Lattice Embedding of Direction-Preserving Correspondence over Integrally Convex Set
(Extended Abstract)

Xi Chen[1] and Xiaotie Deng[2]

[1] Department of Computer Science, Tsinghua University,
Beijing, P.R. China
xichen00@mails.tsinghua.edu.cn

[2] Department of Computer Science, City University of Hong Kong,
Hong Kong SAR
deng@cs.cityu.edu.hk

Abstract. We consider the relationship of two fixed point theorems for direction-preserving discrete correspondences. We show that, for any space of no more than three dimensions, the fixed point theorem [4] of Iimura, Murota and Tamura, on integrally convex sets can be derived from Chen and Deng's fixed point theorem [2] on lattices by extending every direction-preserving discrete correspondence over an integrally convex set to one over a lattice. We present a counter example for the four dimensional space. Related algorithmic results are also presented for finding a fixed point of direction-preserving correspondences on integrally convex sets, for spaces of all dimensions.

1 Introduction

A recent work on discrete fixed point introduced by Iimura [4] has attracted a series of work on related problems. Iimura, Murota and Tamura [6] improved the original proof of Iimura. Chen and Deng presented an alternative discrete fixed point theorem for general domain with a matching algorithmic bound for all finite dimensions [2]. In [8], Laan, Talman and Yang presented an iterative algorithm for the zero point problem. Friedl, Ivanyosy, Santha and Verhoeven obtained a $\sqrt{n}$ upper bound for the dimension two Sperner problem [7], thus a matching bound when combined with the lower bound of Crescenzi and Silvestri [3].

These problems are closely related. The matching bound of Friedl, Ivanyosy, Santha and Verhoeven for the Sperner problem is in some sense a mirror result of an earlier work of Hirsch, Papadimitriou and Vavasis on 2D approximate fixed point [5]. In addition, the higher dimensional query complexity for the Sperner problem of Friedl, Ivanyosy, Santha and Verhoeven, i.e., with query time linear in the separation number of the skeleton graph of the manifold and the size of its boundary, compares closely with the upper bound of Chen and Deng [2], for the query complexity of finding a discrete fixed point.

In this work, we set to understand the relationship between the discrete fixed point theorem of Iimura, Murota and Tamura, and the discrete fixed point

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theorem of Chen and Deng. In both cases, the discussion focuses on direction-preserving correspondences. The main differences are the restriction of the domains for which the theorems could apply. Murota, Iimura and Tamura consider a domain which is integrally convex. Informally, a point in the convex hull of the domain can be represented by a convex combination of integral points in the domain within unit distance from it. The work of Chen and Deng allows the domain not to be convex at all. Moreover, the result of Murota, Iimura and Tamura restricts the correspondence to be bounded in the domain. A more general boundary condition for correspondence is presented in Chen and Deng [2]. It is therefore natural to believe that the work of Murota, Iimura and Tamura can be derived from the seemingly more general version of Chen and Deng.

Indeed, for dimension two and three, we confirm it by embedding an integrally convex set in a lattice so that the bounded correspondence on the integrally convex set can be extended to a bounded and direction-preserving function on the lattice. Any fixed point of this function leads to a fixed point of the original correspondence. Therefore, a claim of existence of a fixed point on the lattice leads to a claim of existence of a fixed point in the integrally convex set. Such a direct extension, however, does not carry over to higher dimensions. We derive an interesting counter example for four-dimensional space.

There is another unsettled issue for the discrete fixed point theorem of Murota, Iimura and Tamura, that of algorithmic issues. In [8], Laan, van der Talman, and Yang presented an iterative algorithm which is shown to terminate with a fixed point. Our extension theorem for two and three dimensional spaces directly answers this problem and derives a matching algorithmic bound. For higher constant dimensional spaces, we need to refine the domain to derive an algorithmic solution.

In section 2, we define a fixed point problem called $\mathbf{FPC}^d$. Previous results are then reviewed in section 3. We formalize the concept of function extension mechanism in section 4. After presenting positive results for both two and three dimension spaces, we derive a counter example for the four-dimensional space in section 5. Section 6 gives a sketch of an algorithm to solve problem $\mathbf{FPC}^d$, for spaces of all dimensions, which implies a matching bound for the time complexity of $\mathbf{FPC}^d$. Finally, we conclude in section 7 with discussions on the difference between the two approaches.

