The Isolation Game: A Game of Distances

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Abstract. We introduce a new multi-player geometric game, which we will refer to as the *isolation game*, and study its Nash equilibria and best or better response dynamics. The isolation game is inspired by the Voronoi game, competitive facility location, and geometric sampling. In the Voronoi game studied by Dürr and Thang, each player's objective is to maximize the area of her Voronoi region. In contrast, in the isolation game, each player's objective is to position herself as far away from other players as possible in a bounded space. Even though this game has a simple definition, we show that its game-theoretic behaviors are quite rich and complex. We consider various measures of farness from one player to a group of players and analyze their impacts to the existence of Nash equilibria and to the convergence of the best or better response dynamics: We prove that it is NP-hard to decide whether a Nash equilibrium exists, using either a very simple farness measure in an asymmetric space or a slightly more sophisticated farness measure in a symmetric space. Complementing to these hardness results, we establish existence theorems for several special families of farness measures in symmetric spaces: We prove that for isolation games where each player wants to maximize her distance to her m^{th} nearest neighbor, for any m, equilibria always exist. Moreover, there is always a better response sequence starting from any configuration that leads to a Nash equilibrium. We show that when $m = 1$ the game is a potential game — no better response sequence has a cycle, but when $m > 1$ the games are not potential. More generally, we study farness functions that give different weights to a player's distances to others based on the distance rankings, and obtain both existence and hardness results when the weights are monotonically increasing or decreasing. Finally, we present results on the hardness of computing best responses when the space has a compact representation as a hypercube.

1 Introduction

In competitive facility location [4,5,7] data clustering [8], and geometric sampling [10], a fundamental geometric problem is to place a set of objects (such as facilities and

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cluster centers) in a space so that they are mutually far away from one another. Inspired by the study of Dürr and Thang [3] on the Voronoi game, we introduce a new multi-player geometric game called *isolation game*.

In an isolation game, there are k players that will locate themselves in a space (Ω, Δ) where $\Delta(x, y)$ defines the pairwise distance among points in Ω. If $\Delta(x, y) = \Delta(y, x)$, for all $x, y \in \Omega$, we say (Ω, Δ) is symmetric. The i^{th} player has a $(k-1)$-place function $f_i(\ldots, \Delta(p_i, p_{i-1}), \Delta(p_i, p_{i+1}), \ldots)$ from the $k-1$ distances to all other players to a real value, measuring the farness from her location p_i to the locations of other players. The objective of player i is to maximize $f_i(\ldots, \Delta(p_i, p_{i-1}), \Delta(p_i, p_{i+1}), \ldots)$, once the positions of other players $(\ldots, p_{i-1}, p_{i+1}, \ldots)$ are given.

Depending on applications, there could be different ways to measure the farness from a point to a set of points. The simplest farness function $f_i()$ could be the one that measures the distance from p_i to its nearest player. Games based on this measure are called *nearest-neighbor games*. Another simple measure is the total distance from p_i to other players. Games based on this measure are called *total distance games*. Other farness measures include the distance of p_i to its m^{th} nearest player, or a weighted combination of the distances from player i to other players.

Isolation games with simple farness measures can be viewed as an approximation of the Voronoi game [1,2,6]. Recall that in the Voronoi game, the objective of each player is to maximize the area of her Voronoi cell in Ω induced by $\{p_1, ..., p_k\}$ — the set of points in Ω that are closer to p_i than to any other player. The Voronoi game has applications in competitive facility location, where merchants try to place their facilities to maximize their customer bases, and customers are assumed to go to the facility closest to them. Each player needs to calculate the area of her Voronoi cell to play the game, which could be expensive. In practice, as an approximation, each player may choose to simply maximize her nearest-neighbor distance or total-distance to other players. This gives rise to the isolation game with these special farness measures.

The generalized isolation games may have applications in product design in a competitive market, where companies' profit may depend on the dissimilarity of their products to those of their competitors, which could be measured by the multi-dimensional features of products. Companies differentiate their products from those of their competitors by playing some kind of isolation games in the multi-dimensional feature space. The isolation game may also have some connection with political campaigns such as in a multi-candidate election, in which candidates, constrained by their histories of public service records, try to position themselves in the multi-dimensional space of policies and political views in order to differentiate themselves from other candidates.

