Faster Algorithms for Approximate Common Divisors: Breaking Fully-Homomorphic-Encryption Challenges over the Integers

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Abstract

At EUROCRYPT '10, van Dijk, Gentry, Halevi and Vaikuntanathan presented simple fully-homomorphic encryption (FHE) schemes based on the hardness of approximate integer common divisors problems, which were introduced in 2001 by Howgrave-Graham. There are two versions for these problems: the partial version (PACD) and the general version (GACD). The seemingly easier problem PACD was recently used by Coron, Mandal, Naccache and Tibouchi at CRYPTO '11 to build a more efficient variant of the FHE scheme by van Dijk *et al.*. We present a new PACD algorithm whose running time is essentially the "square root" of that of exhaustive search, which was the best attack in practice. This allows us to experimentally break the FHE challenges proposed by Coron *et al.* Our PACD algorithm directly gives rise to a new GACD algorithm, which is exponentially faster than exhaustive search: namely, the running time is essentially the 3/4-th root of that of exhaustive search. Interestingly, our main technique can also be applied to other settings, such as noisy factoring and attacking low-exponent RSA.

1 Introduction

Following Gentry's breakthrough work [6], there is currently great interest on fully-homomorphic encryption (FHE), which allows to compute arbitrary functions on encrypted data. Among the few FHE schemes known [6, 25, 5, 2, 8], the simplest one is arguably the one of van Dijk, Gentry, Halevi and Vaikuntanathan [25] (vDGHV), published at EUROCRYPT '10. The security of the vDGHV scheme is based on the hardness of *approximate integer common divisors problems* introduced in 2001 by Howgrave-Graham [12]. In the general version of this problem (GACD), the goal is to recover a secret number p (typically a large prime number), given polynomially many near-multiples $x_0, \ldots, x_m$ of p, that is, each integer x_i is of the hidden form $x_i = pq_i + r_i$ where each q_i is a very large integer and each r_i is a very small integer. In the partial version of this problem (PACD), the setting is exactly the same, except that x_0 is chosen as an exact multiple of p, namely $x_0 = pq_0$ where q_0 is a very large integer chosen such that no non-trivial factor of x_0 can be found efficiently: for instance, [5] selects q_0 as a rough number, *i.e.* without any small prime factor.

By definition, PACD cannot be harder than GACD, and intuitively, it seems that it should be easier than GACD. However, van Dijk *et al.* [25] mention that there is currently no PACD algorithm that does not work for GACD. And the usefulness of PACD is demonstrated by the recent construction [5], where Coron, Mandal, Naccache and Tibouchi built a much more efficient variant of the FHE scheme by van Dijk *et al.* [25], whose security relies on PACD rather than GACD. Thus, it is very important to know if PACD is actually easier than GACD.

The hardness of PACD and GACD depends on how the q_i's and the r_i's are exactly generated. For the generation of [25] and [5], the noise r_i is extremely small, and the best attack known is simply gcd exhaustive

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search: for GACD, this means trying every noise (r_0, r_1) and check whether $\gcd(x_0 - r_0, x_1 - r_1)$ is sufficiently large and allows to recover the secret key; for PACD, this means trying every noise r_1 and check whether $\gcd(x_0, x_1 - r_1)$ is sufficiently large and allows to recover the secret key. In other words, if ρ is the bit-size of the noise r_i, then breaking GACD (resp. PACD) requires $2^{2\rho}$ (resp. 2^{ρ}) polynomial-time operations, for the parameters of [25, 5].

Our results. We present new algorithms to solve PACD and GACD, which are exponentially faster in theory and practice than the best algorithms considered in [25, 5]. More precisely, the running time of our new PACD algorithm is $2^{\rho/2}$ polynomial-time operations, which is essentially the "square root" of that of gcd exhaustive search. This directly leads to a new GACD algorithm running in $2^{3\rho/2}$ polynomial-time operations, which is essentially the 3/4-th root of that of gcd exhaustive search. Our PACD algorithm relies on classical algorithms to evaluate univariate polynomials at many points, whose space requirements are not negligible. We therefore present additional tricks, some of which reduce the space requirements, while still providing substantial speedups. This allows us to experimentally break the FHE challenges proposed by Coron *et al.* in [5], which were assumed to have comparable security to the FHE challenges proposed by Gentry and Halevi in [7]: the latter challenges are based on hard problems with ideal lattices. Table 1 gives benchmarks for our attack on the FHE challenges, and deduces speedups compared to gcd exhaustive search.

