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RESEARCH PAPER

Symplectic system based analytical solution for bending of rectangular orthotropic plates on Winkler elastic foundation

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Abstract This paper analyses the bending of rectangular orthotropic plates on a Winkler elastic foundation. Appropriate definition of symplectic inner product and symplectic space formed by generalized displacements establish dual variables and dual equations in the symplectic space. The operator matrix of the equation set is proven to be a Hamilton operator matrix. Separation of variables and eigenfunction expansion creates a basis for analyzing the bending of rectangular orthotropic plates on Winkler elastic foundation and obtaining solutions for plates having any boundary condition. There is discussion of symplectic eigenvalue problems of orthotropic plates under two typical boundary conditions, with opposite sides simply supported and opposite sides clamped. Transcendental equations of eigenvalues and symplectic eigenvectors in analytical form given. Analytical solutions using two examples are presented to show the use of the new methods described in this paper. To verify the accuracy and convergence, a fully simply supported plate that is fully and simply supported under uniformly distributed load is used to compare the classical Navier method, the Levy method and the new method. Results show that the new technique has good accuracy and better convergence speed than other methods, especially in relation to internal forces. A fully clamped rectangular plate on Winkler foundation is solved to validate application of the new methods, with solutions compared to those produced by the Galerkin method.

Keywords Orthotropic plate · Symplectic space · Winkler elastic foundation · Analytical solution

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1 Introduction

Plates positioned on elastic foundations such as building foundation plates and pavement slabs are widely used in engineering as construction materials. The Winkler model is often used to describe the contact pressure of foundations and plates, and plates often satisfy the Kirchhoff hypothesis [1]. Due to mathematical complexity, analyzing the bending of plates on elastic foundations is limited to definite shape and boundary conditions of the plates. Classical methods like the Navier method and the Levy method can be applied to plates with two opposite sides that are simply supported but can not be applied to plates with other boundary conditions and convergence of internal forces is not satisfactory. Numerical approximations are often employed for other boundary conditions, such as in Selvadurai's study of thin plates on soil-foundation, which uses a finite difference method [2], Kong and Cheung [3] studied rectangular plates by using a finite strip method, Cheung and Zienkiewicz [4] used a finite element method based on the Winkler model to study rectangular plates. Sadecka [5] conducted finite/infinite element analysis of a thick plate on a layered foundation. Silva et al. [6] used a numerical method to analyze plates on elastic foundations. Sladek et al. [7] used the meshless local Petrov–Galerkin method to study orthotropic thick plates.

A new symplectic dual solution method can be used to solve elasticity in symplectic space via separation of variables and eigenfunction expansion [8, 9]. Yao et al. [10] studied an elastic wedge to reveal paradoxical characteristics. Zhong et al. [11] introduced bending moment functions to propose new formulations of Kirchhoff plate bending problem and solve the pure bending of a long plate of semi-infinite dimension in symplectic space. Lim et al. [12, 13] used bending moment functions to provide a benchmark or exact solutions for rectangular thin plates, which were supported at the corners or simply supported on the two opposite sides. Yao et al. [14, 15] applied the symplectic method to obtain solutions for an orthotropic thin plate and a Reissner plate.

Despite many advances, methods used to analyze thin plates can not be applied directly to plates on foundation due to deflection that does not appear in basic variables. This paper applies a new symplectic method to the bending of orthotropic plates, based on the Winkler elastic foundation. To start, this paper describes release of constraint between slope and deflection yields dual equations formed by dual variables in symplectic space. Schemes to separate variables and eigenfunction expansion are implemented. There is discussion follows of symplectic eigenvalue problems for orthotropic plates with typical boundary conditions, namely, two opposite sides simply supported and two opposite sides that are clamped.

To verify accuracy and convergence of the new method presented here, a fully supported plate under uniformly distributed load is compared using the Navier method and Levy method. Results show that the new method has good accuracy and better convergence speed than earlier methods, especially regarding internal forces. A fully clamped rectangular plate on Winkler foundation is solved in order to validate applicability of new methods. Solutions are also compared with the Galerkin method.

2 Fundamental equations

The rectangular domain under consideration is $\Omega = \{-a < x < a, -b < y < b\}$. Directions of positive internal forces on the plate are shown in Fig. 1.

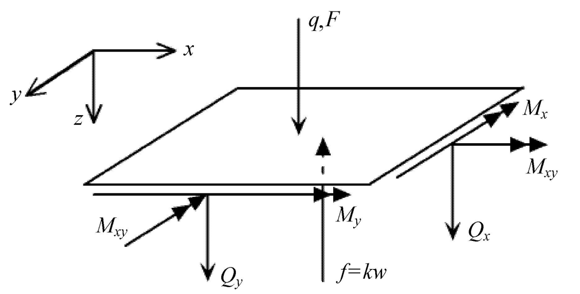

Fig. 1 Directions of positive internal forces on a rectangular plate

The relationship between deflection and bending moments is specified as

$$\begin{Bmatrix} M_x \\ M_y \\ 2M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \partial_{xx} w \\ \partial_{yy} w \\ \partial_{xy} w \end{Bmatrix}, \tag{1}$$

where $D_{11}, D_{12}, D_{22}, D_{66}$ are bending stiffness coefficients of an orthotropic plate. ∂_x and ∂_y denote first order partial differential with respect to variables x and y, respectively, the others are similar in the following derivation.

