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RESEARCH PAPER

A note on the definition of fractional derivatives applied in rheology

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Abstract It is known that there exist obivious differences between the two most commonly used definitions of fractional derivatives—Riemann–Liouville (R–L) definition and Caputo definition. The multiple definitions of fractional derivatives in fractional calculus have hindered the application of fractional calculus in rheology. In this paper, we clarify that the R–L definition and Caputo definition are both rheologically imperfect with the help of mechanical analogues of the fractional element model (Scott–Blair model). We also clarify that to make them perfect rheologically, the lower terminals of both definitions should be put to $-\infty$. We further prove that the R–L definition with lower terminal $a \to -\infty$ and the Caputo definition with lower terminal $a \to -\infty$ are equivalent in the differentiation of functions that are smooth enough and functions that have finite number of singular points. Thus we can define the fractional derivatives in rheology as the R–L derivatives with lower terminal $a \to -\infty$ (or, equivalently, the Caputo derivatives with lower terminal $a \to -\infty$) not only for steady-state processes, but also for transient processes. Based on the above definition, the problems of composition rules of fractional operators and the initial conditions for fractional differential equations are discussed, respectively. As an example we study a fractional oscillator with Scott–Blair model and give an exact solution of this equation under given initial conditions.

Keywords Fractional derivatives · Rheology · Riemann–Liouville definition · Caputo definition

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1 Introduction

During the last two decades fractional calculus has been increasingly applied to physics, especially to rheology [1–4]. In particular, fractional calculus has played an important role in the constitutive modeling of viscoelastic materials. Some characteristics of complex viscoelastic materials can be described by fractional derivatives. The fractional element model (Scott–Blair model), which is the most basic of all the fractional-order models of viscoelastic materials, was introduced by Scott–Blair, and its constitutive equation can be expressed as [5]

$$\sigma(t) = E\lambda^{\alpha} \frac{\mathrm{d}^{\alpha}\varepsilon(t)}{\mathrm{d}t^{\alpha}}, \qquad 0 < \alpha < 1, \tag{1}$$

where E, λ, α are material-dependent constants and $\mathrm{d}^{\alpha}\varepsilon(t)/\mathrm{d}t^{\alpha}$ denotes the time-fractional derivative of strain. By replacing the springs and dashpots of the classical viscoelastic models by the Scott–Blair elements, several fractional models, including the fractional Maxwell model, fractional Voigt model and fractional Kelvin model, have been proposed [1,6]. The experimental results obtained by Hernández–Jiménez et al. [7] show that the behavior of some polymers shows good agreements with that of the fractional Maxwell model. Experimental investigations done by Meral et al. [8] also show that the fractional Voigt model can better simulate the surface wave response of some soft tissue-like materials. These are successful applications of fractional derivatives in rheology.

The applications of fractional calculus in physics is dependent on the definitions of fractional derivatives. The most famous definition is the Riemann–Liouville (R–L) derivative, which can be expressed as [1, 9]

$${}^{\mathrm{R}}_{a}\mathrm{D}^{\alpha}_{t} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}} \mathrm{d}\tau,$$
$$n - 1 \leq \alpha < n, \tag{2}$$

where α is the order of the derivative, a is the lower terminal, n is a nonnegative integer such that $n-1 \leq \alpha < n$ and the superscript "R" represents the R–L fractional derivative. Another most commonly used definition may be the Caputo fractional derivative, which was introduced in the late sixties of the twentieth century [10]. It can be expressed as [9,10]

$$ {}_a^{\mathrm{C}}\mathrm{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha+1-n}}, \quad n-1 < \alpha \leq n, \quad (3) $$

where n is a nonnegative integer such that $n-1 < \alpha \leq n$ and the superscript "C" is used to distinguish the Caputo fractional derivative from the R–L fractional derivative.

Unfortunately, the R–L definition and Caputo definition are not equivalent. One difference between them is that the Caputo derivative of a constant is zero, whereas in the case of a finite value of the lower terminal a the R–L fractional derivative of a constant is not equal to zero, but [9]

$$ {}_a^{\mathrm{R}}\mathrm{D}_t^{\alpha} C = \frac{C(t-a)^{-\alpha}}{\Gamma(1-\alpha)}, \quad \alpha > 0. \quad (4) $$

This means that for the Scott–Blair model, the two definitions will give different stress responses while the strain is equal to a constant. The fact led, for example, Ochmann and Makarov to use the R–L derivatives with the lower terminal set to $-\infty$, because formula (4) gives zero if $a \to -\infty$ [11]. Podlubny pointed out that if $a \to -\infty$ is put in both definitions and reasonable behavior is required for $f(t)$ and its derivatives as $t \to -\infty$, Eqs. (2) and (3) will give the same results [9]. It shows that for studying steady-state dynamical processes the R–L definition and the Caputo definition must give the same results. Podlubny also concluded that the transient effects can not be studied if the lower terminal (i.e. the starting time of the process) is set to $-\infty$ [9]. One of the purposes of this paper is to analyze the validity of the two definitions in the rheological sense and solve the contradictions between the R–L definition and the Caputo definition for transient problems in rheology.

Another significant difference between the two definitions is closely related to the applications of fractional calculus. The solution of a linear fractional differential equation defined in terms of the R–L derivatives will require fractional-order initial conditions which can cause troubles with their physical interpretation, while the solution of a linear fractional differential equation defined in terms of the Caputo derivatives will require regular initial conditions that take on the same form as that for integer-order differential equations [9]. As a result, the Caputo derivatives are more popular with the physicists. Another purpose of this paper is to re-examine the problems of initial conditions of fractional differential equations based on our definition of fractional derivatives.

Both the R–L definition and the Caputo definition are reasonable mathematically, whereas at most only one definition is allowed physically. To analyze the validity of the two definitions in the rheological sense, the mechanical analogues of Scott–Blair model are used. They were developed during the last twenty years. Heymans and Bauwens [12] and Heymans [13] derived the constitute equation of the spring-dashpot fractal network in Fig. 1 using complex modulus and found that the stress is proportional to the 1/2-order derivative of strain. Hu and Zhu [14] also derived the constitutive equation of the tree model with 1/2-order derivative using Heaviside operational calculus, and proved that using the models shown in Fig. 1, springs and dashpots, one can get a multiple spring-dashpot fractal network which describes fractional element models with an arbitrary order derivative between 0 and 1. Schiessel and Blumen [15] presented a ladde–like structure consisting of springs along one of the struts and dashpots along the rungs of the ladder, and proved that the mechanical construction is a fractional element with an arbitrary order derivative between 0 and 1. In the following analysis, the tree model shown in Fig. 1 is used, because it is much simpler than Schiessel's ladder model.

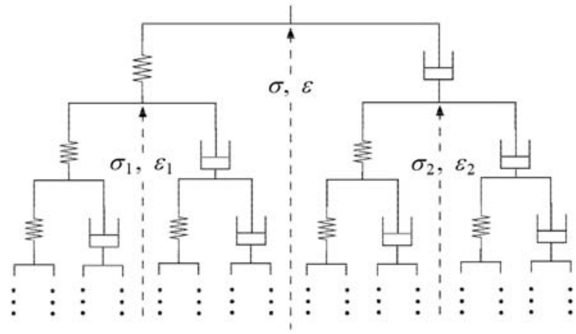

Fig. 1 A self-similar tree model of fractional element

The remainder of this paper is organized as follows. In Sect. 2, the stress responses of the 1/2-order Scott–Blair model to a constant strain and to a strain jump are studied physically using the tree model. The results based on R–L definition and Caputo definition are compared. It shows that the R–L definition and the Caputo definition are both defective rheologically and some revisions are needed. In Sect. 3, we show that to make the two definitions more reasonable rheologically, the lower terminals should be put to $-\infty$ in them. Then the fractional derivatives of smooth functions and functions with finite number of singular points are considered, respectively. We prove that for the fractional differentiation of these functions the R–L definition (lower terminal $a \to -\infty$) and Caputo definition (lower terminal $a \to -\infty$) must give the same results, that is, the two definitions with lower terminals $a \to -\infty$ are equivalent not only in the study of steady-state processes but also in the study of transient processes. Thus we define the fractional derivatives in rheology as the R–L derivatives with lower terminals $a \to -\infty$ (or, equivalently, the Caputo derivatives with lower terminals $a \to -\infty$). In Sect. 4, the composition rules of fractional operators that are of great importance to the application of fractional calculus are studied based on the definition above. In Sect. 5, the problem about the initial conditions is discussed for fractional differential equations defined in terms of the above fractional derivatives with lower terminals $a \to -\infty$. In Sect. 6, we study a linear fractional oscillator with Scott–Blair model and give an analytical solution of the equation under given conditions. Conclusions are finally drawn in Sect. 7.

