Acta Mech. Sin. (2011) 27(6):950–955
DOI 10.1007/s10409-011-0499-8

RESEARCH PAPER

Stress analysis of functionally graded rotating discs: analytical and numerical solutions

Hasan Çallioğlu · Numan Behlül Bektaş · Metin Sayer

Received: 1 December 2010 / Revised: 27 January 2011 / Accepted: 21 March 2011

Abstract This study deals with stress analysis of annular rotating discs made of functionally graded materials (FGMs). Elasticity modulus and density of the discs are assumed to vary radially according to a power law function, but the material is of constant Poisson's ratio. A gradient parameter n is chosen between 0 and 1.0. When $n = 0$, the disc becomes a homogeneous isotropic material. Tangential and radial stress distributions and displacements on the disc are investigated for various gradient parameters n by means of the diverse elasticity modulus and density by using analytical and numerical solutions. Finally, a homogenous tangential stress distribution and the lowest radial stresses along the radius of a rotating disc are approximately obtained for the gradient parameter $n = 1.0$ compared with the homogeneous, isotropic case $n = 0$. This means that a disc made of FGMs has the capability of higher angular rotations compared with the homogeneous isotropic disc.

Keywords Functional graded materials · Stress analysis · Analytical analysis · Finite element analysis (FEA)

1 Introduction

Many materials with mechanical and physical properties that vary continuously as a function of position within the materials occur in nature and they are known as continuously non-homogeneous materials. The mechanical benefits obtained by a material gradient may be significant, as can be seen by the excellent structural performance of some of these materials. These materials are sometimes called functionally graded materials (FGMs) when synthetic materials are graded for specific applications [1]. FGMs are being used as an interfacial zone to improve the bonding strength of layered composites in order to reduce the residual and thermal stresses in bonded dissimilar materials and as wear resistant layers in machine and engine components [2, 3]. Therefore, composites made of FGMs have been attracting considerable attention in recent years.

H. Çallioğlu · N.B. Bektaş · M. Sayer (✉)
Department of Mechanical Engineering,
Pamukkale University, Kinikli Campus,
20020, Denizli, Turkey
e-mail: msayer@pau.edu.tr

With increasing demand to achieve high strength to weight ratios, optimizing the geometrical and physical properties of the disc configuration becomes more significant. A closed form solution for the stress analysis in a homogeneous isotropic rotating disc or a disc under pressure can be found in Ref. [4]. Chiba [5] analytically derived the second-order statistics in an axisymmetrically heated, functionally graded, annular disc with spatially random heat transfer coefficients on the upper and lower surfaces using an integral transform method and a perturbation method. Wang and Noda [6] investigated the fracture behavior of a cracked, functionally graded actuator on a substrate under thermal load. They utilized the integral transform method in the numerical solution. Sladek et al. [7] presented a meshless local boundary integral equation method (LBIEM) for dynamic analysis of an anti-plane crack in functionally graded materials. Peng and Li [8] analyzed thermoelastic problem of a rotating, functionally graded, hollow circular disk. The numerical results obtained by using the Fredholm integral equation were presented graphically to show the effects of gradient parameter, temperature change, angular velocity and thickness of the disk on the distribution of thermal stresses and radial displacement. Sladek et al. [9] derived local integral equations (LIE) for numerical solutions of 3D problems in linear elasticity of functionally graded materials viewed as 2D axisymmetric problems. The convergence and accuracy of these numerical methods are investigated using the exact solution for a functionally graded hollow cylinder subjected to internal pressure. Singh and Ray [10] investigated creep in an orthotropic aluminum-silicon carbide composite rotating disc by using Hill's anisotropic yield. In that study, the results obtained have been compared with the results obtained using von Mises yield criterion for the isotropic, heterogeneous composites.

Much of the work on FGMs has generally been carried out numerically, e.g. using the finite element method (FEM) and the boundary element method (BEM). Nevertheless, the mechanical and mathematical modeling of FGMs is currently an active research area. Analytical solutions to benchmark problems provide invaluable checks on the accuracy of numerical or approximate analytical schemes and allow for widely applicable parametric studies.