We study the Nash equilibria [9] and best or better response dynamics of the isolation games. We consider various measures of farness from one player to a group of players and analyze their impact to the existence of Nash equilibria and to the convergence of best or better response dynamics in an isolation game. For simple measures such as the nearest-neighbor and the total-distance, it is quite straightforward to show that these isolation games are potential games when the underlying space is symmetric. Hence, the game has at least one Nash equilibrium and all better response dynamics converge. Surprisingly, we show that when the underlying space is asymmetric, Nash equilibria may not exist, and it is NP-hard to determine whether Nash equilibria exist in an isolation game. The general isolation game is far more complex even for symmetric spaces, even if we restrict our attention only to uniform anonymous isolation games. We say an isolation game is *anonymous* if for all i, $f_i()$ is invariant under the permutation of its parameters. We say an anonymous isolation game is *uniform* if $f_i() = f_j()$ for all i, j. For instance, the two potential isolation games with the nearest-neighbor or total-distance measure mentioned above are uniform anonymous games. Even these classes of games exhibit different behaviors: some subclass of games always have Nash equilibrium, some can always find better response sequences that converge to a Nash equilibrium, but some may not have Nash equilibrium and determining the existence of Nash equilibrium is NP-complete. We summarize our findings below.

First, We prove that for isolation games where each player wants to maximize her distance to her m^{th} nearest neighbor, equilibria always exist. In addition, there is always a better response sequence starting from any configuration that leads to a Nash equilibrium. We show, however, this isolation game is not a potential game — there are better response sequences that lead to cycles. Second, as a general framework, we model the farness function of a uniform anonymous game by a vector $\boldsymbol{w} = (w_1, w_2, \ldots, w_{k-1})$. Let $\boldsymbol{d_j} = (d_{j,1}, d_{j,2}, \ldots, d_{j,k-1})$ be the *distance vector* of player j in a configuration, which are distances from player j to other $k-1$ players sorted in nondecreasing order, i.e., $d_{j,1} \le d_{j,2} \le \ldots \le d_{j,k-1}$. Then the utility of player j in the configuration is $\boldsymbol{w} \cdot \boldsymbol{d} = \sum_{i=1}^{k-1}(w_i \cdot d_{j,i})$. We show that Nash equilibrium exists for increasing or decreasing weight vectors $\boldsymbol{w}$, when the underlying space (Ω, Δ) satisfies certain conditions, which are different for increasing and decreasing weight vectors. For a particular version of the decreasing weight vectors, namely $(1, 1, 0, \ldots, 0)$, we show that: (a) it is not potential even on a continuous one dimensional circular space; (b) in general symmetric spaces Nash equilibrium may not exist, and (c) it is NP-complete to decide if a Nash equilibrium exists in general symmetric spaces. Combining with the previous NP-completeness result, we see that either a complicated space (asymmetric space) or a slightly complicated farness measure ($(1, 1, 0, \ldots, 0)$ instead of $(1, 0 \ldots, 0)$ or $(0, 1, 0, \ldots, 0)$) would make the determination of Nash equilibrium difficult.

We also examine the hardness of computing best responses in spaces with compact representations such as a hypercube. We show that for one class of isolation games including the nearest-neighbor game as the special case it is NP-complete to compute best responses, while for another class of isolation games, the computation can be done in polynomial time.

The rest of the paper is organized as follows. Section 2 covers the basic definitions and notation. Section 3 presents the results for nearest-neighbor and total-distance isolation games. Section 4 presents results for other general classes of isolation games. Section 5 examines the hardness of computing best responses in isolation games. We conclude the paper in Section 6. The full version of the paper with complete proofs can be found in [11].