Table 1: Time required to break the FHE challenges by Coron *et al.* [5]. Size in bits, running time in seconds for a single 2.27GHz-core with 72Gb of RAM. The implementation parameters of the new attack are suboptimal for Medium and Large challenges.

Name	Toy	Small	Medium	Large
Size(public key)	0.95Mb	9.6Mb	89Mb	802Mb
Size(modulus)	$1.6 * 10^5$	$0.86 * 10^6$	$4.2 * 10^6$	$19 * 10^6$
Size(noise)	17	25	33	40
Expected security level	≥ 42	≥ 52	≥ 62	≥ 72
Running time of gcd-search	2420	8323848	19648616386	17950077079257
	40 mins	96 days	623 years	569193 years
Concrete security level	≈ 42	≈ 54	≈ 65	≈ 75
Running time of the new attack	99	25665	1.635×10^7	6.79×10^{10}
	1.6 min	7.1 hours	190 days	2153 years
Parameters	$d = 2^8$	$d = 2^{12}$	$d = 2^{13}$	$d = 2^{10}$
Speedup	24	324	1202	264
New security level	≤ 37.7	≤ 45.7	≤ 55	≤ 67

We also apply our technique to different settings, such as noisy factoring and attacking low-exponent RSA encryption. A typical example of noisy factoring is the following: assume that p is a divisor of a public modulus N, and that one is given a noisy version p' of p, which differs from p by at most k bits at unknown positions, can one recover p from (p', N) faster than exhaustive search? This may have applications in side-channel attacks. Like in the PACD setting, we obtain a square-root attack. Similarly, we speed up several exhaustive search attacks on low-exponent RSA encryption.

Related work. Multipoint evaluation of univariate polynomials has been used in public-key cryptanalysis before. For instance, it is used in factoring (such as in the Pollard-Strassen factorization algorithm [19, 24] or in ECM speedup [16]), in the folklore square-root attack on RSA with small CRT exponents (mentioned by Boneh and Durfee [1], and described in [20, 17]), as well as in the recent square-root attack [4] by Coron, Joux, Mandal, Naccache and Tibouchi on Groth's RSA Subgroup Assumption [10]. But this does not imply that our attack is trivial, especially since the authors of [5] form a subset of the authors of [4]. In fact, in most cryptanalytic applications (including [4]) of multipoint evaluation, one is actually interested in the following problem: given two lists $\{a_i\}_i$ and $\{b_j\}_j$ of numbers modulo N, find a pair (a_i, b_j) such that $\gcd(a_i - b_j, N)$ is non-trivial. Instead, we use multipoint evaluation differently, as a way to compute certain products of m elements modulo N in $\tilde{O}(\sqrt{m})$ polynomial-time operations, where $\tilde{O}()$ is the usual notation hiding poylogarithmic terms. More precisely, it applies to products $\prod_{i=1}^{m} x_i \bmod N$ which can be rewritten under the form $\prod_{j=1}^{m_1} \prod_{k=1}^{m_2} (y_j + z_k) \bmod N$ where both m_1 and m_2 are $O(\sqrt{m})$. The Pollard-Strassen factorization algorithm [19, 24] can be viewed as a special case of this technique: it computes $m! \bmod N$ to factor N.

Roadmap. In Sect. 2, we describe our square-root algorithm for PACD, and apply it to GACD. In Sect. 3, we discuss implementation issues, present several tricks to speed up the PACD algorithm in practice, and we discuss the impact of our algorithm on the fully-homomorphic challenges of Coron *et al.* [5]. Finally, we apply our main technique to different settings: noisy factoring (Sect. 4) and attacking low-exponent RSA (Sect. 5).

2 A Square-Root Algorithm for Partial Approximate Common Divisors

In this section, we describe our new square-root algorithm for the PACD problem, which is based on evaluating univariate polynomials at many points. In the last subsection, we apply it to GACD.

