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RESEARCH PAPER

The method of variation on parameters for integration of a generalized Birkhoffian system

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Abstract This paper focuses on studying the integration method of a generalized Birkhoffian system. The method of variation on parameters for the dynamical equations of a generalized Birkhoffian system is presented. The procedure for solving the problem can be divided into two steps: the first step, a system of auxiliary equations is constructed and its general solution is given; the second step, the parameters are varied, and the solution of the problem is obtained by using the properties of generalized canonical transformation. The method of variation on parameters for the generalized Birkhoffian system is of universal significance, and we take a nonholonomic system and a nonconservative system as examples to illustrate the application of the results of this paper.

Keywords Birkhoffian system · Method of integration · Variation on parameter · Nonholonomic constraint · Nonconservative system

1 Introduction

The integration problems of a dynamical system are major issues worthy of continuous research [1]. The classical integration methods for dynamical equations of analytical mechanics include Routh reduction method, Whittaker reduction method, Poisson theorem, canonical transformations, Hamilton-Jacobi method, and integral invariants and so on [1–4]. The diversity of forces and the complexity of constraints increase the difficulty of integration. It is an important part of analytical mechanics to research integration methods of complex mechanical systems: one reason is to extend the classical integration methods of holonomic conservative systems to the nonconservative or nonholonomic systems to some extent [5, 6]; the other is to develop modern methods, such as the gradient method and the field method and their extensions [7, 8], Noether theorem and its extensions [9–11], the applications of Lie theory [12–14], Mei symmetry and its applications [15, 16] and so on.

Birkhoffian mechanics is a natural generalization of Hamiltonian mechanics [17, 18] its core is the Pfaff–Birkhoff principle and Birkhoff's equations. Hamilton's principle is a special form of the Pfaff–Birkhoff principle, Hamilton canonical equations are a special form of Birkhoff's equations. Hamilton canonical equations remain the same under a canonical transformation, and become Birkhoff's equations under a general non-canonical transformation. Therefore, the theory of Birkhoffian mechanics is applicable to Hamiltonian mechanics, Lagrangian mechanics and Newtonian mechanics, and also applicable to general holonomic and nonholonomic mechanics [18]. At the same time, Birkhoffian mechanics can also be applied to quantum mechanics, statistical mechanics, atomic and molecular physics, hadron physics, biological physics and engineering and other fields [17]. In 1989, Galiullan [19] pointed out that it is an important developmental direction of modern analytical mechanics to study Birkhoffian mechanics. And the integration theory of the Birkhoffian system is an important aspect for the study of Birkhoffian dynamics [18]. Santilli extended the Hamilton-Jacobi method to the Birkhoffian system on the basis of an in-depth study of the transformation theory of Birkhoff's equations [17]. Galiullan et al. studied the inverse problems of dynamics of the Birkhoffian system, the symmetry and the integral invariants of the Birkhoffian system and so on [19, 20]. Mei et al. [18] established the Pois-

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son theory of the Birkhoffian system, the field method for integrating Birkhoff's equations, and the inverse problems of dynamics of Birkhoffian mechanics and so on. Recently, Zheng and Mei [21] studied the first integrals and reduction of the Birkhoffian system. Zhang [22] extended the Routh method of reduction of holonomic systems to the Birkhoffian system. Ge [23] extended the time-integral theorems of nonholonomic systems to the Birkhoffian system [24]. Li and Mei [1, 25] extended the method of the Jacobi last multiplier for solving dynamical equations of holonomic conservative systems to the generalized Hamilton system [26] and the generalized Birkhoffian system [27]. Liu [28] presented a general method for solving nonholonomic systems with non-Chetaev's constraints, i.e. the method of variation on parameters. Zhang and Han [29] extended the method to a system of generalized classical mechanics. In this paper we will further apply the method of variation on parameters to solve the integration issues of the generalized Birkhoffian system. The method of variation on parameters of the Birkhoffian system is of universal importance. We bring a nonholonomic system and a nonconservative system into the generalized Birkhoffian system, and apply the results of this paper to the nonholonomic system and the nonconservative system.