2 Analysis of the 1/2-order tree model

Through analysis of the stress response of the 1/2-order tree model in Fig. 1 to a constant strain, we will show that there exist obvious deficiencies in R–L definition when used in rheology. We denote the elasticity modulus of the springs and viscosity of the dashpots by E and η, respectively. Springs in the tree model in Fig. 1 obey Hooke's law $\sigma_s = E\varepsilon_s$ and the dashpots obey Newton's law $\sigma_d = \eta \mathrm{d}\varepsilon_d/\mathrm{d}t$. The constitutive equation of the model in Fig. 1 is expressed as [14]

$$\sigma(t) = E\lambda^{1/2}\frac{\mathrm{d}^{1/2}\varepsilon(t)}{\mathrm{d}t^{1/2}}, \tag{5}$$

where $\lambda = \eta/E$ is the relaxation time. When the strain of the model is identically equal to a constant $\varepsilon(t) \equiv \varepsilon_0$, the strain rate of each dashpot in the model is zero. Thus the stresses of the dashpots are equal to zero. There are infinitely many branches in the tree model between its upper and lower ends, while any branch that contains at least one dashpot can resist no stress at all. Therefore, the stresses of the whole model are applied to the leftmost branch of the model, the only branch that contains no dashpot. However, there are an infinite number of springs in series in this branch and the elasticity modulus of each spring in the branch is finite. As a result, the equivalent elasticity modulus of this branch is equal to zero. Then we obtain a zero stress of the model when the strain is a constant

$$\sigma = E\lambda^{1/2}\frac{\mathrm{d}^{1/2}\varepsilon_0}{\mathrm{d}t^{1/2}} = 0. \tag{6}$$

Thus the 1/2-order derivative of a constant should be equal to zero, in accordance with the result obtained from Caputo definition. We can reasonably conclude that the R–L definition has obvious rheological deficiencies.

Further study shows that the Caputo definition has also obvious deficiencies when applied to the Scott–Blair model. To make this point clear, we will study the stress response of the 1/2-order tree model to a strain jump

$$\varepsilon(t) = \varepsilon_0\theta(t) = \begin{cases} 0, & t < 0, \\ \varepsilon_0, & t \geq 0, \end{cases} \tag{7}$$

where $\theta(t)$ is a unit step function. The R–L derivative of $\varepsilon(t)$ is

$$\begin{aligned}\sigma(t) &= E\lambda^{\alpha}\,{}_0^{\mathrm{R}}\mathrm{D}_t^{\alpha}\varepsilon(t) \\ &= \frac{E\lambda^{\alpha}}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t \frac{\varepsilon_0}{(t-\tau)^{\alpha+1-n}}\mathrm{d}\tau \\ &= \frac{E\varepsilon_0(t/\lambda)^{-\alpha}}{\Gamma(1-\alpha)},\end{aligned} \tag{8}$$

while the Caputo derivative of $\varepsilon(t)$ is

$$\begin{aligned}\sigma(t) &= E\lambda^{\alpha}\,{}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha}C \\ &= \frac{E\lambda^{\alpha}}{\Gamma(n-\alpha)}\int_0^t \frac{\varepsilon_0^{(n)}}{(t-\tau)^{\alpha+1-n}}\mathrm{d}\tau = 0.\end{aligned} \tag{9}$$

Let us investigate the internal dynamical behavior of the tree model in Fig. 1. After the strain jump, the dashpots in the model behave in a rigid manner as the strain rate is infinite at this instant. Therefore, the stress of the model goes to infinite when $t \to 0+$. Then the stress set up in the model will gradually relax and fade away as the pistons of the dashpots overcome the resistance of the damping fluid. When $t \to \infty$, the stress of the model will go to zero as the case of a constant strain. Qualitatively, the behavior of the tree model shows good agreement with the result obtained from R–L definition. To calculate the stress response of the model in Fig. 1 quantitatively, we use the L_- Laplace transform given by Lundberg et al. [16]

$$L_-[f(t)] = F(s) = \int_{0_-}^{+\infty} f(t)\mathrm{e}^{-st}\mathrm{d}t, \tag{10}$$

where the domain of integral fully includes the origin and any singularities occurring at that time. We denote the L_- Laplace transform of the stress of the tree model by $L[\sigma]$, and the strain by $L[\varepsilon]$. As the strain of the model is equal to zero before $t = 0$, we have

$$\sigma^{(k)}(0_-) = 0, \quad \varepsilon^{(k)}(0_-) = 0, \quad k = 0, 1, 2, \cdots. \tag{11}$$

Thus we can reasonably assume that the relationship between $L[\sigma]$ and $L[\varepsilon]$ is

$$L[\sigma] = X(s)L[\varepsilon], \tag{12}$$

where $L[\varepsilon] = L_-[\varepsilon_0\theta(t)] = \varepsilon_0/s$. According to the self-similar character of the model in Fig. 1, we also get that

$$L[\sigma_1] = XL[\varepsilon_1], \quad L[\sigma_2] = XL[\varepsilon_2]. \tag{13}$$

Then the tree model can be reduced to the second network shown in Fig. 2.

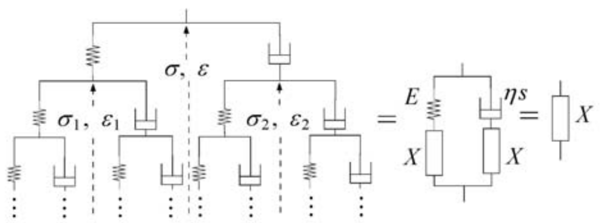

Fig. 2 The reduced model of the fractal tree model

From Fig. 2, we can obtain an equation of X

$$X = \left(\frac{1}{E} + \frac{1}{X}\right)^{-1} + \left(\frac{1}{\eta s} + \frac{1}{X}\right)^{-1}. \tag{14}$$

From the equation above we obtain that

$$X = (E\eta s)^{1/2}. \tag{15}$$

Thus, we get

$$\begin{aligned}\sigma(t) &= L^{-1}[\sigma(s)] \\ &= L^{-1}\left[\varepsilon_0\left(\frac{E\eta}{s}\right)^{1/2}\right] \\ &= \varepsilon_0\left(\frac{E\eta}{t\lambda}\right)^{1/2} \\ &= \frac{E\varepsilon_0(\lambda/t)^{1/2}}{\Gamma(1/2)},\end{aligned} \tag{16}$$

which is in accordance with the result obtained from the R–L definition. This may be seen as the relaxation effect of the Scott–Blair model. In Fig. 3, we plot the stress-relaxation curve for 1/2-order Scott–Blair model. The time is non-dimensionalized as $\tau = t/\lambda$, the strain is non-dimensionalized as $\varepsilon^* = \varepsilon/\varepsilon_0 = \theta(t)$ and the stress is non-dimensionalized as

$$\sigma^*(t) = \sigma(t)/\sigma(\lambda) = (\lambda/t)^{1/2}. \tag{17}$$

We can see that the stress of the model decays with the increase of $(t - a)$, and the stress will decay to zero only when $(t - a) \to \infty$. Here the starting time is $a = 0$.