Equilibrium equations for a thin plate on Winkler elastic foundation are

$$\begin{aligned} &\partial_x Q_x + \partial_y Q_y + q - kw = 0, \\ &\partial_x M_x + \partial_y M_{xy} + Q_x = 0, \\ &\partial_y M_y + \partial_x M_{xy} + Q_y = 0, \end{aligned} \tag{2}$$

where $k > 0$ is the modulus of Winkler foundation and q is distributed load on the plate.

Equations (1) and (2) can be derived from the Hellinger–Reissner variation principle

$$\delta \iint_\Omega \Big(M_x \partial_{xx} w + M_y \partial_{yy} w + 2M_{xy} \partial_{xy} w \\ -U - qw + \frac{1}{2} k w^2 \Big) \mathrm{d}x \mathrm{d}y = 0, \tag{3}$$

where complementary energy density is

$$U = \frac{1}{2(D_{11}D_{22} - D_{12}^2)} \Big[D_{22} M_x^2 + D_{11} M_y^2 - 2D_{12} M_x M_y \\ + \frac{4(D_{11}D_{22} - D_{12}^2)}{D_{66}} M_{xy}^2 \Big]. \tag{4}$$

Assuming external normal and tangent directions of the boundary to be n and s, respectively, (n, s) composes a right-handed coordinate system and total equivalent shear forces on sides of a rectangular plate are

$$V_n = -\partial_s M_{ns} + Q_n = -\partial_n M_n - 2\partial_s M_{ns}. \tag{5}$$

Thus, boundary conditions of a plate can be specified.

In general:

(1) For a free edge, bending moment and total equivalent shear force are

$$M_n = \bar{M}_n, \qquad V_n = \bar{V}_n. \tag{6a}$$

(2) For a simply supported edge, bending moment and deflection are

$$M_n = \bar{M}_n, \qquad w = \bar{w}. \tag{6b}$$

(3) For a clamped edge, the deflection and rotation are

$$w = \bar{w}, \qquad \partial_n w = \bar{\theta}_n. \tag{6c}$$

3 Derivation of symplectic system

Bending moment M_x and equivalent shear force V_x in the x-direction are denoted M and V, respectively. The symbol "$\cdot$" in the following derivation denotes differential with respect to x, i.e. $\dot{w} = \partial_x$.

Introducing constraint

$$\theta = \dot{w}, \tag{7}$$

and Lagrange multiplier V into variation formula Eq. (3), produces the new variation formula

$$\delta \iint_{\Omega} \Big[M\dot{\theta} + M_y \partial_{yy} w + 2M_{xy}\partial_y \theta - qw + \frac{1}{2}kw^2 - U + V(\dot{w} - \theta) \Big] \mathrm{d}x\mathrm{d}y = 0. \tag{8}$$

The variation of Eq. (8) with respect to M_y and M_{xy} are

$$M_y = \frac{D_{12}}{D_{11}} M_x + \Big(D_{22} - \frac{D_{12}^2}{D_{11}}\Big)\partial_{yy} w, \qquad M_{xy} = \frac{D_{66}}{2}\partial_y \theta. \tag{9}$$

Substituting Eq. (9) into Eq. (8) and eliminating M_y and M_{xy} yields a mixed energy variational principle

$$\delta \iint \{V\dot{w} + M\dot{\theta} - H\} \mathrm{d}x\mathrm{d}y = 0, \tag{10}$$

where

$$H = V\theta + qw - \frac{1}{2}kw^2 - \frac{1}{2}\Big(D_{22} - \frac{D_{12}^2}{D_{11}}\Big)(\partial_{yy}w)^2 - \frac{D_{12}}{D_{11}} M\partial_{yy}w - \frac{1}{2}D_{66}(\partial_y\theta)^2 + \frac{1}{2D_{11}}M^2. \tag{11}$$

The stationary requirements of Eq. (10) yield a group of equations that can be written in matrix form

$$\dot{v} = \mathbf{H}\boldsymbol{v} + \boldsymbol{q}, \tag{12}$$

in which the operator matrix is

$$\mathbf{H} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{D_{12}}{D_{11}}\partial_{yy} & 0 & 0 & \dfrac{1}{D_{11}} \\ k + \Big(D_{22} - \dfrac{D_{12}^2}{D_{11}}\Big)\partial_{yyyy} & 0 & 0 & \dfrac{D_{12}}{D_{11}}\partial_{yy} \\ 0 & -D_{66}\partial_{yy} & -1 & 0 \end{bmatrix}, \tag{13}$$

and the nonhomogeneous term $\boldsymbol{q} = \{0 \;\; 0 \;\; -q \;\; 0\}^{\mathrm{T}}$ describes the load acting in the domain. $\boldsymbol{v} = \{w \;\; \theta \;\; V \;\; M\}^{\mathrm{T}}$ is the full state vector.

For the purpose of discussing the property of operator matrix $\mathbf{H}$, the unit symplectic matrix is

$$\boldsymbol{J} = \begin{bmatrix} \mathbf{0} & \boldsymbol{I} \\ -\boldsymbol{I} & \mathbf{0} \end{bmatrix}, \qquad \boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{14}$$

and the symplectic inner product is

$$\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = \int_{-b}^{b} \boldsymbol{v}_1^{\mathrm{T}} \boldsymbol{J} \boldsymbol{v}_2 \mathrm{d}y + D_{66}\Big(w_1 \partial_{xy} w_2 - w_2 \partial_{xy} w_1\Big)_{y=-b}^{b}. \tag{15}$$

Equation (15) satisfies the four conditions of the symplectic inner product [9]. Hence, vector $\boldsymbol{v}$ forms a symplectic geometry space in accordance with the definition of the symplectic inner product (15). Two vectors are symplectically orthogonal if their symplectic inner product is zero. Otherwise, the vectors are symplectic adjoint.