2 Definition of Problem $\mathbf{FPC}^d$

In this section, we will define a fixed point problem called $\mathbf{FPC}^d$. It originates from the fixed point theorem of Iimura, Murota and Tamura [4, 6] concerning direction-preserving correspondences on integrally convex sets.

Definition 1. *Let X be a nonempty finite subset of $\mathbb{Z}^d$ and $\Gamma : X \to\to X$ be a nonempty-valued correspondence (that is, for every $x \in X$, $\Gamma(x) \subset X$).*

A point $x \in X$ is said to be a fixed point of Γ if $x \in \Gamma(x)$.

For each $x \in X$, let $\tau(x) \in \overline{\Gamma(x)}$ denote the projection of x onto $\overline{\Gamma(x)}$, i.e.,

$$||\tau(x) - x||_2 = \min_{y \in \overline{\Gamma(x)}} ||y - x||_2$$

where $||y-x||_2 = (\sum_{i=1}^{d}(y_i - x_i)^2)^{1/2}$.

Definition 2. *A correspondence* $\Gamma : X \rightarrow\rightarrow X$ *where* $X \subset \mathbb{Z}^d$ *is said to be direction-preserving on* X *if for all* $x, x' \in X$ *with* $||x - x'||_\infty \leq 1$, *we have* $(\tau_i(x) - x_i)(\tau_i(x') - x'_i) \geq 0$ *for every* $1 \leq i \leq d$.

We now define two classes of convex sets in $\mathbb{Z}^d$, integrally convex sets and discretely convex sets, which play important roles in the fixed point theorem.

Definition 3. *A finite set* $X \subset \mathbb{Z}^d$ *is integrally convex if for all points* $y \in \overline{X}$, $y \in \overline{X \cap N(y)}$, *where* $N(y) = \{ z \in \mathbb{Z}^d \mid ||z - y||_\infty < 1 \}$.

Definition 4. *A finite set* $X \subset \mathbb{Z}^d$ *is discretely convex if* $X = \overline{X} \cap \mathbb{Z}^d$.

Theorem 1 (Theorem of Iimura, Murota and Tamura [6]). *Let* $X \subset \mathbb{Z}^d$ *be a nonempty integrally convex set. For every nonempty, discretely convex-valued and direction-preserving correspondence* Γ *from* X *to itself, there must exist a fixed point* $x^* \in X$ *such that* $x^* \in \Gamma(x^*)$.

In brief, the task of the fixed point problem $\mathbf{FPC}^d$ is to find a fixed point of correspondence Γ which satisfies all the conditions in Theorem 1. Formally speaking, the input includes both the set X and correspondence Γ. Here X is described by all the extreme points of convex set $\overline{X}$. This representation of X is succinct, according to the following lemma.

Lemma 1. *For every* $d \geq 1$, *there exists an integer* N_d *such that, for all integrally convex sets* $X \subset \mathbb{Z}^d$, *the number of extreme points of* $\overline{X}$ *is less than* N_d.

On the other hand, correspondence Γ looks like a black box to algorithms. We only consider algorithms which are based on correspondence evaluations. Such an algorithm should behaves as follows: It makes up a test point $r_1 \in X$, sends it to the black box and receives $\tau(r_1)$. Based on r_1 and $\tau(r_1)$, it computes a new test point r_2 and evaluate $\tau(r_2)$. It continues until a fixed point of Γ is reached. We assume that each evaluation of τ takes one step.

Diameter of the integrally convex set X, that is, $n = \max_{x,y \in X} ||x - y||_\infty$, is taken as the input size of $\mathbf{FPC}^d$. We are interested in the time complexity $T_d(n)$ of problem $\mathbf{FPC}^d$. Our main result is stated in the following theorem.

Theorem 2. *For every constant* $d \geq 2$, $T_d(n) = \Theta(n^{d-1})$.

Problem $\mathbf{FPC}^d$ is closely related to problems $\mathbf{DFP}^d$ and $\mathbf{AFP}^d$ [2].

3 Previous Results on Fixed Point Problems

In this section, we review both the problem definitions and algorithmic results in [2]. For every $1 \leq k \leq d$, we use e^k to denote the kth unit vector of $\mathbb{Z}^d$. Here $e^k_k = 1$ and $e^k_i = 0$ for all $1 \leq i \neq k \leq d$,.