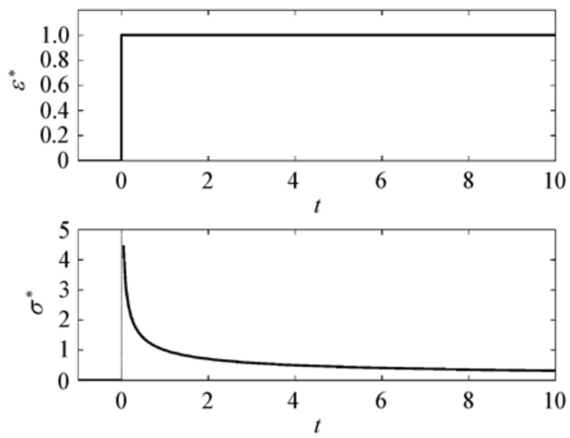

Fig. 3 The stress-relaxation curve for 1/2-order Scott–Blair model

3 Definition of fractional derivatives in rheology

According to the analysis in Sect. 2, the R–L definition has obvious deficiencies in the case of constant functions, while the Caputo definition has obvious deficiencies in the case of step functions. In this section, we attempt to revise the two definitions to make them more reasonable rheologically. From Eqs. (2) and (3) we can see that the R–L derivative and the Caputo derivative of $f(t)$ at the instant of t have nothing to do with the behaviors of $f(t)$ before the lower terminal a. Let us consider the derivatives of constant function $\varepsilon(t) \equiv \varepsilon_0$ and step function $\varepsilon(t) = \varepsilon_0\theta(t)$. We have obtained that ${}_0^{\mathrm{R}}\mathrm{D}_t^\alpha\varepsilon_0 = {}_0^{\mathrm{R}}\mathrm{D}_t^\alpha[\varepsilon_0\theta(t)]$ and ${}_0^{\mathrm{C}}\mathrm{D}_t^\alpha\varepsilon_0 = {}_0^{\mathrm{C}}\mathrm{D}_t^\alpha[\varepsilon_0\theta(t)]$ in Sect. 2. The differences between ε_0 and $\varepsilon_0\theta(t)$ before the starting time $a = 0$ have no influence on the results either in the case of R–L definition or in the case of Caputo definition. But our analysis of the tree model in Sect. 2 has showed that the stress responses of the Scott–Blair model are completely different from each other in the two cases, which means that the behavior of the functions before the starting time $a = 0$ must greatly influence the fractional derivatives. Thus we guess that the loss of the information before the lower terminals may be an important reason accounting for the deficiencies of the two definitions.

Now we will show that the R–L definition and the Caputo definition give the same results not only in the study of constant functions but also in the study of step functions when the lower terminals are put to $-\infty$. Consider the R–L derivative and the Caputo derivative of a step function $H_C(t)$

$$H_C(t) = C\theta(t - b) = \begin{cases} 0, & t < b, \\ C, & t \geq b. \end{cases} \tag{18}$$

For $n - 1 < \alpha < n$ and $t > b$ we get that, the R–L derivative of $H_C(t)$ with lower terminal $a \to -\infty$ is

$$\begin{aligned}{}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^\alpha H_C(t) &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_b^t \frac{C}{(t-\tau)^{\alpha+1-n}}\mathrm{d}\tau \\ &= \frac{C(t-b)^{-\alpha}}{\Gamma(1-\alpha)},\end{aligned} \tag{19}$$

while the Caputo derivative of $H_C(t)$ is

$$\begin{aligned}{}_{-\infty}^{\mathrm{C}}\mathrm{D}_t^\alpha H_C(t) &= \frac{C}{\Gamma(n-\alpha)} \int_{-\infty}^t \frac{\delta^{(n-1)}(\tau - b)}{(t-\tau)^{\alpha+1-n}}\mathrm{d}\tau \\ &= \frac{C(t-b)^{-\alpha}}{\Gamma(1-\alpha)} = {}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^\alpha H_C(t).\end{aligned} \tag{20}$$

This result indicates that the two definitions with lower terminals $a \to -\infty$ may be equivalent not only in the study of steady-state processes (i.e. functions that are smooth enough) but also in the study of the transient problems (i.e. functions with finite number of singular points). We will prove it strictly using the theory of generalized functions in this section. It should point out that the order of the derivative α is always taken as a non-integer number in this section. In the case of an integer order $\alpha = n$, we can easily prove that ${}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^n f(t) = {}_{-\infty}^{\mathrm{C}}\mathrm{D}_t^n f(t) = f^{(n)}(t)$.

3.1 Proof for functions that are smooth enough

Let us suppose that the function $f(t)$ is $(n-1)$-times continuously differentiable in the interval $(-\infty, T]$, $f^{(n)}(t)$ is integrable in $(-\infty, T]$ and the Caputo derivative ${}_{-\infty}^{\mathrm{C}}\mathrm{D}_t^\alpha f(t)$

$(0 \le m-1 < \alpha < m \le n)$ exists for $t < T$ (i.e. integral $\int_{-\infty}^{t} f^{(m)}(\tau)/(t-\tau)^{\alpha-m+1}\mathrm{d}\tau$ is convergent). Thus we have that

$$\lim_{\tau\to-\infty}\frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m}} = 0, \qquad t < T, \tag{21}$$

and we further obtain that

$$\lim_{\tau\to-\infty}\frac{f^{(j)}(\tau)}{(t-\tau)^{\alpha-j}} = 0, \quad j = 0,1,\cdots,m-1, \quad t < T. \tag{22}$$

According to Podlubny [9], the following holds

$$ {}_a^{\mathrm{R}}\mathrm{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)}\int_a^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\mathrm{d}\tau + \sum_{k=0}^{m-1}\frac{f^{(k)}(a)(t-a)^{-\alpha+k}}{\Gamma(k+1-\alpha)}. \tag{23}$$

Then with the help of Eq. (22), we can get the R–L derivative of $f(t)$ with lower terminal $a \to -\infty$

$$\begin{aligned} {}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{\alpha} f(t) &= \lim_{a\to-\infty}\left[\frac{1}{\Gamma(m-\alpha)}\int_a^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}}\mathrm{d}\tau + \sum_{j=0}^{m-1} f^{(j)}(a)\frac{(t-a)^{j-\alpha}}{\Gamma(-\alpha+j+1)}\right] \\ &= \lim_{a\to-\infty}\left[\frac{1}{\Gamma(m-\alpha)}\int_a^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}}\mathrm{d}\tau\right] \\ &= {}_{-\infty}^{\mathrm{C}}\mathrm{D}_t^{\alpha} f(t). \end{aligned} \tag{24}$$

This verifies the conclusion of Podlubny [9] that if the lower terminal is put to $-\infty$ in both definitions, they will give the same results, which shows that for the study of steady-state dynamical processes the R–L definition and the Caputo definition must give the same results.

3.2 Proof for functions with finite number of singular points

To simplify the proof, first let us consider a function which is expressed as

$$\tilde{f}(t) = f(t)\theta(t-a) = \begin{cases} 0, & t < a, \\ f(t), & a \le t \le T, \end{cases} \tag{25}$$

with a singular point at $t = a$. We suppose that the function $f(t)$ is $(n-1)$-times continuously differentiable in the interval $[a, T]$ and $f^{(n)}(t)$ is integrable in $[a, T]$.

Firstly, let us calculate the R–L derivative of $\tilde{f}(t)$. If $0 \le m-1 < \alpha < m \le n$, for $t < a$ we can easily obtain that

$$ {}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{\alpha}\tilde{f}(t) = 0, \tag{26}$$

and for $t > a$ we get that

$$\begin{aligned} {}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{\alpha}\tilde{f}(t) &= {}_a^{\mathrm{R}}\mathrm{D}_t^{\alpha} f(t) \\ &= \frac{1}{\Gamma(m-\alpha)}\int_a^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\mathrm{d}\tau \\ &\quad + \sum_{k=0}^{m-1}\frac{f^{(k)}(a)(t-a)^{-\alpha+k}}{\Gamma(k+1-\alpha)}. \end{aligned} \tag{27}$$

Next, we will consider the Caputo derivative of $\tilde{f}(t)$. If $0 \le m-1 < \alpha < m \le n$, we also obtain that

$$ {}_{-\infty}^{\mathrm{C}}\mathrm{D}_t^{\alpha}\tilde{f}(t) = 0 \tag{28}$$

for $t < a$ and for $t > a$ we get that

$$ {}_{-\infty}^{\mathrm{C}}\mathrm{D}_t^{\alpha}\tilde{f}(t) = \frac{1}{\Gamma(m-\alpha)}\int_{-\infty}^t \frac{\tilde{f}^{(m)}(\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha+1-m}}. \tag{29}$$