Horgan and Chan [11] investigated the stress response on a pressurized hollow cylinder or disk made of functionally graded, isotropic, linearly elastic materials. They investigated a body with Young's modulus varying radially only. Ying and Wang [12] presented an analytical solution for a rotating multiferroic composite hollow cylinder made of radially polarized piezoelectric and piezomagnetic materials. Zenkour [13] dealt with a solution for a rotating annular disk which is assumed to be graded in the radial direction according to a simple exponential-law distribution. Durodola and Attia [14] investigated deformation and stresses in functionally graded rotating disks. They compared two methods, the finite element method (ABAQUS) and direct numerical integration of governing differential equations, with each other. In these papers in general, the modulus varies in radial direction whereas the density remains constant. Çallioğlu [15] studied the stress of functionally graded rotating discs subjected to internal and external pressures.

In the present study, elasticity modulus E and density ρ of the discs are a function of radius r, so that those materials are called functionally graded materials. However, Poisson's ratio is assumed to be constant, because its variation has much less practical significance than that of the elasticity modulus and density. So, the discs can be classified as radial orthotropic material discs. For functionally graded annular rotating discs at constant angular velocity, closed-form solutions using the infinitesimal theory of elasticity are obtained by means of a FORTRAN® program presented. For the numerical solution, a commercial finite element program ANSYS® is used. Functionally graded rotating discs are modeled by axisymmetric element, Plane42 in ANSYS® [16]. In the solutions for both methods, the power law function is used, and the gradient parameter n is chosen between 0 and 1.0. Results obtained from analytical and numerical solutions are compared with each other and are in good agreement.

2 Stress analysis

The governing differential equation of equilibrium for a thin rotating disc is

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \frac{\sigma_r - \sigma_\theta}{r} + \rho(r)\omega^2 r = 0, \tag{1}$$

where $\rho(r)$ is density of the material and it is assumed to vary in the radial direction, and ω is the angular velocity of the disc.

Due to the rotational symmetry, the strain-displacement relations are given by

$$\varepsilon_r = \frac{\mathrm{d}u}{\mathrm{d}r}, \qquad \varepsilon_\theta = \frac{u}{r}, \tag{2}$$

where u is the displacement component in the radial direction. The strain compatibility equation is

$$\varepsilon_r = \frac{\mathrm{d}}{\mathrm{d}r}(r\varepsilon_\theta). \tag{3}$$

The strain–stress relation can be given by

$$\begin{aligned} \varepsilon_r &= \frac{1}{E(r)}(\sigma_r - \nu\sigma_\theta), \\ \varepsilon_\theta &= \frac{1}{E(r)}(\sigma_\theta - \nu\sigma_r). \end{aligned} \tag{4}$$

The material elasticity modulus E is assumed to vary along the radius of the disc. Since variation of Poisson's ratio has much less practical significance than that of the elasticity modulus and density, Poisson's ratio ν is assumed as a constant value.

The equilibrium equation (1) is satisfied by the stress function F defined as

$$\sigma_r = \frac{F}{r}, \qquad \sigma_\theta = \frac{\mathrm{d}F}{\mathrm{d}r} + \rho(r)\omega^2 r^2. \tag{5}$$

Substituting Eqs. (4) and (5) into compatibility Eq. (3) yields

$$\begin{aligned} &r^2F'' + rF'\left(1 - r\frac{E'(r)}{E(r)}\right) - F\left(1 - \nu r\frac{E'(r)}{E(r)}\right) \\ &\quad = -\rho(r)\omega^2 r^3\left(3 + \nu - r\frac{E'(r)}{E(r)}\right) - \rho'(r)\omega^2 r^4. \end{aligned} \tag{6}$$

Now suppose that

$$E(r) = E\left(\frac{r}{b}\right)^n, \qquad \rho(r) = \rho\left(\frac{r}{b}\right)^n, \tag{7}$$

where n is gradient parameter, and then the differential equation (6) reduces to

$$r^2F'' + rF'(1-n) - F(1-\nu n) = -\frac{\rho\omega^2 r^{n+3}(3+\nu)}{b^n}. \tag{8}$$

As $n = 0$, Eq. (8) reduces to the equation of the homogeneous, isotropic disc [3],

$$r^2F'' + rF' - F = -\rho\omega^2 r^3(3+\nu). \tag{9}$$

The stress function, F, can be written as

$$F = C_1 r^{\frac{n+m}{2}} + C_2 r^{\frac{n-m}{2}} + Cr^{n+3}, \tag{10}$$

where, C_1 and C_2 are the integration constants and the positive constant m is

$$m = (n^2 - 4\nu n + 4)^{\frac{1}{2}}, \tag{11}$$

and the term C is

$$C = -\rho\omega^2 \frac{3+\nu}{b^n(8+3n+\nu n)}. \tag{12}$$

The stress components can be obtained from the stress function as

$$\sigma_r = C_1 r^{\frac{n+m-2}{2}} + C_2 r^{\frac{n-m-2}{2}} + Cr^{n+2}, \tag{13}$$

$$\sigma_\theta = \left(\frac{n+m}{2}\right) C_1 r^{\frac{n+m-2}{2}} + \left(\frac{n-m}{2}\right) C_2 r^{\frac{n-m-2}{2}} + (n+3)Cr^{n+2} + \frac{\rho\omega^2 r^{n+2}}{b^n}. \tag{14}$$