Definition 5. *For all $p < q \in \mathbb{Z}^d$, $A_{p,q} = \{\, r \in \mathbb{Z}^d \,\big|\, \forall\ 1 \le i \le d,\ p_i \le r_i \le q_i \,\}$.*

Its boundary is defined as $B_{p,q} = \{\, r \in A_{p,q} \,\big|\, \exists\ 1 \le i \le d,\ r_i = p_i \text{ or } q_i \,\}$.

Definition 6. *Function $f : S \rightarrow \{\, 0, \pm e^1, \pm e^2 ... \pm e^d \,\}$ where $S \subset \mathbb{Z}^d$ is said to be direction-preserving if for all $r^1, r^2 \in S$ which satisfy $||\, r^1 - r^2 \,||_\infty \le 1$, we have $||\, f(r^1) - f(r^2) \,||_\infty \le 1$.*

When $S = A_{p,q}$, f is said to be bounded if $r + f(r) \in A_{p,q}$ for all $r \in B_{p,q}$.

It is proved in [2] that any function f which is both bounded and direction-preserving has a zero point $r^* \in A_{p,q}$ such that $f(r^*) = 0$. The task of problem $\mathbf{DZP}^d$ is to find such a point in $A_{p,q}$. To get information of f, algorithms make up test points and evaluate f at these points. Similarly, we use $T_d^1(n)$ to denote the time complexity of $\mathbf{DZP}^d$, where $n = \max_{1 \le i \le d} (q_i - p_i)$.

Definition 7. *Map $\mathcal{G} : E^d = [0,1]^d \rightarrow \mathbb{R}^d$ satisfies a Lipschitz condition with constant M if $||\, \mathcal{G}(x) - \mathcal{G}(y) \,||_\infty \le M \,||\, x - y \,||_\infty$ for all $x, y \in E^d$.*

We use $L_{M,d}$ to denote the set of all those maps $\mathcal{F} : E^d \rightarrow E^d$ such that $\mathcal{G}(x) = \mathcal{F}(x) - x$ satisfies a Lipschitz condition with constant M.

By Brouwer's fixed point theorem, every map $\mathcal{F} \in L_{M,d}$ has a fixed point $x^* \in E^d$ such that $\mathcal{F}(x^*) = x^*$. Given a map $\mathcal{F} \in L_{M,d}$ and $\epsilon > 0$, the output of problem $\mathbf{AFP}^d$ is an approximate fixed point $x^* \in E^d$ with error bounded by ϵ. More exactly, x^* should satisfy $||\, \mathcal{F}(x^*) - x^* \,||_\infty \le \epsilon$. Similarly, $\mathcal{F}$ looks like a black box to algorithms, which can only be accessed by evaluations. We use $T_d^2(M, \epsilon)$ to denote the time complexity of problem $\mathbf{AFP}^d$.

Theorem 3 ([2]). *For every constant $d \ge 2$,*

$$T_d^1(n) = \Theta(n^{d-1}) \quad \text{and} \quad T_d^2(M, \epsilon) = \Theta\Big(\big(\frac{M}{\epsilon} \big)^{d-1} \Big).$$

In fact, the lower bound of $T_d(n)$ in Theorem 2 can be easily derived from the lower bound of $T_d^1(n)$ above.

4 Extension Mechanism for Low Dimensional Spaces

In this section, we focus on a natural idea to solve problem $\mathbf{FPC}^d$. First, we formalize the concept of function extension mechanism $\mathcal{M}^d$. Its existence gives an algorithm for $\mathbf{FPC}^d$ with time complexity $O(n^{d-1})$. $\mathcal{M}^2$ and $\mathcal{M}^3$ are then constructed and we get the upper bound in Theorem 2 for cases $d = 2$ and 3.

4.1 Definition of Function Extension Mechanism $\mathcal{M}^d$

The discrete approach presented in this section is based on the existence of algorithms for problem $\mathbf{DZP}^d$ with time complexity $O(n^{d-1})$. Let $A_{p,q}$ be the smallest set that contains X which is the domain of Γ and τ. A function extension mechanism $\mathcal{M}^d$ extends map τ to be a direction function f from $A_{p,q}$ to $\{\, 0, \pm e^1, \pm e^2 ... \pm e^d \,\}$ which is both bounded and direction-preserving. We can use any algorithm for problem $\mathbf{DZP}^d$ to find a zero point of f. Properties of $\mathcal{M}^d$ guarantee that, given a zero point of f, one can find a fixed point of map τ (and thus, correspondence Γ) very efficiently.