According to Kanwal, the generalized (in the sense of generalized functions) derivative of $\tilde{f}(t)$ is [17]

$$\begin{aligned} \tilde{f}^{(m)}(t) &= [f(t)\theta(t-a)]^{(m)} \\ &= f_C^{(m)}(t) + \sum_{k=0}^{m-1}\delta^{(m-k-1)}(t-a)f^{(k)}(a), \quad t > a, \end{aligned} \tag{30}$$

where $f_C^{(m)}(t)$ is differentiation of $\tilde{f}(t)$ in the classical sense, expressed as

$$f_C^{(m)}(t) = f^{(m)}(t)\theta(t-a) = \begin{cases} 0, & t < a, \\ f^{(m)}(t), & t \ge a. \end{cases} \tag{31}$$

Therefore we can get

$$\begin{aligned} {}_{-\infty}^{\mathrm{C}}\mathrm{D}_t^{\alpha}\tilde{f}(t) &= \frac{1}{\Gamma(m-\alpha)}\int_{-\infty}^t \frac{\tilde{f}^{(m)}(\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha+1-m}} \\ &= \frac{1}{\Gamma(m-\alpha)} \times \int_{-\infty}^t \frac{f_C^{(m)}(\tau) + \sum_{k=0}^{m-1}\delta^{(m-k-1)}(\tau-a)f^{(k)}(a)}{(t-\tau)^{\alpha+1-m}}\mathrm{d}\tau \\ &= \frac{1}{\Gamma(m-\alpha)}\int_a^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}\mathrm{d}\tau + \frac{1}{\Gamma(m-\alpha)}\int_a^t \sum_{k=0}^{m-1}\frac{f^{(k)}(a)\delta^{(m-k-1)}(\tau-a)}{(t-\tau)^{\alpha+1-m}}\mathrm{d}\tau \\ &= {}_a^{\mathrm{C}}\mathrm{D}_t^{\alpha} f(t) + \sum_{k=0}^{m-1}\frac{f^{(k)}(a)}{\Gamma(m-\alpha)} \times \int_a^t \delta^{(m-k-1)}(\tau-a)(t-\tau)^{m-\alpha-1}\mathrm{d}\tau \\ &= {}_a^{\mathrm{C}}\mathrm{D}_t^{\alpha} f(t) + \sum_{k=0}^{m-1}\frac{(-1)^{m-k-1}f^{(k)}(a)}{\Gamma(m-\alpha)}\left.\frac{\mathrm{d}^{m-k-1}(t-\tau)^{m-\alpha-1}}{\mathrm{d}\tau^{m-k-1}}\right|_{\tau=a} \\ &= {}_a^{\mathrm{C}}\mathrm{D}_t^{\alpha} f(t) + \sum_{k=0}^{m-1} f^{(k)}(a)\frac{(t-a)^{k-\alpha}}{\Gamma(k+1-\alpha)}. \end{aligned} \tag{32}$$

Equations (26)–(28) and (32) lead to

$$ {}_{-\infty}^{\mathrm{C}}\mathrm{D}_t^{\alpha}\tilde{f}(t) = {}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{\alpha}\tilde{f}(t) \tag{33}$$

for $t < T$. This shows that if the lower terminal is put to $-\infty$ in both definitions, they will give the same results in the study of the functions having the form of Eq. (25).

Generally, let us consider an arbitrary function with a singular point at $t = a$, expressed as

$$g(t) = \begin{cases} f_1(t), & t < a, \\ f_2(t), & a \leq t \leq T, \end{cases} \tag{34}$$

where $f_1(t)$, $f_2(t)$ and their derivatives have reasonable behaviors as required. We can find an analytical continuation of $f_1(t)$ as follows

$$F_1(t) = \begin{cases} f_1(t), & t < a, \\ g_1(t), & a \leq t \leq T, \end{cases} \tag{35}$$

where $g_1(t)$ is properly chosen such that $F_1(t)$ is $(n-1)$-times continuously differentiable in the interval $(-\infty, T]$, $F_1^{(n)}(t)$ is integrable in $(-\infty, T]$ and the α-order derivative of $F_1(t)$ exists for $t < T$. Thus the function $g(t)$ can be decomposed as

$$g(t) = F_1(t) + \tilde{g}(t), \tag{36}$$

where

$$\tilde{g}(t) = g(t) - F_1(t) = \begin{cases} 0, & t < a, \\ f_2(t) - g_1(t), & a \leq t \leq T. \end{cases} \tag{37}$$

Then for α $(0 \leq m-1 < \alpha < m \leq n)$ and $t < T$, we obtain

$$\begin{aligned} {}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{\alpha} g(t) &= {}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{\alpha} F_1(t) + {}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{\alpha} \tilde{g}(t) \\ &= {}_{-\infty}^{\mathrm{C}}\mathrm{D}_t^{\alpha} F_1(t) + {}_{-\infty}^{\mathrm{C}}\mathrm{D}_t^{\alpha} \tilde{g}(t) \\ &= {}_{-\infty}^{\mathrm{C}}\mathrm{D}_t^{\alpha} g(t). \end{aligned} \tag{38}$$

Finally, let us consider a function $f(t)$ with k singular points at $a_1, a_2, \cdots, a_{k-1}, a_k$ $(a_1 < a_2 < \cdots < a_{k-1} < a_k)$ and require reasonable behaviors of $f(t)$ and its derivatives in all the intervals $(-\infty, a_1)$, (a_i, a_{i+1}) $(i = 1, 2, \cdots, k-1)$ and (a_k, T). It can be proved that $f(t)$ can be written as the summation of k functions

$$f(t) = \sum_{i=1}^{k} f_i(t), \tag{39}$$

where every function $f_i(t)$ has only one singular point. Then according to Eq. (38), we have

$$\begin{aligned} {}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{\alpha} f(t) &= \sum_{i=1}^{k} {}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{\alpha} f_i(t) \\ &= \sum_{i=1}^{k} {}_{-\infty}^{\mathrm{C}}\mathrm{D}_t^{\alpha} f_i(t) \\ &= {}_{-\infty}^{\mathrm{C}}\mathrm{D}_t^{\alpha} F(t), \end{aligned} \tag{40}$$

showing that the R–L definition and the Caputo definition are equivalent for the study of the functions with finite number of singular points.

Now we can define the α-order fractional derivative as

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}} f(t) := {}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{\alpha} f(t) = {}_{-\infty}^{\mathrm{C}}\mathrm{D}_t^{\alpha} f(t). \tag{41}$$

The original R–L definition and Caputo definition with lower terminals a can be regarded as special cases of definition equation (41). For $n-1 < \alpha < n$ and $t > a$, we define that

$$\begin{aligned} {}_{a}^{\mathrm{R}}\mathrm{D}_t^{\alpha} f(t) &:= \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{-\infty}^{t} \frac{\tilde{f}(\tau)}{(t-\tau)^{\alpha+1-n}} \mathrm{d}\tau \\ &= {}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{\alpha} \tilde{f}(t) = \frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}} \tilde{f}(t), \end{aligned} \tag{42}$$

where

$$\tilde{f}(t) = \begin{cases} 0, & t < a, \\ f(t), & t \geq a, \end{cases} \tag{43}$$

and define that

$$\begin{aligned} {}_{a}^{\mathrm{C}}\mathrm{D}_t^{\alpha} f(t) &:= \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^{t} \frac{\hat{f}^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} \mathrm{d}\tau \\ &= {}_{-\infty}^{\mathrm{C}}\mathrm{D}_t^{\alpha} \hat{f}^{(n)}(t) = \frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}} \hat{f}^{(n)}(t), \end{aligned} \tag{44}$$

where

$$\begin{aligned} \hat{f}(t) &= \begin{cases} \displaystyle\sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{\Gamma(i+1)} (t-a)^i, & t < a, \\ f(t), & t \geq a, \end{cases} \\ &= f(t)\theta(t-a) + \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{\Gamma(i+1)} (t-a)^i [1 - \theta(t-a)], \end{aligned} \tag{45}$$

and

$$\begin{aligned} \hat{f}^{(n)}(t) &= f^{(n)}(t)\theta(t-a) + \sum_{k=0}^{n-1} \delta^{(n-k-1)}(t-a) \\ &\quad \times \left\{ f^{(k)}(a) - \left[\sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{\Gamma(i+1)} (t-a)^i \right]^{(k)} \bigg|_{t=a} \right\} \\ &= f^{(n)}(t)\theta(t-a) \end{aligned} \tag{46}$$

according to Eq. (30). We see that the two definitions correspond to two different extensions of the function $f(t)$ on the interval $(-\infty, a)$.