2.1 Overview

Consider an instance of PACD:

$$\begin{aligned} x_0 &= pq_0 \\ x_i &= pq_i + r_i \text{ where } 0 \le r_i < 2^\rho, 1 \le i \le m \end{aligned}$$

We start with the following basic observation due to Nguyen (as reported in [5, Sect 6.1]):

$$p = \gcd\left(x_0, \prod_{i=0}^{2^\rho - 1} (x_1 - i) \pmod{x_0}\right) \tag{1}$$

At first sight, this observation only allows to replace 2^ρ gcd computations (with numbers of size $\approx \gamma$ bits) with essentially 2^ρ modular multiplications (where the modulus has $\approx \gamma$ bits): the benchmarks of [5] report a speedup of ≈ 5 for the FHE challenges, which is insufficient to impact security estimates.

However, we observe that (1) can be exploited in a much more powerful way as follows. We define the polynomial $f_j(x)$ of degree j, with coefficients modulo x_0:

$$f_j(x) = \prod_{i=0}^{j-1} (x_1 - (x + i)) \pmod{x_0} \tag{2}$$

Letting $\rho' = \lfloor \rho/2 \rfloor$, we notice that:

$$\prod_{i=0}^{2^\rho - 1} (x_1 - i) \equiv \prod_{k=0}^{2^{\rho' + (\rho \bmod 2)} - 1} f_{2^{\rho'}}(2^{\rho'} k) \pmod{x_0}.$$

We can thus rewrite (1) as:

$$p = \gcd\left(x_0, \prod_{k=0}^{2^{\rho' + (\rho \bmod 2)} - 1} f_{2^{\rho'}}(2^{\rho'} k) \pmod{x_0}\right) \tag{3}$$

Clearly, (3) allows to solve PACD using one gcd, $2^{\rho' + (\rho \bmod 2)} - 1$ modular multiplications, and the multi-evaluation of a polynomial (with coefficients modulo x_0) of degree $2^{\rho'}$ at $2^{\rho' + (\rho \bmod 2)}$ points, where $\rho' + (\rho \bmod 2) = \rho - \rho'$. We claim that this costs at most $\tilde{O}(2^{\rho'}) = \tilde{O}(\sqrt{2^\rho})$ operations modulo x_0, which is essentially the square root of gcd exhaustive search. This is obvious for the single gcd and the modular multiplications. For the multi-evaluation part, it suffices to use classical algorithms (see [26, 14]) which evaluate a polynomial of degree d at d points, using at most $\tilde{O}(d)$ operations in the coefficient ring. Here, we also need to compute the polynomial $f_{2^{\rho'}}(x)$ explicitly, which can fortunately also be done using $\tilde{O}(\sqrt{2^\rho})$ operations modulo x_0. We give a detailed description of the algorithms in the next subsection.

2.2 Description

We first recall our algorithm to solve PACD, given as Alg. 1, and which was implicitly presented in the overview.