Definition 8. *Given an input pair (X, Γ) of $\mathbf{FPC}^d$, if the integrally convex set X is non-degenerate, that is, $\overline{X}$ is a d-polytope in $\mathbb{R}^d$, then function extension mechanism $\mathcal{M}^d = (\mathcal{A}^d, \mathcal{B}^d)$ for d-dimensional space constructs a direction function f from $A_{p,q}$ to $\{ 0, \pm e^1, \dots \pm e^d \}$.*

The following five properties should be satisfied:

- $\boldsymbol{P}_1$. *Function f is both bounded and direction-preserving on $A_{p,q}$;*
- $\boldsymbol{P}_2$. *For every $r \in A_{p,q}$, algorithm $\mathcal{A}^d$ takes (r, X, τ) as input and computes $f(r)$ with $O(1)$ (d is viewed as a constant here) running time;*
- $\boldsymbol{P}_3$. *For every $r \in X$, $f(r) = 0$ if and only if $\tau(r) = r$;*
- $\boldsymbol{P}_4$. *For evert $r \in X$ such that $\tau(r) \neq 0$, $f(r) \cdot (\tau(r) - r) > 0$;*
- $\boldsymbol{P}_5$. *For every zero point r of f such that $r \notin X$, algorithm $\mathcal{B}^d$ takes (r, X, τ) as input and computes a fixed point $r' \in X$ of τ with $O(1)$ running time.*

Clearly, once we find a mechanism $\mathcal{M}^d$ for d-dimensional space, we get an algorithm for $\mathbf{FPC}^d$ with time complexity $O(n^{d-1})$ (if X is degenerate, then we exhaustively check every point in X, since $|X| \le n^{d-1}$). From now on, we always assume that X is non-degenerate.

4.2 Function Extension Mechanism $\mathcal{M}^2$ for Case $d = 2$

$\mathcal{M}^2$ is closely related to a map ψ from $A_{p,q}$ to X. For every $r \in X$, $\psi(r) = r$. Otherwise, $\psi(r) = \widetilde{r}$ where

$$|\, r_1 - \widetilde{r}_1 \,| = \min_{r' \in X,\ r_2 = r'_2} |\, r_1 - r'_1 \,|.$$

The construction of function f is described in figure 1.

Properties P_2, P_3 and P_4 are easy to verify. For property P_5, if $r \notin X$ and $f(r) = 0$, then $f(r') = 0$ where $r' = \psi(r)$. With the succinct representation of X, $r' = \psi(r)$ can be computed in $O(1)$ time. Proof of the following lemma is available in the full version [1].

Lemma 2. *f constructed by $\mathcal{M}^2$ is both bounded and direction-preserving.*

Function Extension Mechanism $\mathcal{M}^2$

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for any $r \in X$
    if $\tau(r) = 0$ then $f'(r) = 0$
    else if $\tau_2(r) \neq 0$ then $f'(r) = \mathrm{sign}\,(\tau_2(r))\, e^2$
    else $f'(r) = \mathrm{sign}\,(\tau_1(r))\, e^1$
for any $r \in A_{p,q}$, $f(r) = f'(\psi(r))$
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Fig. 1. Details of the Function Extension Mechanism $\mathcal{M}^2$

4.3 Function Extension Mechanism $\mathcal{M}^3$ for Case $d = 3$

Behavior of the mechanism $\mathcal{M}^3$ is similar to $\mathcal{M}^2$, while the details are a little more complicated. First, we divide $A_{p,q}$ into three pairwise disjoint sets, X, S_1 and S_2 where

$$S_1 = \{\ r \notin X,\ r \in A_{p,q} \ \big|\ \exists\ r' \in X, r_2 = r_2' \text{ and } r_3 = r_3'\ \},$$
$$S_2 = \{\ r \notin X \cup S_1\ , r \in A_{p,q} \ \big|\ \exists\ r' \in X \cup S_1, r_1 = r_1' \text{ and } r_3 = r_3'\ \}.$$

We then define two maps. ψ^1 is from $X \cup S_1$ to X. For all $r \in X$, $\psi^1(r) = r$. For all $r \in S_1$, $\psi^1(r) = \widetilde{r}$ where

$$|\, r_1 - \widetilde{r}_1 \,| = \min_{r' \in X,\ r_2 = r_2',\ r_3 = r_3'} |\, r_1 - r_1' \,|.$$