4 Composition rules of fractional operators

In this section, we cosider the composition rules of fractional operators, which are of great importance to the application of fractional calculus. The rules will be used in our drivation in Sect. 6. In the present work, the Liouville integral is used for our purpose, expressed as [9, 18]

$$\frac{\mathrm{d}^{-\alpha}}{\mathrm{d}t^{-\alpha}} f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} \frac{f(\tau)}{(t-\tau)^{-\alpha+1}} \mathrm{d}\tau, \quad \alpha > 0. \tag{47}$$

And the R–L integral with a finite value of the lower terminals a can be regarded as special cases of definitions (47)

$$
{}_a^{\mathrm{R}}\mathrm{D}_t^{-\alpha} f(t) := \frac{1}{\Gamma(-\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{-\alpha+1}} \mathrm{d}\tau = \frac{1}{\Gamma(-\alpha)} \int_{-\infty}^t \frac{\tilde{f}(\tau)}{(t-\tau)^{-\alpha+1}} \mathrm{d}\tau, \tag{48}
$$

where $\tilde{f}(t) = f(t)\theta(t-a)$. We will study the composition rules of the fractional operators defined in Eqs. (41) and (47). Only functions that are smooth enough and those with finite number of singular points are considered.

Podlubny has proved the composition rule of R–L integrals [9]

$$
{}_a^{\mathrm{R}}\mathrm{D}_t^{-\alpha}\left[{}_a^{\mathrm{R}}\mathrm{D}_t^{-\beta} f(t)\right] = {}_a^{\mathrm{R}}\mathrm{D}_t^{-\alpha-\beta} f(t), \quad \alpha, \beta \ge 0, \tag{49}
$$

and the composition rule of

$$
{}_a^{\mathrm{R}}\mathrm{D}_t^{\alpha}\left[{}_a^{\mathrm{R}}\mathrm{D}_t^{-\beta} f(t)\right] = {}_a^{\mathrm{R}}\mathrm{D}_t^{\alpha-\beta} f(t), \quad \alpha, \beta \ge 0 \tag{50}
$$

in his book. Equations (49) and (50) can be easily generalized to the composition rules of our fractional operators

$$
\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \frac{\mathrm{d}^{-\beta} f}{\mathrm{d}t^{-\beta}} = \frac{\mathrm{d}^{\alpha-\beta} f}{\mathrm{d}t^{\alpha-\beta}}, \quad \alpha \in \mathrm{R}, \quad \beta \ge 0 \tag{51}
$$

for smooth functions and functions with finite number of singular points. In general, the composition rule of fractional derivatives

$$
\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \frac{\mathrm{d}^{\beta} f}{\mathrm{d}t^{\beta}} = \frac{\mathrm{d}^{\alpha+\beta} f}{\mathrm{d}t^{\alpha+\beta}}, \quad \alpha, \beta \ge 0 \tag{52}
$$

is not valid for the R–L definition and Caputo definition in the cases of a finite value of the lower terminals [9,18]. However, we will prove that the composition rule (52) is still valid for the fractional derivatives defined in Eq. (41). To complete the proof, we first prove that

$$
{}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{p}\left[f^{(n)}(t)\right] = {}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{p+n} f(t) - \lim_{b\to-\infty} \sum_{j=0}^{n-1} \frac{f^{(j)}(b)(t-b)^{j-p-n}}{\Gamma(1+j-p-n)}, \quad p \in \mathrm{R}. \tag{53}
$$

In fact, Podlubny proved that [9]

$$
{}_b^{\mathrm{R}}\mathrm{D}_t^{p}\left[f^{(n)}(t)\right] = {}_b^{\mathrm{R}}\mathrm{D}_t^{p+n} f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(b)(t-b)^{j-p-n}}{\Gamma(1+j-p-n)}, \quad p \in \mathrm{R}. \tag{54}
$$

From Eq. (54), we can directly obtain Eq. (53) for functions that are smooth enough. Then for functions $\tilde{f}(t)$ in Eq. (25) we have

$$
{}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{p}[\tilde{f}^{(n)}(t)] = 0 = {}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{p+n} \tilde{f}(t) - \lim_{b\to-\infty} \sum_{j=0}^{n-1} \frac{\tilde{f}^{(j)}(b)(t-b)^{j-p-n}}{\Gamma(1+j-p-n)}, \quad t < a, \tag{55}
$$

and

$$
\begin{aligned}
{}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{p}\left[\tilde{f}^{(n)}(t)\right] &= {}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{p+n}\left[{}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{-n} \tilde{f}^{(n)}(t)\right] \\
&= {}_a^{\mathrm{R}}\mathrm{D}_t^{p+n}\left\{{}_a^{\mathrm{R}}\mathrm{D}_t^{-n} f^{(n)}(t) + {}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{-n}\left[\sum_{k=0}^{n-1} \delta^{(n-k-1)}(t-a) f^{(k)}(a)\right]\right\} \\
&= {}_a^{\mathrm{R}}\mathrm{D}_t^{p+n}\left\{f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)(t-a)^j}{\Gamma(j+1)} + \sum_{k=0}^{n-1} \delta^{(-k-1)}(t-a) f^{(k)}(a)\right\} \\
&= {}_a^{\mathrm{R}}\mathrm{D}_t^{p+n}\left\{f(t) - \sum_{k=0}^{n-1} \frac{f^{(j)}(a)(t-a)^j}{\Gamma(j+1)} + \sum_{k=0}^{n-1} \frac{(t-a)^k}{\Gamma(k+1)} f^{(k)}(a)\right\} \\
&= {}_a^{\mathrm{R}}\mathrm{D}_t^{p+n} f(t) = {}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{p+n} \tilde{f}(t) - \lim_{b\to-\infty} \sum_{j=0}^{n-1} \frac{\tilde{f}^{(j)}(b)(t-b)^{j-p-n}}{\Gamma(1+j-p-n)}, \quad t > a.
\end{aligned} \tag{56}
$$

Thus we can conclude that Eq. (53) is valid for any function that is smooth enough and for functions with finite number of singular points.

We suppose that the derivatives of function $f(t)$ – $\mathrm{d}^\alpha f/\mathrm{d}t^\alpha$, $\mathrm{d}^\beta f/\mathrm{d}t^\beta$ and $\mathrm{d}^{\alpha+\beta} f/\mathrm{d}t^{\alpha+\beta}$ exist for $m-1 < \alpha \le m$ and $n-1 < \beta \le n$. If $\beta = n$ and $\alpha = m$, we can obviously get the composition rule of $\mathrm{d}^\alpha/\mathrm{d}t^\alpha$ and $\mathrm{d}^\beta/\mathrm{d}t^\beta$. If $\beta \ne n$, from the existence of the derivative $\mathrm{d}^\beta f/\mathrm{d}t^\beta$ we obtain

$$
\lim_{\tau\to-\infty} \frac{f^{(j)}(\tau)}{(t-\tau)^{\beta-j}} = 0, \quad j = 0, 1, \cdots, n. \tag{57}
$$

And if $\beta = n$ and $\alpha \ne m$, from the existence of the derivative $\mathrm{d}^{\alpha+\beta} f/\mathrm{d}t^{\alpha+\beta}$ we get

$$
\lim_{\tau\to-\infty} \frac{f^{(j)}(\tau)}{(t-\tau)^{\alpha+\beta-j}} = 0, \quad j = 0, 1, \cdots, n+m. \tag{58}
$$

Then, using Eqs. (51), (53), (57) and (58) we obtain

$$
\begin{aligned}
\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} \frac{\mathrm{d}^{\beta} f(t)}{\mathrm{d}t^{\beta}} &= {}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{\alpha}\left[{}_{-\infty}^{\mathrm{C}}\mathrm{D}_t^{\beta} f(t)\right] \\
&= {}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{\alpha}\left[\frac{1}{\Gamma(n-\beta)} \int_{-\infty}^t \frac{f^{(n)}(\tau)\mathrm{d}\tau}{(t-\tau)^{\beta+1-n}}\right] \\
&= {}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{\alpha}\left[{}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{\beta-n} f^{(n)}(t)\right] = {}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{\alpha+\beta-n}\left[f^{(n)}(t)\right] \\
&= \lim_{b\to-\infty}\left[{}_b^{\mathrm{R}}\mathrm{D}_t^{\alpha+\beta} f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(b)(t-b)^{j-\alpha-\beta}}{\Gamma(1+j-\alpha-\beta)}\right] \\
&= \frac{\mathrm{d}^{\alpha+\beta} f(t)}{\mathrm{d}t^{\alpha+\beta}}.
\end{aligned} \tag{59}
$$