Algorithm 1 Solving PACD by multipoint evaluation of univariate polynomials

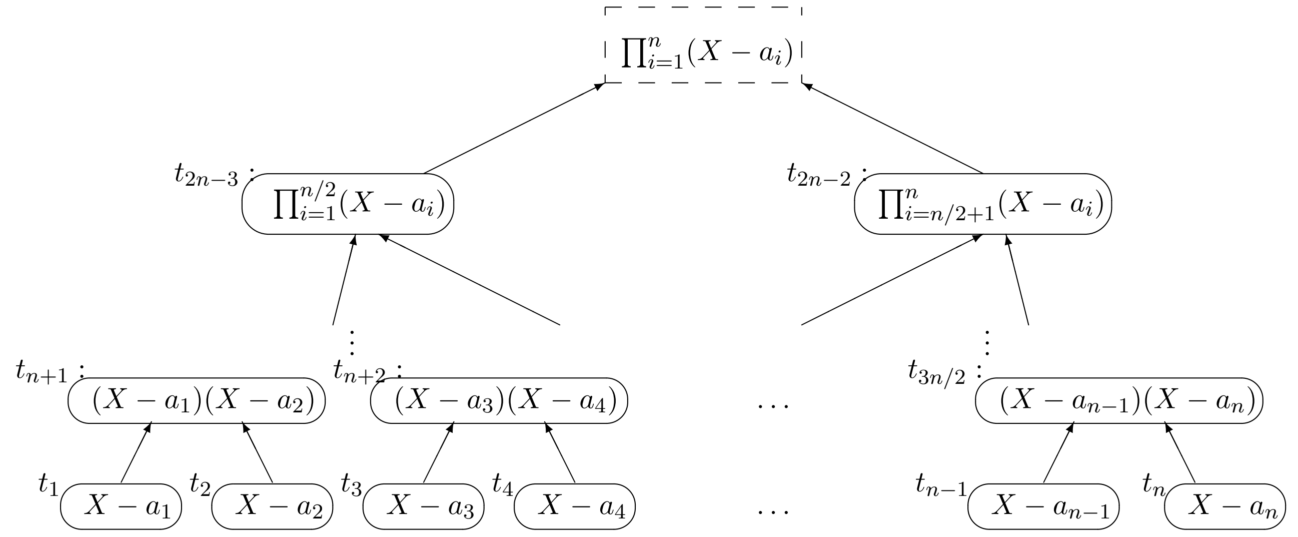
```
Input: An instance $(x_0, x_1)$ of the PACD problem with noise size $\rho$.
Output: The secret number $p$ such that $x_0 = pq_0$ and $x_1 = pq_1 + r_1$ with appropriate sizes.
1: Set $\rho' \leftarrow \lfloor \rho/2 \rfloor$.
2: Compute the polynomial $f_{2^{\rho'}}(x)$ defined by (2), using Alg. 2.
3: Compute the evaluation of $f_{2^{\rho'}}(x)$ at the $2^{\rho'+(\rho \bmod 2)}$ points $0, 2^{\rho'}, \ldots, 2^{\rho'}(2^{\rho'+(\rho \bmod 2)} - 1)$, using $2^{\rho \bmod 2}$
   times Alg. 3 with $2^{\rho'}$ points. Each application of Alg. 3 requires the computation of a product tree, using
   Alg. 2.
```

Alg. 1 relies on two classical subroutines (see [26, 14]):

- a subroutine to (efficiently) compute a polynomial given as a product of n terms, where n is a power of two: Alg. 2 does this in $\tilde{O}(n)$ ring operations, provided that quasi-linear multiplication of polynomials is available, which can be achieved in our case using Fast Fourier techniques. This subroutine is used in Step 2. The efficiency of Alg. 2 comes from the fact that when the algorithm requires a multiplication, it only multiplies polynomials of similar degree.

- a subroutine to (efficiently) evaluate a univariate degree-n polynomial at n points, where n is a power of two: Alg. 3 does this in $\tilde{O}(n)$ ring operations, provided that quasi-linear polynomial remainder is available, which can be achieved in our case using Fast Fourier techniques. This subroutine is used in Step 3, and requires the computation of a tree product, which is achieved by Alg. 2. Alg. 3 is based on the well-known fact that the evaluation of a univariate polynomial at a point α is the same as its remainder modulo $X - \alpha$, which allows to factor computations using a tree.

Figure 1: Polynomial product tree $T = \{t_1, \ldots, t_{2n}\}$ for $\{a_1, \ldots, a_n\}$.


$\prod_{i=1}^{n}(X - a_i)$
t_{2n-3} : $\prod_{i=1}^{n/2}(X - a_i)$
t_{2n-2} : $\prod_{i=n/2+1}^{n}(X - a_i)$
t_{n+1} : $(X - a_1)(X - a_2)$
t_{n+2} : $(X - a_3)(X - a_4)$
$\ldots$
$t_{3n/2}$: $(X - a_{n-1})(X - a_n)$
t_1 $X - a_1$
t_2 $X - a_2$
t_3 $X - a_3$
t_4 $X - a_4$
$\ldots$
t_{n-1} $X - a_{n-1}$
t_n $X - a_n$

Algorithm 2 $[T, D] \leftarrow$`TreeProduct`(A)

```
Output: The polynomial product tree $T = \{t_1, \ldots, t_{2n-1}\}$, corresponding to the evaluation of points $A = \{a_1, \ldots, a_n\}$ as shown in Figure 1.
    $D = [d_1, \ldots, d_{2n-1}]$ descendant indices for non-leaf nodes or 0 for leaf node.
Input: A set of $n = 2^l$ numbers $\{a_1, \ldots, a_n\}$.
for $i = 1 \ldots n$ do
    $t_i \leftarrow X - a_i$ {Initializing leaf nodes}
    $d_j \leftarrow 0$
end for
$i \leftarrow 1$ {Index of lower level}
$j \leftarrow n + 1$ {Index of upper level}
while $j \leqslant 2n - 1$ do
    $t_j \leftarrow t_i \cdot t_{i+1}$
    $d_j \leftarrow i$
    $i \leftarrow i + 2$
    $j \leftarrow j + 1$
end while
```