2 Differential equations of motion of the system

The differential equations of motion of a generalized Birkhoffian system are [30]

$$\Omega_{\mu\nu}\dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} = -\Lambda_\mu, \quad \mu,\nu = 1,2,\cdots,2n, \tag{1}$$

where $B = B(\boldsymbol{a}, t)$ is called a Birkhoffian, $R_\mu = R_\mu(\boldsymbol{a}, t)$ are Birkhoff's functions, and the arbitrary differentiable functions $\Lambda_\mu = \Lambda_\mu(\boldsymbol{a}, t)$ are called additional items and

$$\Omega_{\mu\nu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}, \tag{2}$$

is called Birkhoff's tensor. A mechanical system whose motion can be described by Birkhoff's equations (1), or a physical system whose state can be described by Birkhoff's equations (1), is called a generalized Birkhoffian system [31]. When $\Lambda_\mu = 0$ ($\mu = 1, 2, \cdots, 2n$), Birkhoff's equations (1) become the differential equations of motion of a Birkhoffian system.

Suppose that the system (1) is nonsingular, i.e. $\det(\Omega_{\mu\nu}) \neq 0$, then all of $\dot{a}^\mu$ can be solved from Eq. (1), and we get

$$\dot{a}^\mu = \Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t} - \Lambda_\mu\right), \tag{3}$$

where $\Omega^{\mu\nu}\Omega_{\nu\tau} = \delta_{\mu\tau}$.

3 Method of variation on parameters of the system

The procedure for solving Birkhoff's equation (3) with the method of variation of the parameters can be divided into two steps.

In the first step, we construct an auxiliary system and find its general solution. Let the auxiliary equations corresponding to Eq. (3) be

$$\dot{a}^\mu = \Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right). \tag{4}$$

Assuming that a general solution of Eqs. (4) is

$$a^\mu = a^\mu(\alpha^1, \alpha^2, \cdots, \alpha^{2n}, t), \tag{5}$$

where α^μ are the constants of integration which are the value of a^μ when $t = 0$. Without loss of generality, we take α^μ as new variables, and make a variable substitution by Eq. (5), and choose

$$\begin{aligned} &B^* = B^*(\boldsymbol{\alpha}, t) = \left(B - \frac{\partial a^\nu}{\partial t}R_\nu\right)(\boldsymbol{\alpha}, t),\\ &R^*_\mu = R^*_\mu(\boldsymbol{\alpha}, t) = \left(\frac{\partial a^\nu}{\partial \alpha^\mu}R_\nu\right)(\boldsymbol{\alpha}, t),\\ &\Omega^*_{\mu\nu} = \frac{\partial R^*_\nu}{\partial \alpha^\mu} - \frac{\partial R^*_\mu}{\partial \alpha^\nu},\\ &\Omega^{\mu\nu*}\Omega^*_{\nu\tau} = \delta_{\mu\tau}. \end{aligned} \tag{6}$$

Then we can easily obtain [17]

$$\dot{\alpha}^\mu = \Omega^{\mu\nu*}\left(\frac{\partial B^*}{\partial \alpha^\nu} + \frac{\partial R^*_\nu}{\partial t}\right). \tag{7}$$

The transformation equation (5) is a generalized canonical transformation, and we have [17]

$$\frac{\partial \alpha^\mu}{\partial a^\rho}\Omega^{\rho\sigma}\frac{\partial \alpha^\nu}{\partial a^\sigma} \equiv \Omega^{\mu\nu*}. \tag{8}$$

