cdot Pr[\mathcal{E}(v, j)] \leq l \cdot n^{-\epsilon'} \left((1+\epsilon') \ln n\right)^{k-1} = o(l)$, since k is a fixed integer. This implies that at most $o(l)$ functions induced by the l colors are not total $\{k\}$-dominating functions of G. Therefore, the expected number of total $\{k\}$-dominating functions among $\{f_j \mid j \in [l]\}$ is at least $(1 - o(1))l = (k - \epsilon'')\delta / \ln n$, where $\epsilon'' > 0$ is some constant such that $\epsilon'' \to 0$ as $\epsilon \to 0$. This also proves that $d_t^{\{k\}}(G) \geq (k - \epsilon)\delta(G) / \ln n$ for any fixed $\epsilon > 0$ and sufficiently large n.

To finish the proof of Theorem 5, we need to show that this "coloring" process can be efficiently derandomized. This can be done by the method of conditional probabilities: We assign colors to the kn ccordinates one by one. Each time when a new coordinate needs to be decided, we try all the l possibilities. For each of the possibility, we can compute $Pr[\mathcal{E}(v, j)]$ conditioned on the current partial coloring for all $v \in V$ and $j \in [l]$ using standard combinatorial methods in polynomial time (noting that k is a fixed integer). We then choose a color j so as to minimize $\sum_{v \in V, j \in [l]} Pr[\mathcal{E}(v, j)]$. After all kn colors have been given, we check through the l functions $f_j, j \in [l]$ to find out which are total $\{k\}$-dominating functions. By our method of assigning the colors, the number of total $\{k\}$-dominating functions is at least the expected number that has been obtained before, i.e., $(k - \epsilon'')\delta / \ln n$. This completes the whole proof.

Theorem 5 directly yields the following statement.

Corollary 1. *Let $k \geq 1$ be a fixed integer and $\epsilon > 0$ an arbitrary constant. There is a polynomial time $(\frac{1}{k} + \epsilon) \ln n$ approximation algorithm for the total $\{k\}$-domatic number problem.*

Proof. Use Theorem 5 and the fact that $d_t^{\{k\}}(G) \leq \delta(G)$ for any graph G.

We next show that this approximation ratio is nearly optimal. We need the following lemma which is proved in [9].

Lemma 1 (Propositions 10 and 13 in [9]). *Fix $\epsilon > 0$ and assume $NP \not\subseteq DTIME(n^{O(\log \log n)})$. Given a graph G of order n, we cannot distinguish in polynomial time between the following two cases:*

1. *the size of any dominating set of G is at least r;*
2. *the domatic number of G is at least q,*

where r, q are some known parameters (dependent on the input) satisfying that $q = o(|V|)$ and $qr \geq (1 - \epsilon)|V| \ln |V|$.

Theorem 6. *For every fixed integer $k > 0$ and constant $\epsilon > 0$, if there is a polynomial-time $(\frac{1}{k} - \epsilon)\ln n$ approximation algorithm for the total $\{k\}$-domatic number problem, then $NP \subseteq DTIME(n^{O(\log\log n)})$.*

Proof. Let G be a graph of order n, and q, r be the parameters appeared in Lemma 1. We construct another graph $G' = (V', E')$ as follows. Let $V' = W \cup X \cup Y$, where $X = \{x_v \mid v \in V\}, Y = \{y_v \mid v \in V\}$, and $W = \{w_1, w_2, \ldots, w_q\}$. Let $E' = \{\{x_u, y_v\} \mid \{u, v\} \in E\} \cup \{\{w_i, x_v\} \mid i \in [q], v \in V\}$. This completes the construction of G'.

Consider the two cases stated in Lemma 1. According to our construction, any vertex y_v neighbors precisely those vertices of the form x_u with $(u, v) \in E$. Therefore, if the size of any dominating set of G is at least r, then the size of any total dominating set of G' is at least r. On the other hand, if the domatic number of G is at least q, we claim that G' has a total domatic partition of size at least q. Suppose $S_1, \ldots, S_q$ is a domatic partition of G. Consider the following (partial) partition of G': For every $i \in [q]$, let $S'_i = \{x_v \mid v \in S_i\} \cup \{w_i\}$. It is easy to see that every S'_i forms a total dominating set of G'. Thus, the total domatic number of G' is at least q.

By Lemma 1, assuming $NP \not\subseteq DTIME(n^{O(\log\log n)})$, we can not distinguish between the following two cases in polynomial time: (1) the size of any total dominating set of G' is at least r, and (2) the total domatic number of G' is at least q. By Observation 1, in the former case we have $d_t^{\{k\}}(G) \le k(|V| + q)/r = (1 + o(1))k|V|/r$ since $q = o(|V|)$, and in the latter case we have $d_t^{\{k\}}(G) \ge d_t(G) \ge q$. Therefore, it is hard to approximate the total $\{k\}$-domatic number to a factor of $\frac{q}{(1+o(1))k|V|/r} = \frac{(1-o(1))qr}{k|V|} \ge (\frac{1}{k} - 2\epsilon)\ln|V|$ when $|V|$ is sufficiently large, since $qr \ge (1 - \epsilon)|V|\ln|V|$. This completes the proof of Theorem 6.

Corollary 2. *For every fixed integer $k > 0$, if there is a polynomial time algorithm that computes the total $\{k\}$-domatic number of a given graph, then $NP \subseteq DTIME(n^{O(\log\log n)})$.*

Finally, we note that the $\{k\}$-domatic number problem also has the approximation threshold of $\frac{1}{k}\ln n$. The upper bound is easy since every total $\{k\}$-dominating family of G is also a $\{k\}$-dominating family of G, and the lower bound can be derived via a simpler reduction than that used in the proof of Theorem 6, using $V' = X \cup Y$ as the new graph and ignoring the W part.

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