Integration by parts yields

$$\begin{aligned} \langle \boldsymbol{v}_1, \mathbf{H}\boldsymbol{v}_2 \rangle &= \langle \boldsymbol{v}_2, \mathbf{H}\boldsymbol{v}_1 \rangle \\ &+ \Big\{ w_1 \Big[\Big(D_{22} - \frac{D_{12}^2}{D_{11}}\Big)\partial_{yyy} w_2 + \frac{D_{12}}{D_{11}}\partial_y M_2 + D_{66}\partial_{xy}\theta_2 \Big] \\ &- w_2 \Big[\Big(D_{22} - \frac{D_{12}^2}{D_{11}}\Big)\partial_{yyy} w_1 + \frac{D_{12}}{D_{11}}\partial_y M_1 + D_{66}\partial_{xy}\theta_1 \Big] \\ &- \partial_y w_1 \Big[\Big(D_{22} - \frac{D_{12}^2}{D_{11}}\Big)\partial_{yy} w_2 + \frac{D_{12}}{D_{11}} M_2 \Big] \\ &+ \partial_y w_2 \Big[\Big(D_{22} - \frac{D_{12}^2}{D_{11}}\Big)\partial_{yy} w_1 + \frac{D_{12}}{D_{11}} M_1 \Big] \\ &+ D_{66}[\theta_1 \partial_y(\partial_x w_2 - \theta_2) - \theta_2 \partial_y(\partial_x w_1 - \theta_1)] \Big\}_{y=-b}^{b}. \end{aligned} \tag{16}$$

Hence, if $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ satisfy any of the three corresponding homogeneous conditions of Eq. (6) at $y = \pm b$ and

$$\partial_y(\partial_x w_j - \theta_j) = 0, \quad (j = 1, 2), \quad \text{at} \quad y = \pm b, \tag{17}$$

there is identity

$$\langle \boldsymbol{v}_1, \mathbf{H}\boldsymbol{v}_2 \rangle \equiv \langle \boldsymbol{v}_2, \mathbf{H}\boldsymbol{v}_1 \rangle. \tag{18}$$

Hence, the operator matrix $\mathbf{H}$ is a Hamilton transformation (operator matrix) in the symplectic space.

Vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ in Identity (18) need not satisfy domain differential equations (12). Equation (7) may be untrue, so boundary conditions (17) are needed. But if the vectors satisfy Eq. (12) in the domain, those vectors must also satisfy boundary conditions (17).

4 Symplectic eigenfunction expansion

A homogeneous equation corresponds to Eq. (12)

$$\dot{\boldsymbol{v}} = \mathbf{H}\boldsymbol{v}, \tag{19}$$

Equation (19) can be solved by separating variables, by assuming that

$$\boldsymbol{v} = \zeta(x)\boldsymbol{\psi}(y), \tag{20}$$

and substituting Eq. (20) into Eq. (19) gives

$$\zeta(x) = \exp(\mu x), \tag{21}$$

as well as the symplectic eigenvalue equation

$$\mathbf{H}\boldsymbol{\psi} = \mu\boldsymbol{\psi}, \tag{22}$$

where μ is an eigenvalue, and $\boldsymbol{\psi}(y)$ is an eigenvector that must satisfy boundary conditions at $y = \pm b$.

It can be proven that eigenvalue zero does not exist for Eq. (22) with typical boundary conditions (6). For eigensolutions of nonzero eigenvalues, Eq. (22) is a system of ordinary differential equations with respect to y, which can be solved by determining eigenvalue λ in y-direction. The corresponding equation is

$$\begin{vmatrix} -\mu & 1 & 0 & 0 \\ -\dfrac{D_{12}}{D_{11}}\lambda^2 & -\mu & 0 & \dfrac{1}{D_{11}} \\ k+\Big(D_{22}-\dfrac{D_{12}^2}{D_{11}}\Big)\lambda^4 & 0 & -\mu & \dfrac{D_{12}}{D_{11}}\lambda^2 \\ 0 & -D_{66}\lambda^2 & -1 & -\mu \end{vmatrix} = 0. \tag{23}$$

Expanding the determinant yields eigenvalue equation

$$D_{22}\lambda^4 + (2D_{12} + D_{66})\lambda^2\mu^2 + D_{11}\mu^4 + k = 0. \tag{24}$$