According to the properties of generalized canonical transformation, the transformation equation (5) has an inverse transformation and we assume that

$$\alpha^\mu = \alpha^\mu(a^1, a^2, \cdots, a^{2n}, t). \tag{9}$$

Since the Formula (9) is a first integral of Eqs. (4), we have

$$\frac{\partial \alpha^\mu}{\partial t} + \frac{\partial \alpha^\mu}{\partial a^\nu}\Omega^{\nu\rho}\left(\frac{\partial B}{\partial a^\rho} + \frac{\partial R_\rho}{\partial t}\right) = 0. \tag{10}$$

In the second step, we vary the parameters, and work out a solution of Birkhoff's equation (3). Suppose the solution of Eq. (3) still has the form of the Formula (5), and α^μ here is no more a constant but a function of variables $\boldsymbol{a}$ and time t. Differentiating Eq. (9) with respect to time t, we get

$$\begin{aligned} \dot{\alpha}^\mu &= \frac{\partial \alpha^\mu}{\partial t} + \frac{\partial \alpha^\mu}{\partial a^\nu}\dot{a}^\nu\\ &= -\frac{\partial \alpha^\mu}{\partial a^\nu}\Omega^{\nu\rho}\left(\frac{\partial B}{\partial a^\rho} + \frac{\partial R_\rho}{\partial t}\right)\\ &\quad + \frac{\partial \alpha^\mu}{\partial a^\nu}\Omega^{\nu\rho}\left(\frac{\partial B}{\partial a^\rho} + \frac{\partial R_\rho}{\partial t} - \Lambda_\rho\right)\\ &= -\frac{\partial \alpha^\mu}{\partial a^\nu}\Omega^{\nu\rho}\Lambda_\rho. \end{aligned} \tag{11}$$

By using the relation (8), Eq. (11) can be written as follows

$$\dot{\alpha}^{\mu} = -\Omega^{\mu\sigma*}\frac{\partial a^{\rho}}{\partial \alpha^{\sigma}}\Lambda_{\rho}. \tag{12}$$

Hence we obtain

$$\alpha^{\mu} = \alpha_0^{\mu} - \int \Omega^{\mu\sigma*}\frac{\partial a^{\rho}}{\partial \alpha^{\sigma}}\Lambda_{\rho}\mathrm{d}t. \tag{13}$$

In the discussion above, $\mu = 1, 2, \cdots, 2n$. Substituting the Formula (13) into Eq. (5), we obtain the solution of the differential equations (3) of motion of the generalized Birkhoffian system. Therefore we have:

Proposition 1. For a generalized Birkhoffian system (1), if the Formula (5) is a general solution of auxiliary equations (4), then the general solution of Eq. (3) for the generalized Birkhoffian system can be written as the Formula (5), in which α^{μ} is determined by the Formula (13).

Example 1. Consider a generalized Birkhoffian system, whose Birkhoffian and Birkhoff's functions are

$$\begin{aligned} &B = \frac{1}{2}[(a^3)^2 + (a^4)^2], \\ &R_1 = a^3, \quad R_2 = a^4, \quad R_3 = 0, \quad R_4 = 0, \end{aligned} \tag{14}$$

and the additional items are

$$\begin{aligned} &\Lambda_1 = 0, \quad \Lambda_2 = 0, \quad \Lambda_3 = \frac{1}{b}\arctan bt, \\ &\Lambda_4 = \frac{1}{2b}\ln(1 + b^2t^2). \end{aligned} \tag{15}$$

Let us try to find the motion of the system with the method of variation on parameters.

It can be divided into two steps to solve this problem. First of all, we construct an auxiliary system and find its solution. For this problem, Eq. (4) give

$$\dot{a}^1 = a^3, \quad \dot{a}^2 = a^4, \quad \dot{a}^3 = 0, \quad \dot{a}^4 = 0. \tag{16}$$

The solution of Eq. (16) is

$$\begin{aligned} &a^1 = \alpha^1 + \alpha^3 t, \qquad a^2 = \alpha^2 + \alpha^4 t, \\ &a^3 = \alpha^3, \qquad\qquad a^4 = \alpha^4, \end{aligned} \tag{17}$$

where α^{μ} ($\mu = 1, 2, \cdots, 4$) are the constants of integration.