Algorithm 3 $V \leftarrow$ `RecursiveEvaluation`(f, t_i, D)

```
Input: A polynomial $f$ of degree $n$.
    A polynomial product tree rooted at $t_i$, and whose leaves are $\{X - a_k, \ldots, X - a_m\}$
    An array $D = [d_1, \ldots, d_{2n-1}]$ descendant indices for non-leaf nodes or 0 for leaf node.
Output: $V = \{f(a_k), \ldots, f(a_m)\}$
if $d_i = 0$ then
    return $\{f(a_i)\}$ {When $t_i$ is a leaf, we apply an evaluation directly.}
else
    $g_1 \leftarrow f \mod t_{d_i}$ {left subtree}
    $V_1 \leftarrow$ RecursiveEvaluation$(g_1, t_{d_i}, D)$
    $g_2 \leftarrow f \mod t_{d_i+1}$ {right subtree}
    $V_2 \leftarrow$ RecursiveEvaluation$(g_2, t_{d_i+1}, D)$
    return $V_1 \cup V_2$
end if
```

It follows that the running time of Alg. 1 is $\tilde{O}(2^{\rho'}) = \tilde{O}(\sqrt{2^\rho})$ operations modulo x_0, which is essentially the "square root" of gcd exhaustive search. But the space requirement is $\tilde{O}(2^{\rho'}) = \tilde{O}(\sqrt{2^\rho})$ polynomially many bits: thus, Alg. 1 can be viewed as a time/memory trade-off, compared to gcd exhaustive search.

2.3 Application to GACD

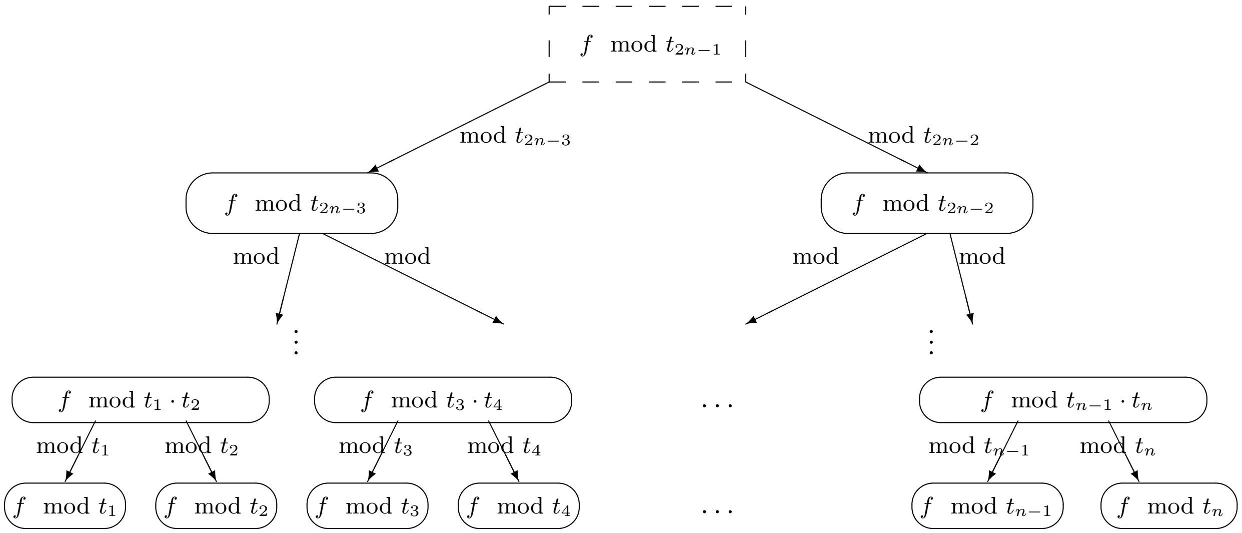
Any PACD algorithm can be used to solve GACD, using the trivial reduction from GACD to PACD based on exhaustive search over the noise r_0. More precisely, for an arbitrary instance of GACD:

$$x_i = pq_i + r_i \text{ where } 0 \leq r_i < 2^\rho, 0 \leq i \leq m$$

we apply our PACD algorithm for all pairs $(x_0 - r_0, x_1)$ where r_0 ranges over $\{0, \ldots, 2^\rho - 1\}$.