This is the composition rule of fractional derivatives we have tried to get. It is of great importance in the application of fractional derivatives. It should note that the composition rule

$$\frac{\mathrm{d}^{-\alpha}}{\mathrm{d}t^{-\alpha}}\frac{\mathrm{d}^{\beta}}{\mathrm{d}t^{\beta}}f(t)=\frac{\mathrm{d}^{-\alpha+\beta}}{\mathrm{d}t^{-\alpha+\beta}}f(t), \qquad \alpha,\beta\geq 0 \tag{60}$$

is not valid in general. However, from the derivation of Eq. (59) we can see that Eq. (60) is valid as long as

$$\lim_{b\to-\infty}\frac{f^{(j)}(b)(t-b)^{j+\alpha-\beta}}{\Gamma(1+j+\alpha-\beta)}=0, \qquad j=0,1,\cdots,n-1. \tag{61}$$

5 Initial conditions for fractional differential equations

As we have pointed in Sect. 1 that solution of a linear fractional differential equation defined in terms of R–L derivatives will require fractional initial conditions, while solution of a linear fractional differential equation defined in terms of Caputo derivatives require only regular initial conditions that are familiar to us. This can also be seen from the formula for the Laplace transform of R–L and Caputo fractional derivatives [9,10]

$$L\left[{}_0^{\mathrm{R}}\mathrm{D}_t^{\alpha}f(t)\right]=s^{\alpha}L[f(t)]-\sum_{k=0}^{n-1}s^{k}\,{}_0^{\mathrm{R}}\mathrm{D}_t^{\alpha-k-1}f(t)\Big|_{t=0}, \quad n-1\leq\alpha<n, \tag{62}$$

and

$$L\left[{}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha}f(t)\right]=s^{\alpha}L[f(t)]-\sum_{k=0}^{n-1}s^{\alpha-k-1}f^{(k)}(0), \quad n-1<\alpha\leq n. \tag{63}$$

As a result, the Caputo derivatives are more popular with the physicists. In this section we discuss the problem about the initial conditions for the differential equations defined in terms of the fractional derivatives of Eq. (41) and give a method to deal with the linear fractional differential equations with arbitrary initial conditions.

It is important to note that to solve the fractional differential equations with lower terminals $a\to-\infty$, we should know the behavior of the solution before the starting time. In other words, the fractional differential equation can be regarded as a system with infinite number of "initial conditions" which are given as the behaviors of the solution at any instant before the starting time. Thus the discussion about the initial conditions at the starting time would be meaningless. Let us consider a fractional differential equation of $f(t)$. The starting time is set to $t=0$ and we assume that the value of $f(t)$ at any instant before $t=0$ is already given. Then the function $f(t)$ can be written as

$$f(t)=\tilde{f}(t)+\bar{f}(t), \tag{64}$$

where

$$\tilde{f}(t)=\begin{cases}0, & t<0,\\ f(t), & t\geq 0,\end{cases} \qquad \bar{f}(t)=\begin{cases}f(t), & t<0,\\ 0, & t\geq 0.\end{cases} \tag{65}$$

The function $\bar{f}(t)$ contains the information about $f(t)$ before the starting time and it is substituted into the original equation as given conditions, which can be viewed as the "initial conditions" of the fractional differential equations. It reflects the influence of the behaviors of $f(t)$ before the starting time to the evolution of the system. Thus the original equation is reduced to the fractional differential equation of $\tilde{f}(t)$ with zero initial conditions at $t=0_-$. In particular, for linear fractional differential equations the Laplace transform method can be used to obtain the solution of the equations

$$L_-\left[\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\tilde{f}(t)\right]=L\left[{}_{-\infty}^{\mathrm{R}}\mathrm{D}_t^{\alpha}\tilde{f}(t)\right]=s^{\alpha}L[\tilde{f}(t)], \tag{66}$$

where the subscript "−" is dropped just for convenience.

6 Discussion of a linear fractional oscillator with Scott–Blair model

Let us study the linear vibration of the system depicted in Fig. 4. The spring is made of the fractional element material whose constitutive equation can be expressed as

$$F=-k\lambda^{\beta}\frac{\mathrm{d}^{\beta}u}{\mathrm{d}t^{\beta}}, \qquad 0<\beta<1, \tag{67}$$

where $u(t)$ denote the displacement of the particle m.

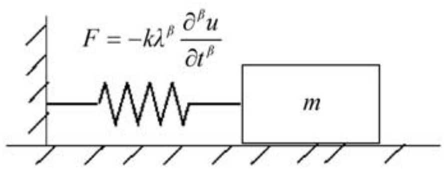

Fig. 4 Force diagram of the particle

Thus we have

$$-k\lambda^{\beta}\frac{\mathrm{d}^{\beta}u}{\mathrm{d}t^{\beta}}=m\frac{\mathrm{d}^{2}u}{\mathrm{d}t^{2}}. \tag{68}$$

Apply operator $\mathrm{d}^{-\beta}/\mathrm{d}t^{-\beta}$ to both sides of Eq. (68)

$$-k\lambda^{\beta}\frac{\mathrm{d}^{-\beta}}{\mathrm{d}t^{-\beta}}\frac{\mathrm{d}^{\beta}u}{\mathrm{d}t^{\beta}}=m\frac{\mathrm{d}^{-\beta}}{\mathrm{d}t^{-\beta}}\frac{\mathrm{d}^{2}u}{\mathrm{d}t^{2}}. \tag{69}$$

In our problem the displacement function $u(t)$ and its derivatives are required to be bounded for $t\to-\infty$. Thus we have

$$\lim_{b\to-\infty}\frac{u^{(j)}(b)(t-b)^{j+\beta-2}}{\Gamma(1+j+\beta-2)}=0, \qquad j=0,1. \tag{70}$$

Then according to Eq. (59) we can obtain

$$\frac{\mathrm{d}^{-\beta}}{\mathrm{d}t^{-\beta}}\frac{\mathrm{d}^{2}u}{\mathrm{d}t^{2}}=\frac{\mathrm{d}^{2-\beta}u}{\mathrm{d}t^{2-\beta}}. \tag{71}$$

Using Eq. (59) we also get

$$\frac{\mathrm{d}^{-\beta}}{\mathrm{d}t^{-\beta}}\frac{\mathrm{d}^{\beta}u}{\mathrm{d}t^{\beta}} = u - \lim_{t\to-\infty} u. \tag{72}$$

Using Eqs. (71) and (72), we finally obtain the fractional vibration equation from Eq. (69)

$$-u + \lim_{t\to-\infty} u = \frac{1}{\omega^{2-\beta}}\frac{\mathrm{d}^{2-\beta}u}{\mathrm{d}t^{2-\beta}}, \tag{73}$$

where $\omega^{2-\beta} = k\lambda^{\beta}/m$. When $\beta = 1$, Eq. (73) represents a exponential decay equation. When $\beta = 0$, Eq. (73) represents a classical harmonic vibration equation, and ω denotes the angular frequency of the system. In this paper, we set the "initial conditions" of the equation to

$$\bar{u}(t) = \begin{cases} 0, & t < -a, \\ u_0, & -a \le t < 0, \end{cases} \qquad a > 0. \tag{74}$$

Then we have $\lim_{t\to-\infty} u = 0$, and Eq. (73) can be reduced to

$$\frac{\mathrm{d}^{2-\beta}u}{\mathrm{d}t^{2-\beta}} + \omega^{2-\beta}u = 0, \qquad 0 < \beta < 1. \tag{75}$$