Second, we make a variation of constants, and find the motion of the generalized Birkhoffian system. Equation (12) give

$$\begin{aligned} &\dot{\alpha}^1 = -\frac{1}{b}\arctan bt, \quad \dot{\alpha}^2 = -\frac{1}{2b}\ln(1 + b^2t^2), \\ &\dot{\alpha}^3 = 0, \qquad\qquad \dot{\alpha}^4 = 0. \end{aligned} \tag{18}$$

Integrating Eq. (18), we have

$$\begin{aligned} &\alpha^1 = \alpha_0^1 - \frac{t}{b}\arctan bt + \frac{1}{2b^2}\ln(1 + b^2t^2), \\ &\alpha^2 = \alpha_0^2 - \frac{1}{b^2}\arctan bt + \frac{t}{b} - \frac{t}{2b}\ln(1 + b^2t^2), \\ &\alpha^3 = \alpha_0^3, \\ &\alpha^4 = \alpha_0^4. \end{aligned} \tag{19}$$

Substituting Eq. (19) into Eq. (17), we obtain

$$\begin{aligned} &a^1 = \alpha_0^1 + \alpha_0^3 t - \frac{t}{b}\arctan bt + \frac{1}{2b^2}\ln(1 + b^2t^2), \\ &a^2 = \alpha_0^2 + \alpha_0^4 t - \frac{1}{b^2}\arctan bt + \frac{t}{b} - \frac{t}{2b}\ln(1 + b^2t^2), \\ &a^3 = \alpha_0^3, \\ &a^4 = \alpha_0^4. \end{aligned} \tag{20}$$

Equation (20) are the general solution of the problem under consideration, which are in correspondence with the results given by Mei using the field method [18].

4 The application of the results

As Birkhoffian mechanics is a generalization of Hamiltonian mechanics, and all holonomic systems and nonholonomic systems can be brought into Birkhoffian systems [18], and the method of variation on parameters is a basic method for solving differential equations, so this method of variation on parameters for generalized Birkhoffian system is of universal significance. Here we only take a nonholonomic system and a nonconservative system as examples to illustrate the application of the above results.

First of all, let us study on a nonholonomic system.

Suppose that the configuration of a mechanical system is determined by n generalized coordinates q_s ($s = 1, 2, \cdots, n$). The system is subjected to g ideal nonholonomic constraints of Chetaev's type

$$f_{\beta}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0, \qquad \beta = 1, 2, \cdots, g. \tag{21}$$

The differential equations of motion of the system can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = \lambda_{\beta}\frac{\partial f_{\beta}}{\partial \dot{q}_s}, \qquad s = 1, 2, \cdots, n, \tag{22}$$

where L is the Lagrangian of the system, λ_{β} are the constraint multipliers. Assuming that the system is non-singular, we can seek λ_{β} as the functions of $t, \boldsymbol{q}, \dot{\boldsymbol{q}}$ before integrating the differential equations of motion [2].

Introducing the generalized momenta and the Hamiltonian

$$p_s = \frac{\partial L}{\partial \dot{q}_s}, \qquad H = p_s\dot{q}_s - L, \tag{23}$$

then Eqs. (22) can be written partially in the canonical form

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \qquad \dot{p}_s = -\frac{\partial H}{\partial q_s} + P_s, \qquad s = 1, 2, \cdots, n, \tag{24}$$

where

$$P_s = P_s(t, \boldsymbol{q}, \boldsymbol{p}) = \lambda_{\beta}(t, \boldsymbol{q}, \boldsymbol{p})\frac{\partial f_{\beta}}{\partial \dot{q}_s}(t, \boldsymbol{q}, \boldsymbol{p}).$$