Assuming that $\mu^4 \neq -k/D_{11}$ and $\mu^4 \neq 4kD_{22}/[(2D_{12} + D_{66})^2 - 4D_{11}D_{22}]$, roots of Eq. (24) must be unequal mutually, i.e. two sets of mutually opposite value. Let

$$\alpha = \sqrt{\frac{1}{2D_{22}}\sqrt{[(2D_{12}+D_{66})^2-4D_{11}D_{22}]\mu^4-4kD_{22}}-\frac{(2D_{12}+D_{66})}{2D_{22}}\mu^2}, \tag{25a}$$

$$\beta = \sqrt{-\frac{1}{2D_{22}}\sqrt{[(2D_{12}+D_{66})^2-4D_{11}D_{22}]\mu^4-4kD_{22}}-\frac{(2D_{12}+D_{66})}{2D_{22}}\mu^2}, \tag{25b}$$

and α, β should satisfy $\mathrm{Re}(\alpha) \geq 0$, $\mathrm{Re}(\beta) \geq 0$, or $\mathrm{Im}(\alpha) \geq 0$ ($\mathrm{Im}(\beta) \geq 0$) when $\mathrm{Re}(\alpha) = 0$ ($\mathrm{Re}(\beta) = 0$). Hence, the general solution of Eq. (22) is

$$\psi = \begin{bmatrix} A_1\mathrm{ch}(\alpha y) + A_2\mathrm{sh}(\alpha y) + A_3\mathrm{ch}(\beta y) + A_4\mathrm{sh}(\beta y) \\ B_1\mathrm{ch}(\alpha y) + B_2\mathrm{sh}(\alpha y) + B_3\mathrm{ch}(\beta y) + B_4\mathrm{sh}(\beta y) \\ C_1\mathrm{ch}(\alpha y) + C_2\mathrm{sh}(\alpha y) + C_3\mathrm{ch}(\beta y) + C_4\mathrm{sh}(\beta y) \\ D_1\mathrm{ch}(\alpha y) + D_2\mathrm{sh}(\alpha y) + D_3\mathrm{ch}(\beta y) + D_4\mathrm{sh}(\beta y) \end{bmatrix}, \tag{26}$$

where constants A_j, B_j, C_j, D_j ($j = 1,2,3,4$) are not all independent. Only four independent constants, e.g. A_j ($j = 1,2,3,4$) are chosen as independent constants. Substituting Eq. (26) into symplectic eigenvalue equation (22) yields relationships between the constants

$$\begin{aligned} &B_j = \mu A_j && (j = 1,2,3,4), \\ &C_j = -\mu(D_{11}\mu^2 + D_{12}\alpha^2 + D_{66}\alpha^2)A_j && (j = 1,2), \\ &C_j = -\mu(D_{11}\mu^2 + D_{12}\beta^2 + D_{66}\beta^2)A_j && (j = 3,4), \\ &D_j = D_{11}\Big(\mu^2 + \frac{D_{12}}{D_{11}}\alpha^2\Big)A_j && (j = 1,2), \\ &D_j = D_{11}\Big(\mu^2 + \frac{D_{12}}{D_{11}}\beta^2\Big)A_j && (j = 3,4). \end{aligned} \tag{27}$$

General solution (26) can divide into two groups: partial solutions relevant to A_j, B_j, C_j, D_j ($j = 1,3$) relate to symmetric deformation on the x-axis and partial solutions relevant to A_j, B_j, C_j, D_j ($j = 2,4$) relate to asymmetric deformation on the x-axis.

In cases where $\mu^4 = -k/D_{11}$ or $\mu^4 = 4kD_{22}/[(2D_{12} + D_{66})^2 - 4D_{11}D_{22}]$, the general solution of Eq. (26) has different forms because Eq. (24) has double roots. Such cases can be discussed similarly to cases discussed in the present paper and are not considered here.

Substituting Eqs. (26) and (27) into homogeneous boundary conditions at $y = \pm b$ yields a homogenous equation for four unknown constants A_j ($j = 1,2,3,4$). Allowing the determinant of its coefficient matrix to vanish gives a transcendental equation for symplectic eigenvalue μ. Solving the transcendental equation and substituting eigenvalue μ_n ($n = 1,2,\cdots$) into the homogenous equation gives the nontrivial solution A_j ($j = 1,2,3,4$), allowing eigenvector ψ_n that corresponds to eigenvalue μ_n to be obtained. Eigenvalue μ_n and expression of eigenvector ψ_n correlates with the specific boundary condition at $y = \pm b$.

After obtaining the eigenvalues and eigenvectors with adjoint symplectic orthogonality property, the general solution of Eq. (12) can be expressed [9]

$$\boldsymbol{v} = \boldsymbol{v}^* + \sum_{n=1}^{\infty}[c_n \exp(\mu_n x)\psi_n], \tag{28}$$

where $\boldsymbol{v}^*$ is a particular solution to transverse load q, which only needs to satisfy Eq. (12) and boundary conditions at $y = \pm b$. Substituting Eq. (28) into boundary conditions at $x = \pm a$, allows constants c_n ($n = 1,2,\cdots$) to be determined and analytical solution to be given.

It is hard to have explicit expression of the particular solution for the complex domain load q or nonhomogeneous boundary conditions at $y = \pm b$. But an expanding form of that expression is possible by applying adjoint symplectic orthogonality property to eigenfunctions [9].