It follows that GACD can be solved in $\tilde{O}(2^{3\rho/2})$ operations modulo x_0, using $\tilde{O}(2^{\rho/2})$ polynomially many bits. This is exponentially faster than the best attack of [25], namely gcd exhaustive search, which required $2^{2\rho}$ gcd operations. Note that in [25], another hybrid attack was described, where one performs exhaustive search over r_0 and factor the resulting number using ECM, but because of the large size of the prime factors (namely, a bit-length $\geq \rho^2$), this attack is not faster: it also requires at least $2^{2\rho}$ operations.

Figure 2: Evaluation on the polynomial tree $T = \{t_1, \ldots, t_{2n-1}\}$ for $\{a_1, \ldots, a_n\}$.



3 Implementation of the Square-Root PACD Algorithm

We implemented our new PACD algorithm using the NTL library [22]. In this section, we present various tricks to implement efficiently the algorithm.

3.1 Obstructions

The main obstruction when implementing Alg. 1 is memory. Consider the Large FHE-challenge from [5]: there, $\rho = 40$, so the optimal parameter is $\rho' = 20$, which implies that $f_{2^{\rho'}}$ is a polynomial of degree 2^{20} with coefficients of size 19×10^6 bits. In other words, simply storing $f_{2^{\rho'}}$ already requires $2^{20} \times 19 \times 10^6$ bits, which is more than 2Tb, while we also need to perform various computations. This means that in practice, we will have to settle for suboptimal parameters.

More precisely, assume that we select an additional parameter d, which is a power of two less than $2^{\rho'}$. We rewrite (3) as:

$$p = \gcd\left(x_0, \prod_{k=0}^{2^\rho/d-1} f_d(dk) \pmod{x_0}\right) \tag{4}$$

This gives rise to a constrained version of Alg. 1, called Alg. 4.

Algorithm 4 Solving PACD by multipoint evaluation of univariate polynomials, using fixed memory

```
Input: An instance $(x_0, x_1)$ of the PACD problem with noise size $\rho$, and a polynomial degree $d$ (which must be a power of two).
Output: The secret number $p$ such that $x_0 = pq_0$ and $x_1 = pq_1 + r_1$ with appropriate sizes.
1: Compute the polynomial $f_d(x)$ defined by (2), using Alg. 2.
2: Compute the evaluation of $f_d(x)$ at the $2^\rho/d$ points $0, d, 2d, \ldots, d(2^\rho/d - 1)$, using $2^\rho/d^2$ times Alg. 3 with $d$ points. Each application of Alg. 3 requires the computation of a product tree, using Alg. 2.
```

The running time of Alg. 4 is $\frac{2^\rho \tilde{O}(d)}{d^2}$ elementary operations modulo x_0, and the space requirement is $\tilde{O}(d)$ polynomially many bits. Note that each of the $2^\rho/d^2$ times applications of Alg. 3 can be done in parallel.

3.2 Tricks

The use of Alg. 4 allows several tricks, which we now present.

3.2.1 Minimizing the Product Tree

Each application of Alg. 3 requires the computation of a product tree, using Alg. 2. But this product tree requires to store $2n-1$ polynomials. Fortunately, these polynomials have coefficients which are in some sense much smaller than the modulus x_0: this is because we evaluate the polynomial $f_d(x)$ at points in $\{0,\ldots,2^\rho-1\}$, which is very small compared to the modulus x_0. However, a naive implementation would not exploit this. For instance, consider the polynomial $(X-a_1)(X-a_2) = X^2-(a_1+a_2)X+a_1a_2$, which belongs to the product tree. In a typical library for polynomial computations, the polynomial coefficients would be represented as positive residues modulo x_0. But if a_1+a_2 is small, then $-(a_1+a_2)+x_0$ is actually big. This means that many coefficients of the product tree polynomials will actually be as big as x_0, if they are represented as positive residues modulo x_0, which drastically reduces the choice of the degree d.