2 Notation

We use (Ω, Δ) to denote the underlying space, where we assume $\Delta(x, x) = 0$ for all $x \in \Omega$, $\Delta(x, y) > 0$ for all $x, y \in \Omega$ and $x \ne y$, and that (Ω, Δ) is bounded — there exists a real value B such that $\varDelta(x, y) \leq B$ for every $x, y \in \varOmega$. In general, $(\varOmega, \varDelta)$ may not be symmetric or satisfy the triangle inequality. We always assume that there are k players in an isolation game and each player's strategy set is the entire $\varOmega$. A *configuration* of an isolation game is a vector $(p_1, p_2, \ldots, p_k)$, where $p_i \in \varOmega$ specifies the position of player i. The utility function of player i is a $(k-1)$-place function $f_i(\ldots, \varDelta(p_i, p_{i-1}), \varDelta(p_i, p_{i+1}), \ldots)$. For convenience, we use $ut_i(c)$ to denote the utility of player i in configuration c.

We consider several classes of weight vectors in the uniform, anonymous isolation game. In particular, the *nearest-neighbor* and *total-distance* isolation games have the weight vectors $(1, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$, respectively; the *single-selection* game has vectors that have exactly one nonzero entry; the *monotonically-increasing* (or *decreasing*) games have vectors whose entries are monotonically increasing (or decreasing).

A *better response* of a player i in a configuration $c = (p_1, \ldots, p_k)$ is a new position $p'_i \neq p_i$ such that the utility of player i in configuration c' by replacing p_i with p'_i in c is larger than her utility in c. In this case, we say that c' is the *result of a better-response move* of player i in configuration c. A *best response* of a player i in a configuration $c = (p_1, \ldots, p_k)$ is a new position $p'_i \neq p_i$ that maximizes the utility of player i while player j remains at the position p_j for all $j \neq i$. In this case, we say that c' is the *result of a best-response move* of player i in configuration c.

A (pure) *Nash equilibrium* of an isolation game is a configuration in which no player has any better response in the configuration. An isolation game is *better-response potential* (or *best-response potential*) if there is a function F from the set of all configurations to a totally ordered set such that $F(c) < F(c')$ for any two configurations c and c' where c' is the result of a better-response move (or a best-response move) of some player at configuration c. We call F a *potential function*. Note that a better-response potential game is also a best-response potential game, but a best-response potential game may not be a better-response potential game. If $\varOmega$ is finite, it is easy to see that any better-response or best-response potential game has at least one Nash equilibrium. Henceforth, we use the shorthand "potential games" to refer to better-response potential games.

3 Nearest-Neighbor and Total-Distance Isolation Games

In this section, we focus on the isolation games with weight vectors $(1, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$. We show that both are potential games when $\varOmega$ is symmetric, but when $\varOmega$ is asymmetric and finite, it is NP-complete to decide whether those games have Nash equilibria.

Theorem 1. *The symmetric nearest-neighbor and total-distance isolation games are potential games.*

The following lemma shows that the asymmetric isolation game may not have any Nash equilibrium for any nonzero weight vector. Thus, it also implies that asymmetric nearest-neighbor and total-distance isolation games may not have Nash equilibria.

Lemma 1. *Consider an asymmetric space $\varOmega = \{v_1, v_2, \ldots, v_{\ell+1}\}$ with the distance function given by the following matrix with $t \geq \ell + 1$. Suppose that for every player i*

her weight vector $\boldsymbol{w}_i$ has at least one nonzero entry. Then, for any $2 \le k \le \ell$, there is no Nash equilibrium in the k-player isolation game.

$$\left(\begin{array}{c|cccccc}
\Delta & v_1 & v_2 & v_3 & \ldots & v_\ell & v_{\ell+1} \\
\hline
v_1 & 0 & t-1 & t-2 & \ldots & t-\ell+1 & t-\ell \\
v_2 & t-\ell & 0 & t-1 & \ldots & t-\ell+2 & t-\ell+1 \\
\vdots & \vdots & & \ddots & & \vdots & \\
v_\ell & t-2 & t-3 & t-4 & \ldots & 0 & t-1 \\
v_{\ell+1} & t-1 & t-2 & t-3 & \ldots & t-\ell & 0
\end{array}\right)$$

Theorem 2. *It is NP-complete to decide whether a finite, asymmetric nearest-neighbor or total-distance isolation game has a Nash equilibrium.*

Proof. We first prove the case of nearest-neighbor isolation game.