Let

$$a^{\mu} = \begin{cases} q_{\mu}, & \mu = 1, 2, \cdots, n, \\ p_{\mu-n}, & \mu = n+1,\ n+2,\ \cdots,\ 2n, \end{cases} \tag{25}$$

$$R_\mu = \begin{cases} p_\mu, & \mu = 1, 2, \cdots, n, \\ 0, & \mu = n+1, n+2, \cdots, 2n, \end{cases} \tag{26}$$

$$\Lambda_\mu = \begin{cases} \tilde{P}_\mu, & \mu = 1, 2, \cdots, n, \\ 0, & \mu = n+1, n+2, \cdots, 2n, \end{cases} \tag{27}$$

where $\tilde{P}_\mu = \tilde{P}_\mu(t, \boldsymbol{a}) = P_\mu(t, \boldsymbol{q}(\boldsymbol{a}), \boldsymbol{p}(\boldsymbol{a}))$, then Eqs. (24) can be written in the contravariant algebraic form

$$\dot{a}^\mu = \Omega^{\mu\nu}\left(\frac{\partial H}{\partial a^\nu} - \Lambda_\nu\right), \quad \mu = 1, 2, \cdots, 2n. \tag{28}$$

The motion of the nonholonomic systems (21) and (22) can be found in the motions of system (28), if only the initial conditions satisfy the nonholonomic constraints equation (21), i.e.

$$\tilde{f}_\beta(t, a_0^\mu) = 0, \quad \beta = 1, 2, \cdots, g. \tag{29}$$

The auxiliary equations corresponding to Eq. (28) is

$$\dot{a}^\mu = \Omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}, \quad \mu = 1, 2, \cdots, 2n. \tag{30}$$

Therefore we have:

Proposition 2. For a nonholonomic systems (21) and (22), if the Formula (5) is a general solution of auxiliary equations (30), then the general solution of equations for the nonholonomic system can be written as the Formula (5), in which α^μ is determined by the Formula (13) and the initial conditions satisfy the condition equation (29).

Example 2. As an example, we study the inertial motion of Chaplygin sled [32]. The Lagrangian of the system is

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}k\dot{\varphi}^2, \tag{31}$$

where k is a constant. The nonholonomic constraint equation is

$$\dot{y} = \dot{x}\tan\varphi. \tag{32}$$

Let $q_1 = x$, $q_2 = y$, $q_3 = \varphi$, the differential equations (22) of motion give

$$\ddot{q}_1 = -\lambda\tan q_3, \quad \ddot{q}_2 = \lambda, \quad k\ddot{q}_3 = 0. \tag{33}$$

From Eqs. (32) and (33), we can easily find the multiplier as

$$\lambda = \dot{q}_1\dot{q}_3. \tag{34}$$

The generalized momenta and the Hamiltonian of the system are, respectively,

$$\begin{aligned} &p_1 = \dot{q}_1, \quad p_2 = \dot{q}_2, \quad p_3 = k\dot{q}_3, \\ &H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2k}p_3^2. \end{aligned} \tag{35}$$

The forces corresponding to the nonholonomic constraint (32) are

$$\begin{aligned} &P_1 = -\frac{1}{k}p_1 p_3\tan q_3, \\ &P_2 = \frac{1}{k}p_1 p_3, \\ &P_3 = 0. \end{aligned} \tag{36}$$

Let $a^1 = q_1$, $a^2 = q_2$, $a^3 = q_3$, $a^4 = p_1$, $a^5 = p_2$, $a^6 = p_3$, we have

$$\begin{aligned} &H = \frac{1}{2}[(a^4)^2 + (a^5)^2] + \frac{1}{2k}(a^6)^2, \\ &R_1 = a^4, \qquad R_2 = a^5, \\ &R_3 = a^6, \qquad R_4 = 0, \\ &R_5 = 0, \qquad R_6 = 0, \\ &\Lambda_1 = -\frac{1}{k}a^4a^6\tan a^3, \qquad \Lambda_2 = \frac{1}{k}a^4a^6, \\ &\Lambda_3 = 0, \qquad \Lambda_4 = 0, \\ &\Lambda_5 = 0, \qquad \Lambda_6 = 0. \end{aligned} \tag{37}$$