Map ψ^2 is from $A_{p,q}$ to $X \cup S_1$. For all $r \in X \cup S_1$, $\psi^2(r) = \psi^1(r)$. For all point $r \in S_2$, $\psi^2(r) = \widetilde{r}$ where

$$|\, r_2 - \widetilde{r}_2 \,| = \min_{r' \in X \cup S_1,\ r_1 = r_1',\ r_3 = r_3'} |\, r_2 - r_2' \,|.$$

Given a map τ, $\mathcal{M}^3$ first convert it into a direction function f' from X to $\{0, \pm e^1, \pm e^2, \pm e^3\}$. After extending f' to be f'' on $X \cup S_1$ using map ψ^1, we employ map ψ^2 to extend f'' onto $A_{p,q}$. The difficulty here is that, to keep the direction-preserving property, we must be careful when dealing with some boundary points of X.

Definition 9. *Point $r \in X$ is said to be a left (or right) boundary point of X if $(r_1 - 1, r_2, r_3) \notin X$ (or $(r_1 + 1, r_2, r_3) \notin X$). We use L_X (or R_X) to denote the set of left (or right) boundary points of X.*

From the definition of integrally convex sets, we get the following lemma.

Lemma 3. *For all points $r^1, r^2 \in L_X$ (or R_X) which satisfy $|\, r_2^1 - r_2^2 \,| \le 1$ and $|\, r_3^1 - r_3^2 \,| \le 1$, we have $|\, r_1^1 - r_1^2 \,| \le 2$.*

Furthermore, if $|\, r_1^1 - r_1^2 \,| = 2$, then $|\, r_2^1 - r_2^2 \,| = |\, r_3^1 - r_3^2 \,| = 1$.

Definition 10. *Pair (r^1, r^2) where $r^1, r^2 \in L_X$ (or $r^1, r^2 \in R_X$) is said to be a bad pair of X if $|\, r_2^1 - r_2^2 \,| = |\, r_3^1 - r_3^2 \,| = 1$ and $|\, r_1^1 - r_1^2 \,| = 2$. We use B_X to denote the set of bad pairs of X.*

$r \in X$ is said to be bad if there exists $r' \in X$ such that $(r, r') \in B_X$.

Each bad pair (r^1, r^2) of X gives a supporting hyperplane $H_{r^1,r^2} = (u, a)$ of $\overline{X}$ where $|\, u_i \,| = 1$, for all $1 \le i \le 3$. For example, if $r^1 = (0,0,0)$ and $r^2 = (2,1,1)$ are two left boundary points, then one can prove both $(1,1,0)$ and $(1,0,1)$ belong to L_X. These points together define a hyperplane $H_{r^1,r^2} = (-1,1,1,0)$. With $H_{r^1,r^2} = (u, a)$, we define $S_{r^1,r^2} = \{-u_1 e^1, -u_2 e^2, -u_3 e^3\}$.

On the other hand, for a bad point $r \in X$, there might be more than one point r' such that $(r, r') \in B_X$. We define $S_r = \bigcap_{(r,r') \in B_X} S_{r,r'}$ which has the following property.

Lemma 4. *For every bad point $r \in X$, $1 \leq |S_r| \leq 3$.*

If $S_r = \{+e^k\}$ (or $S_r = \{-e^k\}$) where $1 \leq k \leq 3$, then $r_k = \min_{r' \in X} r'_k$ (or $r_k = \max_{r' \in X} r'_k$). Furthermore, if $k \neq 1$, then there are exactly two points $r' \in X$ such that $(r, r') \in B_X$.

If $|S_r| > 1$ and $\tau(r) \neq r$, then there exists a unit vector $ce^k \in S_r$ such that $ce^k \cdot (\tau(r) - r) > 0$.

For every bad point r such that $S_r = \{ce^k\}$ where $k \neq 1$ and $|c| = 1$, we define vectors v_L, $v_R \in \{\pm e^1, \pm e^2, \pm e^3\}$ based to the value of $\tau(r)$ and the shape of X around r. Only the case for $c = 1$ and $k = 3$ is described below, as other cases are similar.