5 Plates with two opposite sides simply supported

In a typical orthotropic plate with two opposite sides that are simply supported and clamped, bending stiffness coefficients are $D_{11} = D$, $D_{12} = 0.31D$, $D_{22} = 11.1D$, $D_{66} = 2.3D$. For a plate with two opposite sides $y = \pm b$ that are simply supported, boundary conditions in terms of a full state vector are

$$w = 0, \quad \frac{D_{12}}{D_{11}}M_x + \Big(D_{22} - \frac{D_{12}^2}{D_{11}}\Big)\partial_{yy}w = 0, \quad \text{at} \quad y = \pm b, \tag{29}$$

This problem divides into two sets, symmetric and asymmetric solutions with respect to the x-axis. The symmetric solution is

$$\psi = \left\{ \begin{array}{c} A_1\mathrm{ch}(\alpha y) + A_3\mathrm{ch}(\beta y) \\ B_1\mathrm{ch}(\alpha y) + B_3\mathrm{ch}(\beta y) \\ C_1\mathrm{ch}(\alpha y) + C_3\mathrm{ch}(\beta y) \\ D_1\mathrm{ch}(\alpha y) + D_3\mathrm{ch}(\beta y) \end{array} \right\}, \tag{30}$$

where constants B_j, C_j, $D_j (j = 1, 3)$ are determined by Eq. (27). Substituting Eq. (30) into the homogeneous boundary condition equation (29) gives

$$\begin{aligned} &\mathrm{ch}(\alpha b)A_1 + \mathrm{ch}(\beta b)A_3 = 0, \\ &(D_{12}\mu^2 + D_{22}\alpha^2)\mathrm{ch}(\alpha b)A_1 \\ &\quad +(D_{12}\mu^2 + D_{22}\beta^2)\mathrm{ch}(\beta b)A_3 = 0. \end{aligned} \tag{31}$$

The determinant of coefficient matrix vanishes in order to allow a nontrivial solution. The transcendental equation of nonzero eigenvalues for symmetric plate deformation with two opposite sides that are simply supported is

$$\mathrm{ch}(\alpha b)\mathrm{ch}(\beta b) = 0. \tag{32}$$

The roots of Eq. (32) are

$$\mu_n b = \pm d \pm \mathrm{i}e, \tag{33}$$

where

$$d = \sqrt{\frac{2D_{12} + D_{66}}{4D_{11}}\left(l + \frac{1}{2}\right)^2 \pi^2 + \frac{1}{2}\sqrt{\frac{D_{22}}{D_{11}}\left[\left(l + \frac{1}{2}\right)^4 \pi^4 + m^4\right]}}, \tag{34a}$$

$$e = \sqrt{d^2 - \frac{2D_{12} + D_{66}}{2D_{11}}\left(l + \frac{1}{2}\right)^2 \pi^2}, \tag{34b}$$

$$m = \sqrt[4]{kb^4/D_{22}}. \tag{34c}$$

For every given nonnegative integer $l (= 0, 1, 2, \cdots)$, Eq. (33) gives one group of four eigenvalues in different quadrants.

Simultaneously, a set of nontrivial solution of A_1, A_3 is specified by

$$A_1 = \mathrm{ch}(\beta b), \quad A_3 = -\mathrm{ch}(\alpha b). \tag{35}$$

Substituting Eq. (35) into Eqs. (26) and (27) produces corresponding eigenvector ψ_n.

For antisymmetric deformation

$$\psi = \left\{ \begin{array}{c} A_2\mathrm{sh}(\alpha y) + A_4\mathrm{sh}(\beta y) \\ B_2\mathrm{sh}(\alpha y) + B_4\mathrm{sh}(\beta y) \\ C_2\mathrm{sh}(\alpha y) + C_4\mathrm{sh}(\beta y) \\ D_2\mathrm{sh}(\alpha y) + D_4\mathrm{sh}(\beta y) \end{array} \right\}, \tag{36}$$

where constants B_j, C_j, D_j $(j = 2, 4)$ are determined by Eq. (27). Substituting Eq. (36) into the homogeneous boundary conditions (29) gives

$$\begin{aligned} &\mathrm{sh}(\alpha b)A_2 + \mathrm{sh}(\beta b)A_4 = 0, \\ &(D_{12}\mu^2 + D_{22}\alpha^2)\mathrm{sh}(\alpha b)A_2 \\ &\quad +(D_{12}\mu^2 + D_{22}\beta^2)\mathrm{sh}(\beta b)A_4 = 0. \end{aligned} \tag{37}$$

The determinant of coefficient matrix vanishes to produce the nontrivial solution. The transcendental equation of nonzero eigenvalues for antisymmetric plate deformation with both opposite sides that are simply supported is

$$\mathrm{sh}(\alpha b)\mathrm{sh}(\beta b) = 0. \tag{38}$$

Roots of the above equation are specified by

$$\mu_{an} b = \pm f \pm \mathrm{i}g, \tag{39}$$

where

$$f = \sqrt{\frac{2D_{12} + D_{66}}{4D_{11}} l^2\pi^2 + \frac{1}{2}\sqrt{\frac{D_{22}}{D_{11}}(l^4\pi^4 + m^4)}}, \tag{40a}$$

$$g = \sqrt{f^2 - \frac{2D_{12} + D_{66}}{2D_{11}} l^2\pi^2}. \tag{40b}$$

For every given positive integer l $(= 1, 2, 3, \cdots)$, Eq. (39) gives a group of four eigenvalues in different quadrants.

Substituting every root μ_{an} into Eq. (37) gives a nontrivial solution of A_2, A_4

$$A_2 = \mathrm{sh}(\beta b), \qquad A_4 = -\mathrm{sh}(\alpha b). \tag{41}$$

Other constants are determined by Eq. (27), allowing the corresponding eigenvector ψ_{an}.