Suppose that the size of Ω is n. Then the distance function Δ has n^2 entries. The decision problem is clearly in *NP*. The NP-hardness can be proved by a reduction from the Set Packing problem. An instance of the Set Packing problem includes a set $I = \{e_1, e_2, \ldots, e_m\}$ of m elements, a set $S = \{S_1, \ldots, S_n\}$ of n subsets of I, and a positive integer k. The decision problem is to decide whether there are k disjoint subsets in S. We now give the reduction.

The space Ω has $n+k+1$ points, divided into a left set $L = \{v_1, v_2, \ldots, v_n\}$ and a right set $R = \{u_1, u_2, \ldots, u_{k+1}\}$. For any two different points $v_i, v_j \in L$, $\Delta(v_i, v_j) = 2n$ if $S_i \cap S_j = \emptyset$, and $\Delta(v_i, v_j) = 1/2$ otherwise. The distance function on R is given by the distance matrix in Lemma 1 with $\ell = k$ and $t = k+1$. For any $v \in L$ and $u \in R$, $\Delta(v, u) = \Delta(u, v) = 2n$. Finally, the isolation game has $k+1$ players.

We now show that there exists a Nash equilibrium for the nearest-neighbor isolation game on Ω iff there are k disjoint subsets in the Set Packing instance.

First, suppose that there is a solution to the Set Packing instance. Without loss of generality, assume that the k disjoint subsets are $S_1, S_2, \ldots, S_k$. Then we claim that configuration $c = (v_1, v_2, \ldots, v_k, u_1)$ is a Nash equilibrium. In this configuration, it is easy to verify that every player's utility is $2n$, the largest possible pairwise distance. Therefore, c is a Nash equilibrium.

Conversely, suppose that there is a Nash equilibrium in the nearest-neighbor isolation game. Consider the set R. If there is a Nash equilibrium c, then the number of players positioned in R is either $k+1$ or at most 1 because of Lemma 1. If there are $k+1$ players in R, then every player has utility 1, and thus every one of them would want to move to points in L to obtain a utility of $2n$. Therefore, there cannot be $k+1$ players positioned in R, which means that there are at least k players positioned in L.

Without loss of generality, assume that these k players occupy points $v_1, v_2, \ldots, v_k$ (which may have duplicates). We claim that subsets $S_1, S_2, \ldots, S_k$ form a solution to the Set Packing problem. Suppose, for a contradiction, that this is not true, which means there exist S_i and S_j among these k subsets that intersect with each other. By our construction, we have $\Delta(v_i, v_j) = 0$ or $1/2$. In this case, players at point v_i and v_j would want to move to some free points in R, since that will give them utilities of at least 1. This contradicts the assumption that c is a Nash equilibrium. Therefore, we found a solution for the Set Packing problem given a Nash equilibrium c of the nearest-neighbor isolation game.

The proof for the case of total-distance isolation game is essentially the same, with only changes in players' utility values. □

4 Isolation Games with Other Weight Vectors

In this section, we study several general classes of isolation games. We consider symmetric space (Ω, Δ) in this section.

4.1 Single-Selection Isolation Games

Theorem 3. *A Nash equilibrium always exists in any single-selection symmetric game.*

Although Nash equilibria always exist in the single-selection isolation games, the following lemma shows that they are not potential games.

Lemma 2. *Let* $\Omega = \{A, B, C, D, E, F\}$ *contain six points on a one-dimensional circular space with* $\Delta(A, B) = 15$, $\Delta(B, C) = 11$, $\Delta(C, D) = 14$, $\Delta(D, E) = 16$, $\Delta(E, F) = 13$, *and* $\Delta(F, A) = 12$. *The five-player single-selection game with the weight vector* $(0, 1, 0, 0)$ *on* Ω *is not potential.*

Proof. Let the five players stand at A, B, C, D, and E respectively in the initial configuration. Their better response dynamics can iterate forever as shown in Figure 1. Hence this game is not a potential game. □