> **Case 1:** (r, r^L), $(r, r^R) \in B_X$ where $r^L = (r_1 - 2, r_2 - 1, r_3 - 1)$, $r^R = (r_1 + 2, r_2 + 1, r_3 - 1)$. If $\tau_3(r) < 0$, then $v_L = v_R = -e^3$. Otherwise, we have $\tau_1(r) = \tau_2(r)$. If $\tau_1(r) > 0$, then $v_L = +e^1$ and $v_R = +e^2$, or else $v_L = -e^2$ and $v_R = -e^1$.
>
> **Case 2:** (r, r^L), $(r, r^R) \in B_X$ where $r^L = (r_1 - 2, r_2 + 1, r_3 - 1)$, $r^R = (r_1 + 2, r_2 - 1, r_3 - 1)$. If $\tau_3(r) < 0$, then $v_L = v_R = -e^3$. Otherwise, we have $\tau_1(r) = -\tau_2(r)$. If $\tau_1(r) > 0$, then $v_L = +e^1$ and $v_R = -e^2$, or else $v_L = +e^2$ and $v_R = -e^1$.

In both cases, we have $v_L \cdot (\tau(r) - r) > 0$, $v_L \in S_{r,r^L}$, $v_R \cdot (\tau(r) - r) > 0$ and $v_R \in S_{r,r^R}$. Details of the mechanism $\mathcal{M}^3$ are described in figure 2. Similarly, properties P_2, P_3, P_4 and P_5 are easy to verify. Proof of the following lemma is available in the full version.