To avoid this problem, we instead slightly modify the polynomial $f_d(X)$, in order to evaluate at small negative numbers inside $\{0,\ldots,1-2^\rho\}$, so that each polynomial of the product tree has "small" positive coefficients. This drastically reduces the storage of the product tree. More precisely, we rewrite (4) as:

$$p = \gcd\left(x_0, \prod_{k=0}^{2^\rho/d^2-1}\prod_{\ell=0}^{d-1} f'_{d,k}(-\ell d) \pmod{x_0}\right) \tag{5}$$

where

$$f'_{d,k}(x) = \prod_{i=0}^{d-1}(x_1-2^\rho-x+dk-i) \pmod{x_0} \tag{6}$$

Each product $\prod_{\ell=0}^{d-1} f'_{d,k}(-\ell)(\mathrm{mod} x_0)$ is computed by applying Alg. 3 once, using the d points $0,-d,-2d,\ldots,-d(d-1)$.

3.2.2 Powers of Two

We need to compute the polynomial $f'_{d,k}(x)$ defined by (6) before each application of Alg. 3, using a simplified version of Alg. 2, which only computes the root rather than the whole product tree. However, notice that the degree of each polynomial of the product tree is exactly a power of two, which is the worst case for the polynomial multiplication implemented in the NTL library [22]. For instance, in NTL, multiplying two 512-degree polynomials with Medium-FHE coefficients takes 50% more time than multiplying two 511-degree polynomials with Medium-FHE coefficients.

To circumvent threshold phenomenons, we notice that each polynomial of the product tree is a monic polynomial, except the leaves (for which the leading coefficient is -1). But the product of two monic polynomials whose degree is a power of two can be derived efficiently from the product of two polynomials with degree strictly less than the power of two, using:

$$(X^n+P(X))\times(X^n+Q(X)) = X^{2n}+X^n(P(X)+Q(X))+P(X)Q(X).$$

We apply this trick to speed up the computation of the polynomial $f'_{d,k}(x)$.

3.2.3 Precomputations

Now that we use (5), we change several times the polynomial $f'_{d,k}(x)$, but we keep the same evaluation points $0,-d,-2d,\ldots,-d(d-1)$, and therefore the same product tree. This allows to perform precomputations to speed up Alg. 3. Indeed, the main operation of Alg. 3 is computing the remainder of a polynomial with one of the product tree polynomials, and it is well-known that this can be sped up using precomputations depending on the modulus polynomial. One classical way to do this is to use Newton's method for remainder (Alg. 5). This algorithm requires the following notation: for any polynomial f of degree n and for any integer $m \geqslant n$, we define the m-degree polynomial $\mathrm{rev}(f, m)$ as $\mathrm{rev}(f, m) = f(1/X) \cdot X^m$. In Alg. 5, Line 1 is independent of f. Therefore, whenever one needs to compute many remainders with respect to the same modulus g, it is more efficient to precompute and store h, so that Line 1 does not need to be reexecuted. Hence, in an offline phase, we precompute and store (on a hard disk) the polynomial $\bar{g}$ of Line 1 for each product tree polynomial. And for each remainder required by Alg. 3, we execute the last two lines of Alg. 5.

Algorithm 5 Remainder using Newton's method (see [14, Sect 7.2])

```
Input: Polynomials $f \in \mathbb{R}[X]$ of degree $2n-1$, $g \in \mathbb{R}[X]$ of degree $n$.
Output: The polynomial $h = f \mod g$
1: $\bar{g} \leftarrow \mathrm{Inverse}(\mathrm{rev}(g, n)) \mod X^n$
2: $s \leftarrow \mathrm{rev}(f, 2n-1) \cdot \bar{g} \mod X^n$
3: $h \leftarrow f - \mathrm{rev}(s, n-1) \cdot g$
```

It follows that each remainder operation of Alg. 3 is reduced to two polynomial multiplications.

The NTL library also contains routines for doing remainders with precomputations, but Alg. 5 turns out to be more efficient for our setting. This is because many factors impact the performance of polynomial arithmetic, such as the size of the modulus and the degree.