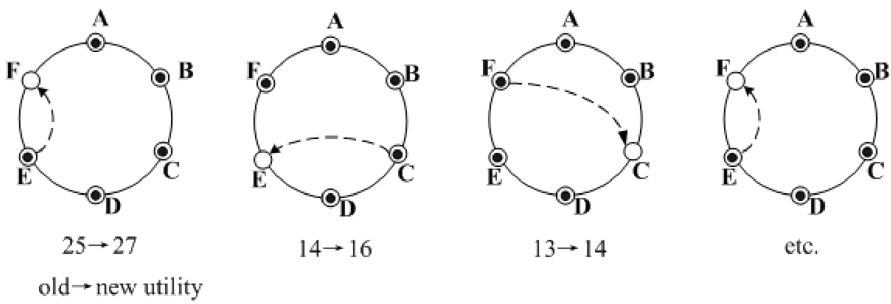

Fig. 1. An example of a better-response sequence that loops forever for a five-player isolation game with weight vector $(0, 1, 0, 0)$ in a one dimensional circular space with six points

Surprisingly, the following theorem complements the previous lemma.

Theorem 4. *If* Ω *is finite, then for any single-selection game on* Ω *and any starting configuration* c*, there is a better-response sequence in the game that leads to a Nash equilibrium.*

Proof. Suppose that the nonzero weight entry is the m^{th} entry in the k-player single-selection isolation game with $m > 1$ (the case of $m = 1$ is already covered in nearest-neighbor isolation game). For any configuration $c = (p_1, \ldots, p_k)$, the utility of player i is the distance between player i and her m^{th} nearest neighbor. Let vector $\boldsymbol{u}(c) = (u_1, u_2, \ldots, u_k)$ be the vector of the utility values of all players in c sorted in nondecreasing order, i.e., $u_1 \leq u_2 \leq \ldots \leq u_k$. We claim that for any configuration c, if c is not a Nash equilibrium, there must exist a finite sequence of configurations $c = c_0, c_1, c_2, \ldots, c_t = c'$, such that c_{i+1} is the result of a better-response move of some player in c_i for $i = 0, 1, \ldots, t-1$ and $\boldsymbol{u}(c) < \boldsymbol{u}(c')$ in lexicographic order.

We now prove this claim. Since the starting configuration $c_0 = c$ is not a Nash equilibrium, there exists a player i that can make a better response move to position p, resulting in configuration c_1. We have $ut_i(c_0) < ut_i(c_1)$. Let S_i be the set of player i's $m-1$ nearest neighbors in configuration c_1. We now repeat the following steps to find configurations $c_2, \ldots, c_t$. When in configuration c_j, we select a player a_j such that $ut_{a_j}(c_j) < ut_i(c_1)$ and move a_j to position p, the same position where player i locates. This gives configuration c_{j+1}. This is certainly a better-response move for a_j because $ut_{a_j}(c_{j+1}) = ut_i(c_{j+1}) = ut_i(c_1) > ut_{a_j}(c_j)$, where the second equality holds because we only move the $m-1$ nearest neighbors of player i in c_1 to the same position as i, so it does not affect the distance from i to her m^{th} nearest neighbor. The repeating step ends when there is no more such player a_j in configuration c_j, in which case $c_j = c_t = c'$.

We now show that $\boldsymbol{u}(c) < \boldsymbol{u}(c')$ in lexicographic order. We first consider any player $j \notin S_i$, either her utility does not change ($ut_j(c) = ut_j(c')$), or her utility change must be due to the changes of her distances to player i and players $a_1, a_2, \ldots, a_{t-1}$, who have moved to position p. Suppose that player j is at position q. Then $\Delta(p, q) \geq ut_i(c_1)$ because $j \notin S_i$. This means that if j's utility changes, her new utility $ut_j(c')$ must be at least $\Delta(p, q) \geq ut_i(c_1)$. For a player $j \in S_i$, if she is one of $\{a_1, \ldots, a_{t-1}\}$, then her new utility $ut_j(c') = ut_i(c') = ut_i(c_1)$; if she is not one of $\{a_1, \ldots, a_{t-1}\}$, then by definition $ut_j(c') \geq ut_i(c_1)$. Therefore, comparing the utilities of every player in c and c', we know that either her utility does not change, or her new utility is at least $ut_i(c') = ut_i(c_1) > ut_i(c)$, and at least player i herself strictly increases her utility from $ut_i(c)$ to $ut_i(c')$. With this result, it is straightforward to verify that $\boldsymbol{u}(c) < \boldsymbol{u}(c')$. Thus, our claim holds.