The auxiliary equations (30) give

$$\begin{aligned} &\dot{a}^1 = a^4, \quad \dot{a}^2 = a^5, \quad \dot{a}^3 = \frac{1}{k}a^6, \\ &\dot{a}^4 = 0, \quad \dot{a}^5 = 0, \quad \dot{a}^6 = 0. \end{aligned} \tag{38}$$

Equation (38) have a following solution

$$\begin{aligned} &a^1 = \alpha^1 + \alpha^4 t, \qquad a^2 = \alpha^2 + \alpha^5 t, \\ &a^3 = \alpha^3 + \frac{1}{k}\alpha^6 t, \qquad a^4 = \alpha^4, \\ &a^5 = \alpha^5, \qquad a^6 = \alpha^6, \end{aligned} \tag{39}$$

where α^μ ($\mu = 1, 2, \cdots, 6$) are the constants of integration.

Equation (12) give

$$\begin{aligned} &\dot{\alpha}^1 = \frac{t}{k}\alpha^4\alpha^6\tan\left(\alpha^3 + \frac{1}{k}\alpha^6 t\right), \\ &\dot{\alpha}^2 = -\frac{t}{k}\alpha^4\alpha^6, \\ &\dot{\alpha}^3 = 0, \\ &\dot{\alpha}^4 = -\frac{1}{k}\alpha^4\alpha^6\tan\left(\alpha^3 + \frac{1}{k}\alpha^6 t\right), \\ &\dot{\alpha}^5 = \frac{1}{k}\alpha^4\alpha^6, \\ &\dot{\alpha}^6 = 0. \end{aligned} \tag{40}$$

Integrating Eqs. (40), we have

$$\begin{aligned} \alpha^1 = {}& \alpha_0^1 + \frac{k\alpha_0^4}{\alpha_0^6\cos\alpha_0^3}\sin\left(\alpha_0^3 + \frac{\alpha_0^6}{k}t\right) \\ & - \frac{\alpha_0^4 t}{\cos\alpha_0^3}\cos\left(\alpha_0^3 + \frac{\alpha_0^6}{k}t\right) - \frac{k\alpha_0^4}{\alpha_0^6}\tan\alpha_0^3, \\ \alpha^2 = {}& \alpha_0^2 - \frac{k\alpha_0^4}{\alpha_0^6\cos\alpha_0^3}\cos\left(\alpha_0^3 + \frac{\alpha_0^6}{k}t\right) \\ & - \frac{\alpha_0^4 t}{\cos\alpha_0^3}\sin\left(\alpha_0^3 + \frac{\alpha_0^6}{k}t\right) + \frac{k\alpha_0^4}{\alpha_0^6}, \\ \alpha^3 = {}& \alpha_0^3, \\ \alpha^4 = {}& \frac{\alpha_0^4}{\cos\alpha_0^3}\cos\left(\alpha_0^3 + \frac{\alpha_0^6}{k}t\right), \end{aligned} \tag{41}$$

$$\alpha^5 = \alpha_0^5 + \frac{\alpha_0^4}{\cos\alpha_0^3}\sin\left(\alpha_0^3 + \frac{\alpha_0^6}{k}t\right) - \alpha_0^4\tan\alpha_0^3,$$
$$\alpha^6 = \alpha_0^6.$$