Function Extension Mechanism $\mathcal{M}^3$

```
for any $r \in X$
    if $\tau(r) = 0$ then $f'(r) = 0$
    else if $r$ is a bad point of $X$ and $|S_r| > 1$ then
        there must exist $k$ such that $ce^k \in S_r$ and $c\tau_k(r) > 0$, set $f'(r) = ce^k$
    else let $k$ be the largest integer satisfies $\tau_k(r) \neq 0$, set $f'(r) = \mathrm{sign}(\tau_k(r))\, e^k$
for any $r \in X \bigcup S_1$
    if $r \in X$ then $f''(r) = f'(r)$
    else if $f'(\psi^1(r)) = 0$ then $f''(r) = 0$
    else if $\psi^1_1(r) = \min_{r' \in X} r'_1$ then $f''(r) = +e^1$
    else if $\psi^1_1(r) = \max_{r' \in X} r'_1$ then $f''(r) = -e^1$
    else if $r' = \psi^1(r)$ is a bad point of $X$ and $S_{r'} = \{ce^k\}$ where $k \neq 1$ then
        if $r_1 < r'_1$ then $f''(r) = v_L$
        else $f''(r) = v_R$
    else $f''(r) = f'(\psi^1(r))$
for any $r \in A_{p,q}$, $f(r) = f''(\psi^2(r))$
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Fig. 2. Details of the Function Extension Mechanism $\mathcal{M}^3$

Lemma 5. *f constructed by $\mathcal{M}^3$ is both bounded and direction-preserving.*

5 A Counter Example for 4-Dimensional Space

Although function extension mechanism $\mathcal{M}^d$ does exist for cases $d = 2$ and 3, we find great difficulty in designing $\mathcal{M}^d$ for higher dimensional spaces. In this section, we construct a set of maps S in the 4-dimensional space and prove the non-existence of mechanism $\mathcal{M}^4$.

The domain of maps in S is

$$X = \left\{ r \in \mathbb{Z}^4 \;\middle|\; \forall\, 1 \le i \le d,\ r_i \ge 0 \text{ and } r_1 + r_2 + r_3 + r_4 \le n \right\}$$

which can be divided into layers $X = X_1 \cup X_2 \ldots \cup X_n = Y \cup Z$. Here set $X_i = \{ r \in X \mid r_4 = i \}$, $Y = X_n \cup X_{n-1} \ldots \cup X_{n-5}$ and $Z = X - Y$. For every $r \in Z$, we construct a map τ_r as follows, and $S = \{ \tau_r \mid r \in Z \}$.

For every two maps $\tau_r, \tau_{r'} \in S$, $\tau_r(p) = \tau_{r'}(p)$ for all $p \in Y$. Values of τ, where $\tau \in S$, on the first four layers X_n, X_{n-1}, X_{n-2} and X_{n-3} are described in figure 3. In this figure, an arrow ce^k on point r means $\Gamma(p) = \{ p + ce^k \}$ and $\tau(p) = p + ce^k$. For every $p \in X_{n-4}$, if $\|p - (2, 0, 0, n-4)\|_\infty > 1$, then $\tau(p) = p - e^4$. If $p = (2, 0, 0, n-4)$, then $\tau(p) = p - e^1$. Otherwise, $\tau(p) - p = \tau((p_1, p_2, p_3, p_4{+}1)) - (p_1, p_2, p_3, p_4{+}1)$. Finally, $\tau(p) = p - e^4$ for every $p \in X_{n-5}$,.

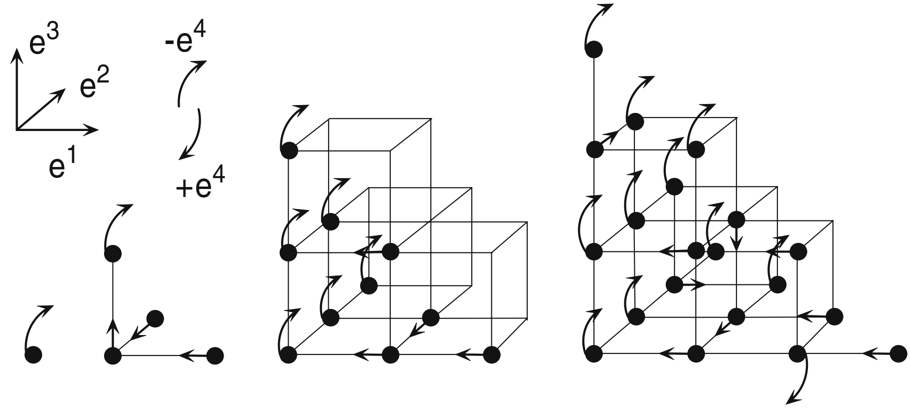

Fig. 3. A Counter Example

Values of τ_r on Z are described as follows. For every $p \in Z$, if $p = r$, then $\tau_r(p) = p$. Otherwise, we have two cases. If $\|r\|_1 > \|p\|_1$ where $\|r\|_1 = \sum_{i=1}^{4} r_i$, letting k be an integer such that $r_k > p_k$, then $\tau_r(p) = p + e^k$. If $\|r\|_1 \le \|p\|_1$, letting k be an integer such that $r_k < p_k$, then $\tau_r(p) = p - e^k$. One can prove the following property of maps in S.

Lemma 6. *$\tau_r : X \to X$ is direction-preserving and r is its only fixed point.*

Now we prove the non-existence of mechanism $\mathcal{M}^4$. Let's make a reduction to absurdity, considering that there exists a mechanism $\mathcal{M}^4 = (\mathcal{A}^d, \mathcal{B}^d)$, however, satisfies all the five properties P_1, $P_2 \dots P_5$, then for every map $\tau_r \in S$, it constructs a direction-preserving function f_r. By property P_4, we have $f_r(r') = \tau_r(r') - r'$ for every $r' \in X$. Since f_r is direction-preserving, we must have $f_r(r^*) = 0$ where $r^* = (1, 1, 1, n-2)$.

Let's pick a map $\tau_r \in S$ arbitrarily and run $\mathcal{B}^d$ with input (r^*, X, τ_r). After constant steps, it should output a fixed point r' of τ_r according to P_5. By Lemma 6, we have $r' = r$. This means that maps in S can be recognized within constant steps, which contradicts with the fact that $|S| = \Theta(n^4)$. As a result, our assumption is wrong and no such mechanism exists.

6 An Algorithm for Problem FPC^d

In this section, we briefly describe an algorithm for $\mathbf{FPC}^d$ and prove the upper bound in Theorem 2, for spaces of all dimensions.

Definition 11. *For every point $r \in \mathbb{Z}^d$, we define a hypercube $C_{r,n} \subset \mathbb{R}^d$ as*

$$C_{r,n} = \left\{ \, x \in \mathbb{R}^d \;\middle|\; r_i \le x_i \le r_i + n, \text{ for all } 1 \le i \le d \, \right\}.$$

Let (Γ, X) be an input instance of $\mathbf{FPC}^d$, then we use $C_{r,n}$ to denote the smallest hypercube containing X. Starting from Γ, we build a map $\mathcal{F}$ from $C_{r,n}$ to itself. Details of the construction can be found in the full version. We give the following lemmas without proof.

Lemma 7. *Given an input instance (Γ, X) of problem $\mathbf{FPC}^d$, for every point $x \in C_{r,n}$, $\mathcal{F}(x)$ can be computed in $O(1)$ time.*

Lemma 8. *For every constant $d \ge 2$, there exists a constant D_d such that, for every input instance (Γ, X) of problem $\mathbf{FPC}^d$, map $\mathcal{F}$ belongs to $L_{D_d,d}$.*

Lemma 9. *For every point $x^* \in \overline{X}$ such that $||\mathcal{F}(x^*) - x^*||_\infty < 1/(d+1)^2$, there must exist a fixed point of correspondence Γ in $N(x^*) \cap X$. Recall that $N(x^*) = \{ r \in \mathbb{Z}^d \mid ||r - x^*||_\infty < 1 \}$.*

Lemma 10. *For every $x \in C_{r,n}$ such that $||\mathcal{F}(x) - x||_\infty < 1/(d^{1/2}(d+1)^2)$, point $x^* = \Psi_{\overline{X}}(x)$ must satisfy $||\mathcal{F}(x^*) - x^*||_\infty < 1/(d+1)^2$. Here $\Psi_{\overline{X}}$ is the projection onto $\overline{X}$ where $||x - \Psi_{\overline{X}}(x)||_2 = \min_{y \in \overline{X}} ||x - y||_2$.*

$\mathcal{F}$ can be scaled to be a map $\mathcal{F}'$ from $E^d = [0,1]^d$ to itself as follows. For every point $x \in E^d$, $\mathcal{F}'(x) - x = (\mathcal{F}(nx + r) - (nx + r))/n$.

The reason we build $\mathcal{F}$ and $\mathcal{F}'$ is to find a fixed point of Γ. By Lemma 8, one can prove that the new map $\mathcal{F}'$ also belongs to $L_{D_d,d}$, thus we can use an algorithm for $\mathbf{AFP}^d$ to compute an $\epsilon = 1/(d^{1/2}(d+1)^2 n)$ approximate fixed point x of $\mathcal{F}'$, and $x^* = nx + r$ must be an $1/(d^{1/2}(d+1)^2)$ approximate fixed