The general solution in the form of Eq. (28) with unknown coefficients $c_n (n = 1, 2, \cdots)$ can be given analytically. Unknown coefficients can be determined by substituting Eq. (28) into boundary conditions at $x = \pm a$.

In practical applications, it is only necessary to solve the first N terms in Eq. (28) [9]

$$\boldsymbol{v} = \boldsymbol{v}^* + \sum_{n=1}^{N} [c_n \exp(\mu_n x)\psi_n]. \tag{42}$$

Expression (42) strictly satisfies basic equations in the domain and boundary conditions at $y = \pm b$, but does not satisfy boundary conditions at $x = \pm a$, so that finite terms can be selected. Only when $N \to \infty$, boundary conditions at $x = \pm a$ in point-point can be satisfied strictly. Here, unknown constants $c_n (n = 1, 2, \cdots, N)$ can be determined by the variation equation of boundary conditions at $x = \pm a$.

Eigenvalues should be selected in ascending order of the modulus at the same time that complex conjugate eigenvalues are selected.

Example 1 A rectangular plate that is fully and simply supported on a Winkler elastic foundation can be solved under uniformly distributed load q. Ratio of the side length is $a/b = 1.5$ and the modulus of Winkler foundation is $k = 200D/b^4$.

A special solution caused by distributed load q in the domain is selected

$$w^*(y) = a_1\mathrm{ch}(ty)\cos(ty) + a_2\mathrm{sh}(ty)\sin(ty) + \frac{q}{k}, \tag{43}$$

where

$$t = \sqrt[4]{k/4D_{22}}, \tag{44}$$

and coefficients a_1 and a_2 are determined by satisfying requirements of the boundary conditions

$$w^*(b) = 0, \qquad M_y^*(b) = 0. \tag{45}$$

The problem is symmetric with respect to the x-axis and the expanded expression can only be constructed from symmetric eigen-solutions of nonzero eigenvalues (30) and (33). Substituting general solutions (42) and (43) into the variational formula for the boundary conditions at $x = \pm a$,

$$\int_{-b}^{b} (w\delta V - M\delta\theta)_{x=-a}^{a} \mathrm{d}y = 0. \tag{46}$$

gives a set of algebraic equations for unknown constants c_n ($n = 1, 2, \cdots, N$), providing analytical solution.

The exact solution of the Navier method

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \tag{47}$$

and the exact solution of the Levy method

$$w = \sum_{m=1}^{\infty} Y_m \sin \frac{m\pi x}{a}. \tag{48}$$

can be applied to solve this problem analytically. Table 1 lists solutions of orthotropic plates given by the Navier method, the Levy method and the new method presented in this paper. The solutions are obtained below, respectively, different expansion terms. The result of the Navier method with 500×500 expansion terms is regarded as a benchmark. Results show that solutions produced by the new method presented in this paper converge more quickly than solutions produced by the Navier method or the Levy method, especially for internal forces. Solutions produced by the new method presented in this paper using $N = 8$ (two groups of eigenvalues) are quite satisfying and results produced by using $N = 12$ (three groups of eigenvalues) are more precise than the Navier method with $N = 80 \times 80$ and the Levy method with $N = 80$, especially for internal forces.

Table 1 Analytical solutions of a plate that is supported fully and simply under uniformly distributed load

	Number of expansion terms	$Dw(0,0)/qb^4$		$M_x(0,0)/qb^2$		$M_y(0,0)/qb^2$		$Dw(a/2,b/2)/qb^4$	
		Solution	Error/%	Solution	Error/%	Solution	Error/%	Solution	Error/%
Present method	4	0.004 766 50	−0.000 056 14	−0.002 428 82	0.004 507 74	−0.116 824 77	−0.000 530 41	0.003 443 48	−0.006 884 22
	8	0.004 766 51	0.000 000 00	−0.002 428 71	0.000 025 42	−0.116 825 39	0.000 000 38	0.003 443 71	−0.000 007 76
	12	0.004 766 51	0.000 000 00	−0.002 428 71	0.000 024 20	−0.116 825 39	0.000 000 37	0.003 443 71	−0.000 000 02
	16	0.004 766 51	0.000 000 00	−0.002 428 71	0.000 024 20	−0.116 825 39	0.000 000 37	0.003 443 71	0.000 000 00
Levy's method	10	0.004 766 35	−0.003 343 94	−0.002 357 28	−2.941 120 00	−0.116 803 25	−0.018 953 71	0.003 443 93	0.006 191 71
	20	0.004 766 50	−0.000 107 45	−0.002 419 68	−0.372 041 81	−0.116 822 59	−0.002 397 47	0.003 443 71	−0.000 206 61
	40	0.004 766 51	−0.000 003 38	−0.002 427 58	−0.046 616 88	−0.116 825 04	−0.000 300 21	0.003 443 71	−0.000 006 58
	80	0.004 766 51	−0.000 000 11	−0.002 428 57	−0.005 810 05	−0.116 825 35	−0.000 037 23	0.003 443 71	−0.000 000 20
Navier's method	10×10	0.004 766 34	−0.003 401 84	−0.002 356 45	−2.975 194 91	−0.116 772 37	−0.045 385 54	0.003 443 93	0.006 305 63
	20×20	0.004 766 50	−0.000 109 34	−0.002 419 57	−0.376 503 07	−0.116 818 63	−0.005 786 63	0.003 443 71	−0.000 210 22
	40×40	0.004 766 51	−0.000 003 44	−0.002 427 57	−0.047 185 94	−0.116 824 54	−0.000 728 10	0.003 443 71	−0.000 006 70
	80×80	0.004 766 51	−0.000 000 11	−0.002 428 57	−0.005 881 85	−0.116 825 29	−0.000 090 94	0.003 443 71	−0.000 000 21
	500×500	0.004 766 51	—	−0.002 428 71	—	−0.116 825 39	—	0.003 443 71	—