We may call the better-response sequence found in the above claim an epoch. We can apply the above claim to concatenate new epochs such that at the end of each epoch the vector $\boldsymbol{u}$ strictly increases in lexicographic order. Since the space Ω is finite, the vector $\boldsymbol{u}$ has an upper bound. Therefore, after a finite number of epochs, we must be able to find a Nash equilibrium, and all these epochs concatenated together form the better-response sequence that leads to the Nash equilibrium. This is clearly true when starting from any initial configuration. □

4.2 Monotonically-Increasing Games

For monotonically-increasing games, we provide the following general condition that guarantees the existence of a Nash equilibrium. We say that a pair of points $u, v \in \Omega$ is a pair of *polar points* if for any point $w \in \Omega$, the inequality $\Delta(u, w) + \Delta(w, v) \leq$

$\Delta(u, v)$ holds. Spaces with polar points include one-dimensional circular space, two-dimensional sphere, n-dimensional grid with L_1 norm as its distance function, etc.

Theorem 5. *If Ω has a pair of polar points, then any monotonically-increasing isolation game on Ω has a Nash equilibrium.*

4.3 Monotonically-Decreasing Games

Monotonically-decreasing games are more difficult to analyze than the previous variants of isolation games, and general results are not yet available. In this section, we first present a positive result for monotonically-decreasing games on a continuous one-dimensional circular space. We then present some hardness result for a simple type of weight vectors in general symmetric spaces.

The following theorem is a general result with monotonically-decreasing games as its special cases.

Theorem 6. *In a continuous one-dimensional circular space Ω, the isolation game on Ω with weight vector $\boldsymbol{w} = (w_1, w_2, \ldots, w_{k-1})$ always has a Nash equilibrium if $\sum_{t=1}^{k-1}(-1)^t w_t \leq 0$.*

A monotonically-decreasing isolation game with weight vector $\boldsymbol{w} = (w_1, w_2, \ldots, w_{k-1})$ automatically satisfies the condition $\sum_{t=1}^{k-1}(-1)^t w_t \leq 0$. Hence we have the following corollary.

Corollary 1. *In a continuous one-dimensional circular space Ω, any monotonically-decreasing isolation game on Ω has a Nash equilibrium.*

We now consider a simple class of monotonically-decreasing games with weight vector $\boldsymbol{w} = (1, 1, 0, \ldots, 0)$ and characterize the Nash equilibria of the isolation game in a continuous one-dimensional circular space Ω. Although the game has a Nash equilibrium in a continuous one-dimensional circular space according to the above corollary, it is not potential, as shown by the following lemma.

Lemma 3. *Consider $\Omega = \{A, B, C, D, E, F\}$ that contains six points in a one-dimensional circular space with $\Delta(A, B) = 13$, $\Delta(B, C) = 5$, $\Delta(C, D) = 10$, $\Delta(D, E) = 10$, $\Delta(E, F) = 11$, and $\Delta(F, A) = 8$. The five-player monotonically-decreasing game on Ω with weight vector $\boldsymbol{w} = (1, 1, 0, 0)$ is not best-response potential (so not better-response potential either). This implies that the game on a continuous one-dimensional circular space is not better-response potential.*

If we extend from the one-dimensional circular space to a general symmetric space, there may be no Nash equilibrium for isolation games with weight vector $\boldsymbol{w} = (1, 1, 0, \ldots, 0)$ at all, as shown in the following lemma.

Lemma 4. *There is no Nash equilibrium for the four-player isolation game with weight vector $\boldsymbol{w} = (1, 1, 0)$ in the space with five points $\{A, B, C, D, E\}$ and the following distance matrix, where $N > 21$ (note that this distance function also satisfies triangle inequality).*

$$\begin{pmatrix} \Delta & A & B & C & D & E \\ \hline A & 0 & N-6 & N-11 & N-1 & N-6 \\ B & N-6 & 0 & N-8 & N-10 & N-1 \\ C & N-11 & N-8 & 0 & N-1 & N-6 \\ D & N-1 & N-10 & N-1 & 0 & N-10 \\ E & N-6 & N-1 & N-6 & N-10 & 0 \end{pmatrix}$$

Using the above lemma as a basis, we further show that it is NP-complete to decide whether an isolation game with weight vector $(1, 1, 0, \ldots, 0)$ on a general symmetric space has a Nash equilibrium. The proof is by a reduction from 3-Dimensional Matching problem.