For $t > 0$, we have

$$\begin{aligned}\frac{\mathrm{d}^{2-\beta}u(t)}{\mathrm{d}t^{2-\beta}} &= {}^{\mathrm{R}}_{-\infty}\mathrm{D}_t^{2-\beta}u(t) \\ &= {}^{\mathrm{R}}_{-\infty}\mathrm{D}_t^{2-\beta}\tilde{u}(x,t) + \frac{1}{\Gamma(\beta)}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{-a}^{0}\frac{u_0}{(t-\tau)^{1-\beta}}\mathrm{d}\tau \\ &= {}^{\mathrm{R}}_{-\infty}\mathrm{D}_t^{2-\beta}\tilde{u}(x,t) - u_0\frac{t^{\beta-2} - (t+a)^{\beta-2}}{\Gamma(\beta-1)},\end{aligned} \tag{76}$$

from which we can obtain the L–Laplace transform of $\mathrm{d}^{2-\beta}u(x,t)/\mathrm{d}t^{2-\beta}$ as

$$L\left[\frac{\mathrm{d}^{2-\beta}u(t)}{\mathrm{d}t^{2-\beta}}\right] = s^{2-\beta}L[\tilde{u}(t)] - s^{1-\beta}u_0 + L\left[\frac{(t+a)^{\beta-2}}{\Gamma(\beta-1)}\right]u_0. \tag{77}$$

Then the Laplace transform of Eq. (74) is

$$s^{2-\beta}L[\tilde{u}(t)] - s^{1-\beta}u_0 + L\left[\frac{(t+a)^{\beta-2}}{\Gamma(\beta-1)}\right]u_0 + \omega^{2-\beta}L[\tilde{u}(t)] = 0. \tag{78}$$

Thus we have

$$L[\tilde{u}(t)] = \frac{s^{1-\beta}}{s^{2-\beta}+\omega^{2-\beta}}u_0 - \frac{1}{s^{2-\beta}+\omega^{2-\beta}}L\left[\frac{(t+a)^{\beta-2}}{\Gamma(\beta-1)}\right]u_0. \tag{79}$$

Using the property [9]

$$\begin{aligned} L[t^{\alpha-1}E_{\alpha,\alpha}(-at^{\alpha})] &= \frac{1}{s^{\alpha}+a}, \\ L[E_{\alpha}(-at^{\alpha})] &= \frac{s^{\alpha-1}}{s^{\alpha}+a}, \end{aligned} \tag{80}$$

where the Mittag–Leffler function $E_{\alpha,\gamma}$ and E_{α} is defined as [9]

$$E_{\alpha,\gamma}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\gamma)}, \qquad E_{\alpha}(z) = E_{\alpha,1}(z), \tag{81}$$

we can obtain the inverse Laplace transform of Eq. (79)

$$\begin{aligned}\tilde{u}(x,t) &= u_0E_{2-\beta}\left[-(\omega t)^{2-\beta}\right] \\ &\quad -u_0\left\{t^{1-\beta}E_{2-\beta,2-\beta}\left[-(\omega t)^{2-\beta}\right]\right\}\frac{(t+a)^{\beta-2}}{\Gamma(\beta-1)} \\ &= u_0\left\{E_{2-\beta}\left[-(\omega t)^{2-\beta}\right] - \int_0^t(t-\tau)^{1-\beta}E_{2-\beta,2-\beta}\right. \\ &\quad \left.\times\left[-\omega^{2-\beta}(t-\tau)^{2-\beta}\right]\frac{(\tau+a)^{\beta-2}}{\Gamma(\beta-1)}\mathrm{d}\tau\right\}.\end{aligned} \tag{82}$$

It can be easily verify that $\tilde{u}(0) = E_{2-\beta}(0)u_0 = u_0$, indicating that the solution is continuous at $t = 0$. We also obtain the partial derivative of $\tilde{u}(x,t)$ with respect to time

$$\begin{aligned}\frac{\mathrm{d}\tilde{u}(t)}{\mathrm{d}t} &= u_0\left\{-(2-\beta)\omega^{2-\beta}t^{1-\beta}E^{(1)}_{2-\beta}\left[-(\omega t)^{2-\beta}\right]\right. \\ &\quad -\int_0^t(t-\tau)^{-\beta}E_{2-\beta,1-\beta}\left[-\omega^{2-\beta}(t-\tau)^{2-\beta}\right] \\ &\quad \left.\times\frac{(\tau+a)^{\beta-2}}{\Gamma(\beta-1)}\mathrm{d}\tau\right\}.\end{aligned} \tag{83}$$

Thus, for $0 < \beta < 1$, we have

$$\begin{aligned}\left.\frac{\mathrm{d}\tilde{u}}{\mathrm{d}t}\right|_{t=0} &= -\lim_{t\to 0}u_0\int_0^t\frac{(t-\tau)^{-\beta}}{\Gamma(1-\beta)}\frac{a^{\beta-2}}{\Gamma(\beta-1)}\mathrm{d}\tau \\ &= \lim_{t\to 0}\frac{a^{\beta-2}}{\Gamma(\beta-1)}\left.\frac{(t-\tau)^{1-\beta}}{\Gamma(2-\beta)}\right|_0^t u_0 = 0.\end{aligned} \tag{84}$$

We see that we can get the same values of $\tilde{u}$ and $\mathrm{d}\tilde{u}/\mathrm{d}t$ at $t = 0$ for solutions with different values of a. This is an important conclusion. From Eq. (63) we know that the solution of the linear fractional differential equation (75) defined in terms of the original Caputo derivative is uniquely determined by the initial values of $\tilde{u}$ and $\mathrm{d}\tilde{u}/\mathrm{d}t$ at the starting time. However, here we see that to solve the equation uniquely, it is not enough that we only know the initial values at the starting time. This verifys the statement in Sect. 5 that the discussion about the initial conditions at the starting time for a fractional system is meaningless and that infinite number of "initial conditions" are needed to determine the solution uniquely.

As motivation for the general case, let us first consider three special cases of solution (82). For $\beta = 1$, we can get

$$\begin{aligned}\tilde{u}(t) &= \left\{E_1(-\omega t) - \int_0^t E_1[-\omega(t-\tau)]\frac{(\tau+a)^{-1}}{\Gamma(0)}\mathrm{d}\tau\right\}u_0 \\ &= \mathrm{e}^{-\omega t}u_0,\end{aligned} \tag{85}$$

where

$$E_1(-\omega t) = \sum_{k=0}^{\infty}\frac{(-\omega t)^k}{\Gamma(k+1)} = \mathrm{e}^{-\omega t}, \qquad \frac{1}{\Gamma(0)} = 0. \tag{86}$$

For $\beta = 0$, we can get

$$\tilde{u}(t) = \left\{E_2[-(\omega t)^2] - \int_0^t(t-\tau)E_{2,2}[-\omega^2(t-\tau)^2]\right.$$

$$\frac{(\tau + a)^{-2}}{\Gamma(-1)} \mathrm{d}\tau \Big\} u_0 = u_0 \cos \omega t, \tag{87}$$

where

$$E_2[-(\omega t)^2] = \sum_{k=0}^{\infty} \frac{(-1)^k (\omega t)^{2k}}{\Gamma(2k+1)} = \cos \omega t, \quad \frac{1}{\Gamma(-1)} = 0. \tag{88}$$

We can see that Eqs. (85) and (87) are the solutions of exponential decay equation and classical harmonic vibration equation, respectively. Finally we consider the case of $a \to +\infty$

$$\begin{aligned} \tilde{u}(t) = \lim_{a \to +\infty} \Big\{ & E_{2-\beta}[-(\omega t)^{2-\beta}] - \int_0^t (t-\tau)^{1-\beta} E_{2-\beta, 2-\beta} \\ & \times [-\omega^{2-\beta} (t-\tau)^{2-\beta}] \frac{(\tau + a)^{\beta - 2}}{\Gamma(\beta - 1)} \mathrm{d}\tau \Big\} u_0 \\ = {} & u_0 E_{2-\beta}[-(\omega t)^{2-\beta}]. \end{aligned} \tag{89}$$

Because we have proved $\tilde{u}(0) = u_0$ and $\mathrm{d}\tilde{u}(0)/\mathrm{d}t = 0$, in this case we can get $\mathrm{d}^\alpha u/\mathrm{d}t^\alpha = {}_0^{\mathrm{C}}\mathrm{D}_t^\alpha u$ from Eq. (44). In fact our solution (89) is consistent with that obtained using Caputo derivative ${}_0^{\mathrm{C}}\mathrm{D}_t^\alpha u(t)$. It should note that if we take $a = 0$, we have

$$\frac{\mathrm{d}^\alpha u(t)}{\mathrm{d}t^\alpha} = {}_0^{\mathrm{R}}\mathrm{D}_t^\alpha u(t). \tag{90}$$

However, the improper integral in Eq. (82) is divergent at $\tau = 0$ when $a = 0$. Thus Eq. (75) can not be solved using R–L definition ${}_0^{\mathrm{R}}\mathrm{D}_t^\alpha u(t)$.