The integration constants, C_1 and C_2, can be obtained from the boundary conditions. Since the discs are subjected to angular velocity only, the radial stresses at the inner and the outer surfaces of the disc are equal to zero, i.e. $\sigma_r = 0$ at $r = a$ and $r = b$. Here a and b are, respectively, inner and outer radii of disc illustrated in Fig. 1. By using these conditions, C_1 and C_2 are determined as

$$C_1 = \frac{C\left(a^{\frac{n+m+6}{2}} - b^{\frac{n+m+6}{2}}\right)}{b^m - a^m}, \quad C_2 = \frac{C\left(a^m b^{\frac{n+m+6}{2}} - b^m a^{\frac{n+m+6}{2}}\right)}{b^m - a^m}. \tag{15}$$

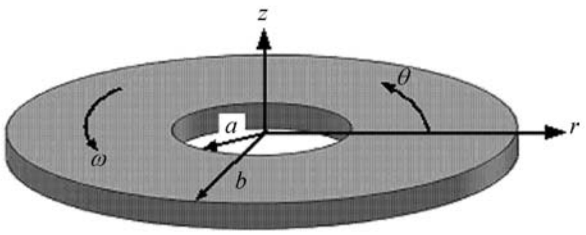

Fig. 1 A functionally graded rotating disc

2.1 Radial displacement component

Radial displacement, u, in the elastic solution for small deformation can be determined from Eq. (2) as

$$u = \frac{r}{E(r)}(\sigma_\theta - \nu\sigma_r). \tag{16}$$

3 Analytical and numerical solutions

In this work, a stress analysis is carried out on annular rotating discs made of FGMs by using analytical and numerical methods. The inner and outer radii of the disc are $a = 40$ mm and $b = 100$ mm, and the thickness of the disc is ignored since it is very small ($t \ll r$), as shown in Fig. 1. For the numerical method, the disc is modeled and meshed by an axisymmetric element (Plane42 2D Structural Solid) in ANSYS®, which is a commercial finite element program, as can be seen in Fig. 2. In mesh refinement, an element is utilized per millimeter of the radius. This means that 60 axisymmetric elements are utilized for 60 mm in the radial direction, as seen in Fig. 2b. The disc material can be assumed as a functionally graded material, which can be made of various contents of Al and ceramic powder particles using powder metallurgy and also called an aluminum metal matrix composite. It is considered that elasticity modulus and density vary in the radial direction in accordance with Eq. (7). So that, each mesh element has different material properties. Thus, in order to apply the theory, an aluminum alloy (7075-T6) is selected as a disc material and its elasticity modulus is $E = 72\,000$ MPa, and mass density is $\rho = 2.8\,\mathrm{t/m^3}$. The gradient parameter n is chosen as 0, 0.5 and 1.0. For $n = 0$, the disc becomes an isotropic, homogeneous disc. Stresses, displacements, variations of elasticity modulus and density in the radial direction are obtained by using both analytical and numerical solutions for angular velocity, $\omega = 15\,000\,\mathrm{min}^{-1}$.