When finite expanding terms are selected in the Navier method (47) and the Levy method (48), the Navier solution can strictly satisfy the boundary condition that is fully and simply supported and the Levy solution can satisfy the boundary condition that is simply supported on opposite sides but can not strictly satisfy the basic differential equations in the domain. Convergence rates are very slow, especially for internal forces. In contrast, the solution described in the present paper has finite expanding terms that can strictly satisfy the domain differential equation and boundary conditions at $y = \pm b$, but does not strictly satisfy boundary conditions at $x = \pm a$. Fortunately, with more and more expanding terms are selected, the influence ignored eigen-solutions degrades rapidly due to the existence of exponential term in eigen-solutions.

6 Plates with two opposite sides clamped

For a plate with two opposite sides $y = \pm b$ clamped, boundary conditions in terms of full state vector are

$$w = 0, \quad \partial_y w = 0, \quad \text{at} \quad y = \pm b. \tag{49}$$

This problem divides into two sets, symmetric and asymmetric solutions with the x-axis. Substituting symmetric general solution (30) and formula (27) into the boundary conditions (49) gives

$$\begin{aligned}&\mathrm{ch}(\alpha b)A_1 + \mathrm{ch}(\beta b)A_3 = 0,\\ &\alpha\mathrm{sh}(\alpha b)A_1 + \beta\mathrm{sh}(\beta b)A_3 = 0.\end{aligned} \tag{50}$$

The determinant of coefficient matrix vanishes, allowing the nontrivial solution. Hence, the transcendental equation of nonzero eigenvalues for symmetric plate deformation with two opposite sides clamped is

$$\beta\mathrm{ch}(\alpha b)\mathrm{sh}(\beta b) - \alpha\mathrm{sh}(\alpha b)\mathrm{ch}(\beta b) = 0. \tag{51}$$

Roots μ_n $(n = 1, 2, \cdots)$ of transcendental equation (51) do not have analytic expression as Eqs. (33) and (34), but can be obtained by numerical technique [16]. Substituting root μ_n into Eq. (50) gives nontrivial solution A_1, A_3. Expression for A_1, A_3 is still Eq. (35); eigenvectors of symmetric plate deformation with two opposite sides clamped are still Eq. (30) with expressions of Eqs. (27) and (35), but there are different eigenvalues.

For modulus of Winkler foundation $k = 10D/b^4$, the first eigenvalues of symmetric plate deformation with two opposite sides clamped are in Table 2, with roots in the first quadrant are listed. Each μ_n has a corresponding symplectic adjoint eigenvalue $-\mu_n$ and there are a total of four complex conjugate eigenvalues. Equation (51) shows that these nonzero eigenvalues are all single roots.

Table 2 Eigenvalues of symmetric deformation when opposite sides are clamped

n	1	2	3	4
$\mathrm{Re}(\mu_n b)$	3.425 7	8.283 9	13.145 5	18.008 0
$\mathrm{Im}(\mu_n b)$	2.687 7	5.684 3	8.721 9	11.760 7

The asymmetric transcendental equation and eigenvector for a plate with two opposite sides clamped are left to readers.

Example 2 A fully clamped rectangular plate on Winkler elastic foundation is solved under uniformly distributed load q.

Table 3 Analytical solutions of a fully clamped plate under different modulus of Winkler foundation

		Number of expansion terms	kb^4/D = 200	150	100	20	10	0.01
$\frac{Dw(0,0)}{qb^4}$	Present method	4	0.002 390	0.002 639	0.002 946	0.003 613	0.003 718	0.003 828
		8	0.002 390	0.002 639	0.002 946	0.003 613	0.003 718	0.003 828
		12	0.002 390	0.002 639	0.002 946	0.003 613	0.003 718	0.003 828
	Galerkin method	3×3	0.002 389	0.002 637	0.002 946	0.003 612	0.003 717	0.003 827
		5×5	0.002 389	0.002 639	0.002 946	0.003 613	0.003 718	0.003 828
		6×6	0.002 389	0.002 639	0.002 946	0.003 613	0.003 718	0.003 828
$\frac{M_x(0,0)}{qb^2}$	Present method	4	−0.002 398	−0.002 722	−0.003 146	−0.004 157	−0.004 326	−0.004 507
		8	−0.002 396	−0.002 720	−0.003 144	−0.004 155	−0.004 324	−0.004 505
		12	−0.002 396	−0.002 720	−0.003 144	−0.004 155	−0.004 324	−0.004 505
	Galerkin method	3×3	−0.002 374	−0.002 697	−0.003 119	−0.004 128	−0.004 297	−0.004 478
		5×5	−0.002 386	−0.002 711	−0.003 136	−0.004 150	−0.004 319	−0.004 501
		6×6	−0.002 398	−0.002 721	−0.003 144	−0.004 154	−0.004 322	−0.004 504
$\frac{M_y(0,0)}{qb^2}$	Present method	4	−0.102 011	−0.113 823	−0.128 340	−0.159 938	−0.164 893	−0.170 136
		8	−0.102 008	−0.113 821	−0.128 338	−0.159 939	−0.164 895	−0.170 137
		12	−0.102 008	−0.113 822	−0.128 339	−0.159 939	−0.164 895	−0.170 138
	Galerkin method	3×3	−0.1013 24	−0.113 154	−0.127 690	−0.159 330	−0.164 292	−0.169 542
		5×5	−0.101 943	−0.113 763	−0.128 288	−0.159 905	−0.164 864	−0.170 109
		6×6	−0.102 024	−0.113 834	−0.128 346	−0.159 938	−0.164 893	−0.170 135