Theorem 7. *In a finite symmetric space (Ω, Δ), it is NP-complete to decide the existence of Nash equilibrium for isolation game with weight vector $\boldsymbol{w} = (1, 1, 0, \ldots, 0)$.*

5 Computation of Best Responses in High Dimensional Spaces

We now turn to the problem of computing the best response of a player in a configuration. A brute-force search on all points in the space can be done in $O(k \log k \sqrt{D})$, where D is the size of the distance matrix. This is fine if the distance matrix is explicitly given as input. However, it could become exponential if the space has a compact representation, such as an n-dimensional grid with the L_1 norm as the distance function. In this section, we present results on an n-dimensional hypercube $\{0, 1\}^n$ with the Hamming distance, a special case of n-dimensional grids with the L_1 norm.

Theorem 8. *In a $2n$-dimensional hypercube $\{0, 1\}^{2n}$, it is NP-complete to decide whether a player could move to a point so that her utility is at least $n - 1$ in the k-player nearest-neighbor isolation game with k bounded by $poly(n)$.*

The above theorem leads to the following hardness result in computing best responses for a general class of isolations games, with nearest-neighbor game as a special case.

Corollary 2. *It is NP-hard to compute a best response for an isolation game in the space $\{0, 1\}^{2n}$ with weight vector $\boldsymbol{w} = (\underbrace{*, \ldots, *, 1}_{c}, 0, \ldots, 0)$ where c is a constant and $*$ is either 0 or 1.*

Contrasting to the above corollary, if the weight vector has only nonzero entries towards the end of the vector, it is easy to compute the best response, as shown in the following theorem.

Theorem 9. *A best response for a k-player isolation game in the space $\{0, 1\}^n$ with $\boldsymbol{w} = (0, \ldots, 0, \underbrace{1, *, \ldots, *}_{c})$ can be computed in polynomial time where c is a constant, k is bounded by $poly(n)$ and $*$ is either 0 or 1.*

6 Final Remarks

The isolation game is very simple by its definition. However, as shown in this paper, the behaviors of its Nash equilibria and best response dynamics are quite rich and complex. This paper presents the first set of results on the isolation game and lays the ground work for the understanding of the impact of the farness measures and the underlying space to some basic game-theoretic questions about the isolation game. It remains an open question to fully characterize the isolation game. In particular, we would like to understand for what weight vectors, the isolation game on simple spaces, such as d-dimensional grids, hypercubes, and torus grid graphs, has potential functions, has Nash equilibria, or has converging best (better) response sequences. What is the impact of distance functions, such as L1-norm or L2-norm to these questions? We would like to know whether it is NP-hard to determine if Nash equilibria exist in these special spaces when the input is the weight vector. What can we say about other continuous spaces such as squares, cubes, balls, and spheres? For example, is there a sequence of better response dynamics that converge to a Nash equilibrium in the isolation game on the sphere with $\boldsymbol{w} = (1, 1, 1, 0, \ldots, 0)$? What can we say about approximate Nash equilibria?

More concretely, In Lemma 2 we show an example in which a single-selection game with weight vector $(0, 1, 0, \ldots, 0)$ is not better-response potential in one dimensional circular space. However, we verify that the game is best-response potential. This phenomenon of being best-response potential but not better-response potential is rarely seen in other type of games. Moreover, our experiments lead us to conjecture that all games on continuous one dimensional circular space with weight vector $(0, 1, 0, \ldots, 0)$ is best-response potential. So far, we are only able to prove that in such games starting from any configuration there is always an acyclic sequence of best responses that either converge to a Nash equilibrium or is infinitely long. If the conjecture is true, we will find a large class of games that are best-response potential but not better-response potential (latter is implied by Lemma 2 for the continuous one dimensional space), an interesting phenomenon not known in other common games.

Another line of research is to understand the connection between the isolation game and the Voronoi game.

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