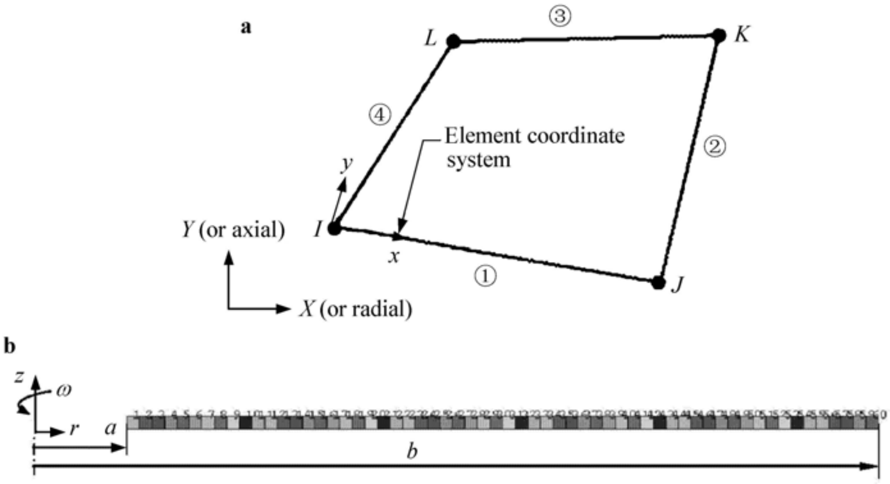

Fig. 2 **a** Plane42 geometry and **b** Cross-section of disc meshed by 60 axisymmetric elements

As for boundary conditions of the disc, they are assumed to be free at the inner and outer radii.

Figure 3 illustrates variations of the elasticity modulus $E(r)$ in the radial direction of the disc for $n = 0$, $n = 0.5$ and $n = 1.0$. For $n = 0$, elasticity modulus is a constant value in the radial direction. This indicates a homogeneous, isotropic material case. For $n = 0.5$ and $n = 1.0$, the elasticity modulus decreases approximately linearly throughout, from outer surface to inner surface of the disc depending on the function, $E(r) = E(r/b)^n$. They are used for analytical and numerical solutions.

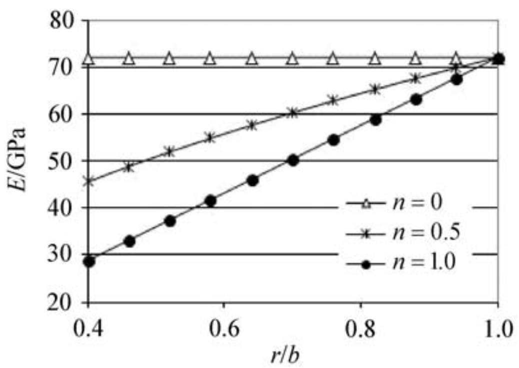

Fig. 3 Variations of elasticity modulus $E(r)$ in the radial direction of disc for $n = 0, n$=0.5 and $n = 1.0$

Figure 4 illustrates variations of the density $\rho(r)$ in the radial direction of the disc for $n = 0$, $n = 0.5$, $n = 1.0$. For $n = 0$, the density is a constant value in the radial direction. For $n = 0.5$ and $n = 1.0$, the density decreases approximately linearly from the outer surface to the inner surface of the disc depending on the function, $\rho(r) = \rho(r/b)^n$. Values of density for $n = 1.0$ are lower than the others. They are also used for analytical and numerical solutions. If it is given as $n = -0.5$ and $n = -1.0$, elasticity moduli and densities of the discs are going to be increased about linearly from the inner surface to outer surface.

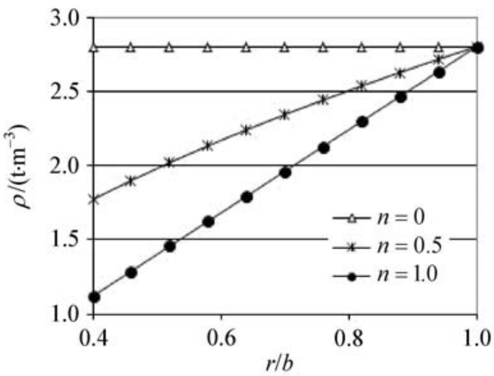

Fig. 4 Variations of density $\rho(r)$ in the radial direction of disc for $n = 0, n$=0.5 and $n = 1.0$

The radial stress, tangential stress and radial displacements obtained by ANSYS® are shown on a half model of a disc in color contour graphs in Fig. 5 for $n = 0.5$. It can be seen from Figs. 6–8 that radial stress, tangential stress and radial displacement obtained from analytical and numerical solutions are in good agreement.