A special solution caused by distributed load q in the domain is still Eq. (43) with expression (44), but coefficients a_1 and a_2 are determined by boundary conditions

$$w^*(b) = 0, \quad \theta_y^*(b) = 0. \tag{52}$$

Since the problem is symmetric with respect to the x-axis, the expanded expression can only be constructed from symmetric eigen-solutions (30) with expressions (27) and (35) for nonzero eigenvalues (51). Substituting general solutions (42) and (43) into the following variational formula under the boundary conditions at $x = \pm a$,

$$\int_{-b}^{b} (w\delta V + \theta\delta M)_{x=-a}^{x=a} \mathrm{d}y = 0, \tag{53}$$

gives a set of algebraic equations for unknown constants c_n ($n = 1, 2, \cdots, N$) and an analytical solution.

An approximated Galerkin method [17] with trial function

$$w = (x^2 - a^2)^2 (y^2 - b^2)^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} x^{2m} y^{2n} \tag{54}$$

is used to compare with the method presented in this paper.

For a plate with different modulus of Winkler foundation with length-width ratio $a/b = 1.5$, solutions using the new method described in this paper by using $N = 4, 8, 12$ and solutions using the Galerkin method are in Table 3. Using the new method with $N = 4, 8, 12$ and the Galerkin method, numerical solutions of a plate with Winkler foundation be $k = 10D/b^4$ are listed in Table 4, in which different length-width ratios are considered. Numerical results show excellent agreement. The success of the present analysis indicates that the new method described in this paper can be applied to the clamped boundary condition.

Table 4 Analytical solutions of a fully clamped plate with different length-width ratios

		Number of expansion terms	a/b = 1.0	1.5	2.0	2.5	3.0
$\frac{Dw(0,0)}{qb^4}$	Present method	4	0.003 725	0.003 718	0.003 651	0.003 645	0.003 647
		8	0.003 725	0.003 718	0.003 651	0.003 645	0.003 647
		12	0.003 725	0.003 718	0.003 651	0.003 645	0.003 647
	Galerkin method	3 × 3	0.003 725	0.003 717	0.003 650	0.003 631	0.003 598
		5 × 5	0.003 725	0.003 718	0.003 650	0.003 645	0.003 648
		6 × 6	0.003 725	0.003 718	0.003 651	0.003 646	0.003 647
$\frac{M_x(0,0)}{qb^2}$	Present method	4	−0.011 280	−0.004 326	−0.004 279	−0.004 501	−0.004 521
		8	−0.011 282	−0.004 324	−0.004 279	−0.004 501	−0.004 521
		12	−0.011 283	−0.004 324	−0.004 279	−0.004 501	−0.004 521
	Galerkin method	3 × 3	−0.011 237	−0.004 297	−0.004 246	−0.004 191	−0.003 775
		5 × 5	−0.011 285	−0.004 319	−0.004 263	−0.004 492	−0.004 537
		6 × 6	−0.011 280	−0.004 322	−0.004 282	−0.004 510	−0.004 530
$\frac{M_y(0,0)}{qb^2}$	Present method	4	−0.166 333	−0.164 893	−0.161 740	−0.161 531	−0.161 617
		8	−0.166 357	−0.164 895	−0.161 739	−0.161 531	−0.161 617
		12	−0.166 357	−0.164 895	−0.161 739	−0.161 531	−0.161 617
	Galerkin method	3 × 3	−0.166 131	−0.164 292	−0.160 827	−0.159 868	−0.158 366
		5 × 5	−0.166 362	−0.164 864	−0.161 613	−0.161 291	−0.161 310
		6 × 6	−0.166 343	−0.164 893	−0.161 763	−0.161 613	−0.161 771

7 Conclusions

Based on a symplectic system, the new analytical method for rectangular orthotropic plates on Winkler elastic foundation presented in this paper is superior to the methods of Navier and Levy, which can only be applied to plates with opposite sides simply supported. The new approach is more complex than other methods, but can be used in any combination of conventional boundary conditions. Numerical examples show two merits of the new method:

(1) Analytical solutions of symplectic expansion form have good convergence and precision, especially for internal forces. These solutions leave ample room for authentication of benchmarks produced by numerical or approximation methods. Solutions of the new method by using $N = 8$ (two groups of eigenvalues are selected) are particularly striking.

(2) The symplectic method can solve not only the bending of an orthotropic plate with two opposite sides supported simply but also any other boundary condition. Besides, the new method is also effective for dynamic and stable problems of plates.

The present work expands the application of symplectic system and proves that symplectic methodology is a valid analytical method. Besides, this method can also be applied to free to force vibration of plates and shells, it will be reported in the future.

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