Now let us study the evolution of solution (82) through numerical simulation. The variables $\tilde{u}$, t and a are now, respectively, expressed as the non-dimensional variables $u^* = \tilde{u}/u_0$, $t^* = \omega t$ and $a^* = \omega a$. We first study the influence of the value of β on the evolution of $u^*(t)$. For $a \to +\infty$, the $u - t$ curve is plotted in Fig. 5. In the case of $\beta = 1$, the solution is an exponential decay, while a harmonic oscillation appears in the system with $\beta = 0$. For $0 < \beta < 1$, the behavior of u is similar to that of damped vibration. The smaller the value of β, the more obvious the oscillation of u and the slower the speed of the decay of u. Similar phenomenon is also found in the case where the value of a is finite (see Figs. 6 and 7). In Fig. 7, a phenomenon of overshoot of $u^*(t)$ in the neighbor of $t = 0$ is observed in the system. It should be pointed out that the slopes of curves $\beta = 0.85$, $\beta = 0.5$ and $\beta = 0.05$ in Fig. 8 are all equal to zero (see Fig. 9).

Then we study the influence of the value of a^* on the evolution of $u^*(t)$. Let us consider the case of $\beta = 0.5$. The influence of the value of a^* on the behavior of u^* can be observed in Fig. 8. We can find a critical value a_{cr} of a^*. For $a^* < a_{\mathrm{cr}}$, the behavior of u^* is still similar to that of damped vibration. However, for $a^* > a_{\mathrm{cr}}$, the phenomenon of overshoot of $u^*(t)$ in the neighbor of $t = 0$ is observed in the system. In fact, the behavior of $u^*(t)$ is determined by the combination of memory effect and relaxation effect of the fractional-order system. If $a^* < a_{\mathrm{cr}}$, the system still has a strong memory of the displacement jump at $t = -a$ and thus the material shows an overshoot effect due to inertia; otherwise, the relaxation effect play a dominant role and the displacement of the material monotonically decreases in the neighbor of $t = 0$. Our calculation shows that $a_{\mathrm{cr}} \approx 0.4$ in the case of $\beta = 0.5$.

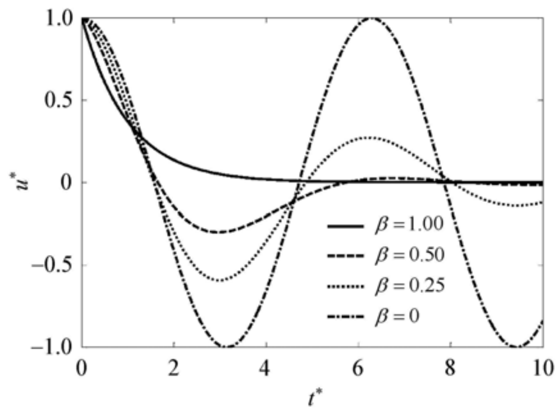

Fig. 5 Evolution of the variable u^* for $a^* \to +\infty$

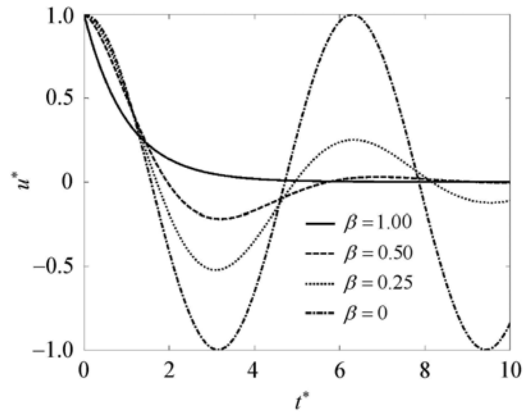

Fig. 6 Evolution of the variable $u^*(t)$ for $a^* = 1$

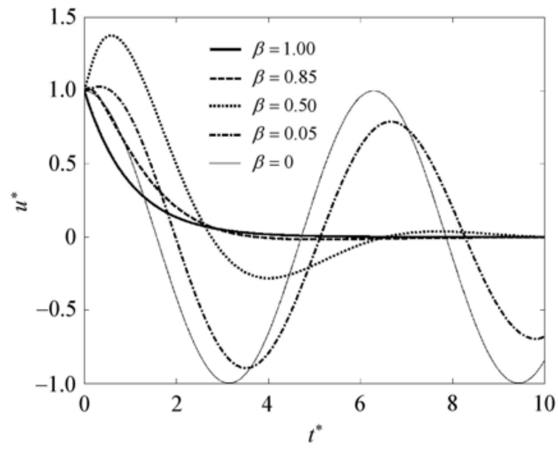

Fig. 7 Evolution of the variable $u^*(t)$ for $a^* = 0.1$

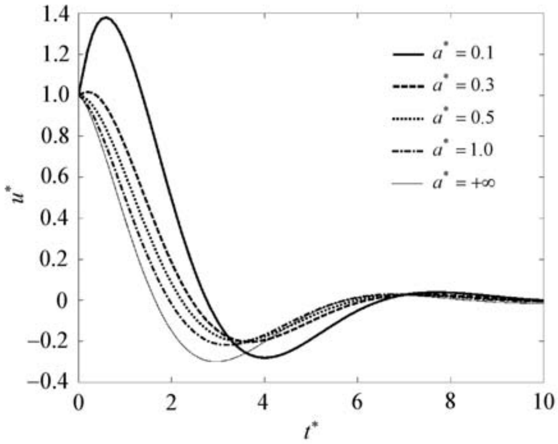

Fig. 8 Evolution of the variable $u^*(t)$ for $\beta = 0.5$

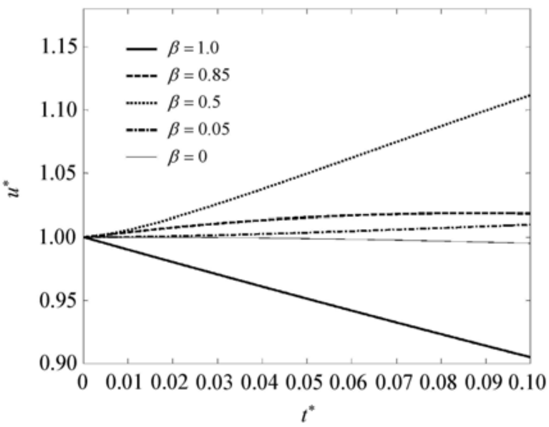

Fig. 9 Partial enlargement of Fig. 8

7 Conclusions

In this study by analyzing the stress responses of the 1/2-order tree model to a constant strain and to a strain jump respectively, the R–L definition and Caputo definition are both found to be defective when used in rheology. We clarify that the main reason that cause the two definitions' deficiencies is the loss of the information of the function (denoted by $f(t)$) before the lower terminals. Thus in the definition of fractional derivatives, the lower terminals should be put to $-\infty$ to include all the information of $f(t)$ from $-\infty$ to t. We further prove that the R–L definition with lower terminal $a \to -\infty$ and the Caputo definition with lower terminal $a \to -\infty$ are equivalent not only in the differentiation of the functions that are smooth enough, but also in the differentiation of those with finite number of singular points. Thus we define the α-order fractional derivative in rheology as the R–L derivative with lower terminal $a \to -\infty$ (or Caputo derivative with lower terminal $a \to -\infty$) not only for steady-state processes, but also for transient processes. The composition rules of fractional operators are also studied and a proof of the composition rule of fractional derivatives is given. Based on the new definition of fractional derivatives, we discuss the problems about the initial conditions for fractional differential equations As an example we study a linear fractional oscillator with Scott–Blair model and give an analytical solution of the equation under given conditions.

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