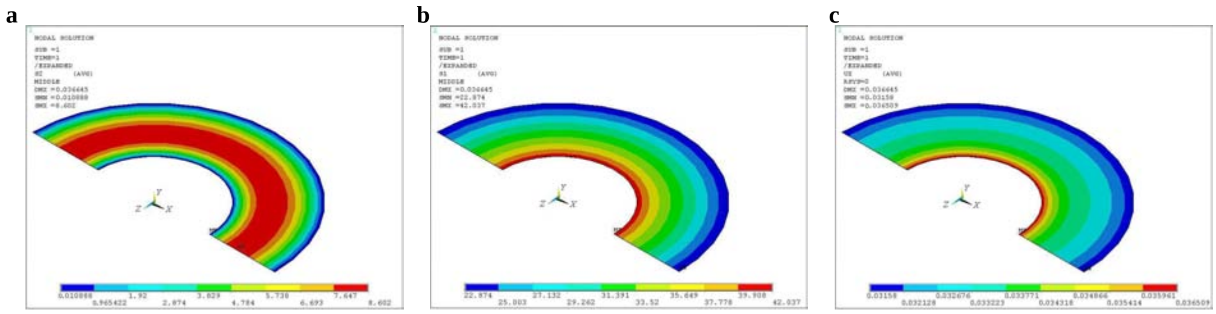

Fig. 5 Variations of the **a** Radial stress σ_r; **b** Tangential stress σ_θ; **c** Radial displacement u in the radial direction of disc for $n = 0.5$ from ANSYS® solution

In Fig. 6, variations of radial stresses σ_r obtained from analytical and numerical solutions, are shown in the radial direction of the disc for $n = 0$, $n = 0.5$ and $n = 1.0$. They are zero at the inner and outer surfaces but they are higher at radius $r \cong (a+b)/2$ for $n = 0.5$, $n = 1.0$ and $r = \sqrt{ab}$ for $n = 0$. They decrease gradually with an increase in the gradient parameter n. Radial stresses have lower values for $n = 1.0$ when compared with the results for $n = 0$ and $n = 0.5$. This is the result of being FGMs of the disc. On the other hand, it increases rotating capacity of the disc.

In Fig. 7, variations of tangential stresses σ_θ are shown in the radial direction of the disc for $n = 0$, $n = 0.5$ and $n = 1.0$. They decrease gradually at the inner surface, while increasing very little at the outer surface with increasing gradient parameter n. They are also higher at the inner surface than those at the outer surface. Tangential stresses get nearly constant values in the radial direction of the disc for $n = 1.0$, and they are also lower values until $r/b \cong 0.85$ when compared with the results for $n = 0$ and $n = 0.5$. This means that the rotating capacity of disc can be increased approximately twice as much.

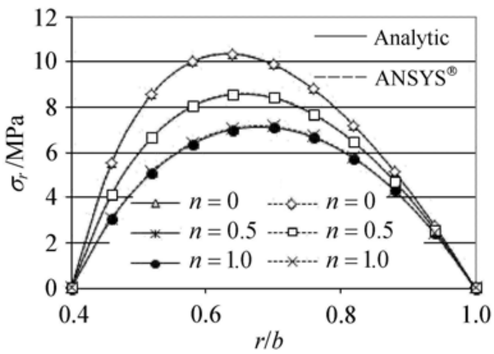

Fig. 6 Variations of radial stress σ_r in the radial direction of disc for $n = 0, n$=0.5 and $n = 1.0$

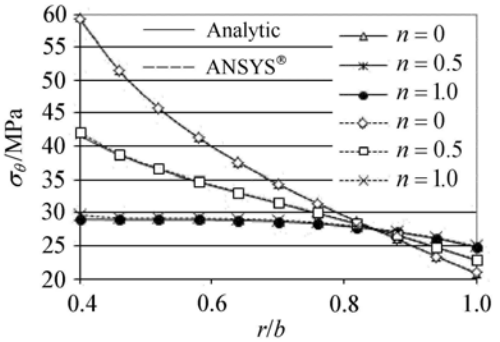

Fig. 7 Variations of tangential stress σ_θ in the radial direction of disc for $n = 0, n$=0.5 and $n = 1.0$

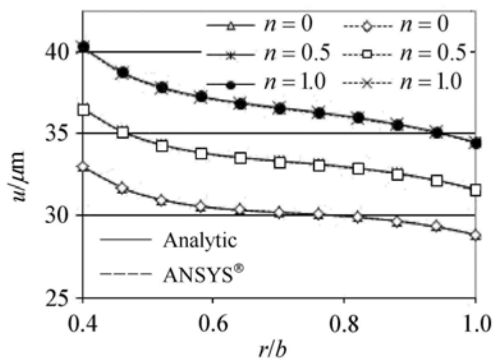

Fig. 8 Variations of displacement u in the radial direction of disc for $n = 0, n$=0.5 and $n = 1.0$

Figure 8 shows variations of displacement u in the radial direction of a disc for $n = 0$, $n = 0.5$ and $n = 1.0$. All displacements are higher at the inner surface, and vary with parallel curves in the radial direction of the disc. They increase gradually with an increase in the gradient parameter n. Values of displacements for $n = 1.0$ are higher than the others.