Substituting Eqs. (41) into Eqs. (39), we obtain

$$\begin{aligned}
a^1 &= \alpha_0^1 + \frac{k\alpha_0^4}{\alpha_0^6\cos\alpha_0^3}\sin\left(\alpha_0^3 + \frac{\alpha_0^6}{k}t\right) - \frac{k\alpha_0^4}{\alpha_0^6}\tan\alpha_0^3,\\
a^2 &= \alpha_0^2 + \alpha_0^5 t - \frac{k\alpha_0^4}{\alpha_0^6\cos\alpha_0^3}\cos\left(\alpha_0^3 + \frac{\alpha_0^6}{k}t\right)\\
&\quad + \frac{k\alpha_0^4}{\alpha_0^6} - \alpha_0^4 t\tan\alpha_0^3,\\
a^3 &= \alpha_0^3 + \frac{1}{k}\alpha_0^6 t,\\
a^4 &= \frac{\alpha_0^4}{\cos\alpha_0^3}\cos\left(\alpha_0^3 + \frac{\alpha_0^6}{k}t\right),\\
a^5 &= \alpha_0^5 + \frac{\alpha_0^4}{\cos\alpha_0^3}\sin\left(\alpha_0^3 + \frac{\alpha_0^6}{k}t\right) - \alpha_0^4\tan\alpha_0^3,\\
a^6 &= \alpha_0^6.
\end{aligned} \tag{42}$$

The condition equation (32) gives

$$\alpha_0^5 = \alpha_0^4\tan\alpha_0^3. \tag{43}$$

Therefore, the solution of the problem is given by Formulae (42) and (43), in which there are five independent constants. This solution is in correspondence with the results given by Ref. [32].

Second, let us study on a nonconservative system.

The differential equations of motion of the system can be written as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q_s} - \frac{\partial L}{\partial q_s} = Q_s'', \qquad s = 1, 2, \cdots, n. \tag{44}$$

Equation (44) can still be written in the contravariant algebraic form

$$\dot a^\mu = \Omega^{\mu\nu}\left(\frac{\partial H}{\partial a^\nu} - \Lambda_\nu\right), \qquad \mu = 1, 2, \cdots, 2n, \tag{45}$$

where

$$\begin{aligned}
&\Lambda_s = \tilde Q_s'' = \tilde Q_s''(t, \boldsymbol{a}) = Q_s''(t, \boldsymbol{q}(\boldsymbol{a}), \boldsymbol{p}(\boldsymbol{a})),\\
&\Lambda_{n+s} = 0, \qquad s = 1, 2, \cdots, n.
\end{aligned} \tag{46}$$

The auxiliary equations corresponding to Eq. (45) are

$$\dot a^\mu = \Omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}, \quad \mu = 1, 2, \cdots, 2n. \tag{47}$$

Therefore we have:

Proposition 3. For a nonconservative system (44), if the Formula (5) is a general solution of auxiliary equations (47), then the general solution of Eqs. (45) for the nonconservative system can be written as the Formula (5), in which α^μ is determined by the Formula (13).

Example 3. Let us study a nonconservative system as follows [7]

$$L = \frac{1}{2}(\dot q_1^2 + \dot q_2^2), \qquad Q_1'' = -\dot q_2, \qquad Q_2'' = -\dot q_1. \tag{48}$$

Let us try to find the motion of the system with the above method.

From Eq. (48), the generalized momenta and the Hamiltonian of the system are

$$p_1 = \dot q_1, \qquad p_2 = \dot q_2, \qquad H = \frac{1}{2}(p_1^2 + p_2^2). \tag{49}$$

The nonconservative forces are

$$Q_1'' = -p_2, \qquad Q_2'' = -p_1. \tag{50}$$

Let $a^1 = q_1$, $a^2 = q_2$, $a^3 = p_1$, $a^4 = p_2$, then we have

$$\begin{aligned}
&H = \frac{1}{2}[(a^3)^2 + (a^4)^2],\\
&R_1 = a^3, && R_2 = a^4,\\
&R_3 = 0, && R_4 = 0,\\
&\Lambda_1 = \tilde Q_1'' = -a^4, && \Lambda_2 = \tilde Q_2'' = -a^3,\\
&\Lambda_3 = 0, && \Lambda_4 = 0.
\end{aligned} \tag{51}$$