```
Algorithm for Problem FPC$^d$
 1: Let $(\Gamma, X)$ be the input instance of problem FPC$^d$
 2: Let $\mathcal{F}$ and $\mathcal{F}'$ be the two maps constructed
 3: Use an algorithm for AFP$^d$ to find an $\epsilon$ approximate fixed point $x$ of $\mathcal{F}'$
 4: compute $x^* = nx + r$
 5: if $x^* \in \overline{X}$, then
 6:     query $\Gamma$ for every point in $N(x^*) \cap X$ and output a fixed point of $\Gamma$
 7: else
 8:     compute $x' = \Psi_{\overline{X}}(x^*)$
 9:     query $\Gamma$ for every point in $N(x') \cap X$ and output a fixed point of $\Gamma$
10: endif
```

Fig. 4. The Algorithm for Fixed Point Problem **FPC**^{d}

point of $\mathcal{F}$. Lemma 9 and 10 together show that, once we get x^*, a fixed point of Γ can be located easily.

The algorithm is described in figure 4. Let's analyze its time complexity. For every test point $x \in E^d$ which is queried by the **AFP**^{d} algorithm, constant steps are sufficient to compute $\mathcal{F}'(x)$ according to Lemma 7. By Theorem 3, the time used by the **AFP**^{d} algorithm in line 3 is $O\left((D_d/\epsilon)^{d-1}\right) = O\left(n^{d-1}\right)$. This gives us the upper bound of time complexity $T_d(n)$ in Theorem 2.

7 Concluding Remarks

In this paper, we described two different approaches to solve the discrete fixed point problem **FPC**^{d}. In the discrete approach, we try to extend map τ to be a direction-preserving function f on lattice $A_{p,q}$. In the continuous approach, we construct a Lipschitz map $\mathcal{F}_3$ from $C_{r,n}$ to itself. While the former only works for low dimensional spaces, the latter solves problem **FPC**^{d} for spaces of all dimensions. But how does the algorithm for problem **AFP**^{d} work? Actually, it samples map $\mathcal{F}_3$ with a suitable interval, builds a direction function which is both bounded and direction-preserving, and employs an algorithm for problem **DZP**^{d} to find an zero point which is also an approximate fixed point of $\mathcal{F}_3$ [2].

Thus in both approaches, we construct (explicitly or implicitly) a bounded and direction-preserving function on some lattice. The difference is that, the lattice of the continuous approach has much higher density than the one of the discrete approach. While no function extension mechanism exists for high dimensional spaces, we can always construct a direction-preserving function on a denser lattice implicitly using the continuous method.

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