4 Conclusions

In the present study, tangential and radial stress equations are derived from the governing differential equation of equilibrium for a thin rotating disc including elasticity modulus E and mass density ρ assumed as a function of r which represents FGMs. To apply the theory, an aluminum alloy material with FGM is chosen. Results obtained are concluded as following:

(1) It is found that the stresses and displacements obtained from the analytical and numerical solutions are in good agreement.
(2) Radial stress σ_r and tangential stress σ_θ are varying in the radial direction of the disc depending on the power law function for elasticity modulus and mass density. Accordingly, radial stresses are zero at the inner and outer surfaces but they are of higher values at the interior parts of the discs. They get lower values for $n = 1.0$ when compared with the results for $n = 0$ and $n = 0.5$. As for tangential stresses, they are higher at the inner surface for $n = 0$ and $n = 0.5$ than at the outer surface. For $n = 1.0$ they get nearly constant values in the radial direction of the disc, and they are of also lower values when compared with the results for $n = 0$ and $n = 0.5$.
(3) The radial displacements increase gradually with a parallel curve with increasing the gradient parameter n. As a result of decreasing the tangential stresses, rotating capacity of the disc is increased nearly twice as much.

Acknowledgements The authors would like to thank Pamukkale University Scientific Research Council supporting this study under Project Contract No. 2008FBE006 and 2010FBE096.

References

1 Neubrand, A., Rödel, J.: Gradient materials: An overview of a novel concept. Zeit. Metall. **88**(5), 358–371 (1997)
2 Erdoğan, F.: Fracture mechanics of functionally graded materials. Compos. Eng. **5**(7), 753–770 (1995)
3 Pindera, M.J., Aboudi, J., Glaeser, A.M, et al.: Use of Composites in Multi-phased and Gunctionally Graded Materials. Composites Part B-Eng **28**(1/2), 1–175 (1997)
4 Timoshenko, S.P., Goodier, J.N.: Theory of Elasticity. (3rd edn). McGraw Hill Book Company, California (1970)
5 Chiba, R.: Stochastic heat conduction analysis of a functionally graded annular disc with spatially random heat transfer coefficients. Appl. Math. Model **33**(1), 507–523 (2009)
6 Wang, B.L., Noda, N.: Thermally induced fracture of a smart functionally graded composite structure. Theor. Appl. Fract. Mec. **35**(2), 93–109 (2001)

7 Sladek, J., Sladek, V., Zhang, C.: A meshless local boundary integral equation method for dynamic anti-plane shear crack problem in functionally graded materials. Eng. Anal. Bound. Elem. **29**(4), 334–342 (2005)
8 Peng, X.L., Li, X.F.: Thermal stress in rotating functionally graded hollow circular disks. Compos. Struct. **92**(8), 1896–1904 (2010)
9 Sladek, V., Sladek, J., Zhang, C.: Local integral equation formulation for axially symmetric problems involving elastic FGM. Eng. Anal. Bound. Elem. **32**(12), 1012–1024 (2008)
10 Singh, S.B., Ray. S.: Modeling the anisotropy and creep in orthotropic aluminum-silicon carbide composite rotating disc. Mech. Mater. **34**(6), 363–372 (2002)
11 Horgan, C.O., Chan, A.M.: The pressurized hollow cylinder or disk problem for functionally graded isotropic linearly elastic materials. J. Elasticity **55**(1), 43–59 (1999)
12 Ying, J., Wang, H.M.: Magnetoelectroelastic fields in rotating multiferroic composite cylindrical structures. J. Zhejiang Univ. Sci. A **10**(3), 319–326 (2009)
13 Zenkour, A.M.: Elastic deformation of the rotating functionally graded annular disk with rigid casing. J. Mater. Sci. **42**(23), 9717–9724 (2007)
14 Durodola, J.F., Attia, O.: Deformation and stresses in functionally graded rotating disks. Compos. Sci. Technol. **60**(7), 987–995 (2000)
15 Çallioğlu, H.: Stress analysis of functionally graded isotropic rotating discs. Adv. Compos. Lett. **17**(5), 147–153 (2008)
16 ANSYS® 10.0 Procedures. Engineering Analysis System Verification Manual, vol. 1. Houston, PA, USA: Swanson Analysis System Inc. (1993)