The auxiliary equations corresponding to Eq. (47) give

$$\begin{aligned}
&\dot a^1 = a^3, \qquad \dot a^2 = a^4,\\
&\dot a^3 = 0, \qquad \dot a^4 = 0.
\end{aligned} \tag{52}$$

The solution of Eqs. (52) is

$$\begin{aligned}
&a^1 = \alpha^1 + \alpha^3 t, \qquad a^2 = \alpha^2 + \alpha^4 t,\\
&a^3 = \alpha^3, \qquad a^4 = \alpha^4,
\end{aligned} \tag{53}$$

where α^μ $(\mu = 1, 2, 3, 4)$ are the constants of integration.

Equation (12) gives

$$\begin{aligned}
&\dot\alpha^1 = \alpha^4 t, \qquad \dot\alpha^2 = \alpha^3 t,\\
&\dot\alpha^3 = -\alpha^4, \qquad \dot\alpha^4 = -\alpha^3.
\end{aligned} \tag{54}$$

Integrating Eq. (54), we have

$$\begin{aligned}
\alpha^1 &= \alpha_0^1 - \alpha_0^3(t\mathrm{ch}t - \mathrm{sh}t) + \alpha_0^4(1 - \mathrm{ch}t + t\mathrm{sh}t),\\
\alpha^2 &= \alpha_0^2 + \alpha_0^3(1 - \mathrm{ch}t + t\mathrm{sh}t) - \alpha_0^4(t\mathrm{ch}t - \mathrm{sh}t),\\
\alpha^3 &= \alpha_0^3\mathrm{ch}t - \alpha_0^4\mathrm{sh}t,\\
\alpha^4 &= -\alpha_0^3\mathrm{sh}t + \alpha_0^4\mathrm{ch}t.
\end{aligned} \tag{55}$$

Substituting Eq. (55) into Eq. (53), we obtain

$$\begin{aligned}
a^1 &= \alpha_0^1 + \alpha_0^3\mathrm{sh}t + \alpha_0^4(1 - \mathrm{ch}t),\\
a^2 &= \alpha_0^2 + \alpha_0^3(1 - \mathrm{ch}t) + \alpha_0^4\mathrm{sh}t,\\
a^3 &= \alpha_0^3\mathrm{ch}t - \alpha_0^4\mathrm{sh}t,\\
a^4 &= \alpha_0^4\mathrm{ch}t - \alpha_0^3\mathrm{sh}t.
\end{aligned} \tag{56}$$

The Formulae (56) are the general solution of the problem. This solution is in correspondence with the results given by Vujanović using the gradient method in Ref. [7]. For this problem, our method is simpler than the gradient method.

5 Conclusion

The method of variation on parameters for integrating a generalized Birkhoffian system is presented. With the method of variation on parameters, the integration of a generalized Birkhoffian system can be carried out in two steps. The first step, an auxiliary system is constructed and solved. The corresponding free Birkhoffian system is chosen as an auxiliary system, and its general solution is found. The second step, the variation on parameters is given, and the problem comes down to solving Eq. (12). The method of variation on parameters is a basic method for solving differential equations. A Birkhoffian system is a more universal dynamical system than a Hamiltonian system, and the integration theory of a Birkhoffian system is not only suitable for Hamiltonian and Lagrangian systems, also suitable for the general holonomic systems and the nonholonomic systems. In this paper, we bring nonholonomic systems and nonconservative systems into generalized Birkhoffian systems, and apply the method of variation on parameters for a generalized Birkhoffian systems to nonholonomic systems and nonconservative systems. The method and results of this paper are of general significance, which can be further used in other constrained systems, such as the system of generalized mechanics, the system of electromechanical dynamics and so on.

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