3.3 Further Tricks

Since memory is the main obstruction for choosing d, it is very important to minimize RAM requirements. Since Alg. 3 can be reduced to multiplications using precomputations, one may consider the use of special multiplication algorithms which require less memory than standard algorithms, such as in-place algorithms. We note that there has been recent work [21, 11] in this direction, but we did not implement these algorithms. This suggests that our implementation is unlikely to be optimal, and that there is room for improvement.

Another potential improvement comes from the following remark: it is easy to rewrite our attack in such a way that we evaluate the same polynomial along an arithmetic progression, and there are tricks for this special case (see [16]), compared to general multipoint evaluation.

3.4 New Security Estimates for the FHE Challenges

Table 1 reports benchmarks for our implementation on the fully-homomorphic-encryption challenges of Coron *et al.* [5], which come in four flavours: Toy, Small, Medium and Large. The security level ℓ is defined in [5] is defined as follows: the best attack should require at least 2^ℓ clock cycles on a standard single core. The row "Expected security level" is extracted from [5].

Our timings refer to a single 2.27GHz-core with 72Gb of RAM. First, we assessed the cost of gcd exhaustive search, by measuring the running time of the (quasi-linear) gcd routine of the widespread gmp library, which is used in NTL [22]: timings were measured for each modulus size of the four FHE-challenges. This gives the "concrete security level" row, which is slightly higher than the expected security level of [5].

We also report timings for our implementation of our square-root PACD algorithm: these timings are below the expected security level, which breaks all four FHE-challenges of [5]. For the Toy and Small challenges, the parameter d was optimal, and we did not require much memory: the speedup is respectively 24 and 324, compared to gcd exhaustive search. For the Medium and Large challenges, we had to use a suboptimal parameter d, due to RAM constraints: we used $d = 2^{13}$ (resp. $d = 2^{10}$) for Medium (resp. Large), instead of the optimal $d = 2^{16}$ (resp. $d = 2^{20}$). But the speedups are already significant: 1202 for Medium, and 264 for Large. The timings are obtained by suitably multiplying the running time of a single execution of Alg. 3 and

Alg. 2: for instance, in the Large case, this online phase took between 64727s to 65139.4s, for 5 executions, and the precomputation storage was 21Gb.

Using a more optimized implementation, and possibly a computer with larger RAM (today, one can already buy servers with 4-Tb RAM), we believe it is possible to obtain larger speedups, so the "New security level" row should only be interpreted as an upper bound. Anyway, our implementation is already sufficient to show that the FHE-challenges of [5] fall short of the expected security level. Already, our running time for the Large challenge (whose public key is close to 1Gb) is similar to the effort spent for the factorization of RSA-768 [13], which was estimated to be around 2000 core-years for a single 2.2 GHz-core.

Hence, one needs to increase the parameters of the FHE scheme of [5], which makes it less competitive with the FHE implementation of [9].

4 Applications to Noisy Factoring

Consider a typical "balanced" RSA modulus $N = pq$ where $p, q \leq 2\sqrt{N}$. A celebrated lattice-based cryptanalysis result of Coppersmith [3] states that if one is given half of the bits of p, either in the most significant positions, or the least significant positions, then one can recover p and q in polynomial time. Although this attack has been extended in several works (see [15] for a survey), all these lattice-based results require that the unknown bits are consecutive, or spread across extremely few blocks. This decreases its potential applications to side-channel attacks where errors are likely to be spread unevenly.

This suggests the following setting, which we call noisy factoring. Assume that one is given a noisy version p' of the prime factor p, which differs from p by at most k bits, not necessarily consecutive, under either of the following two cases:

- If the k positions of the noisy bits are known, we can recover p (and therefore q) by exhaustive search using at most 2^k polynomial-time operations: we stress that in this case, we assume that we do not know if each of the k bits has been flipped, otherwise no search would be necessary.
- If instead, none of the positions is known, but we know that exactly k bits have been modified, we can recover p by exhaustive search using at most $\left(\begin{array}{c} n \\ k \end{array}\right)$ polynomial-time operations, where n is the bit-length of p. If we only know an upper bound on the number of modified bits, we can simply repeat the attack with decreasing values of k.

These running times do not require that p and q are balanced.

In this section, we show that our previous technique for PACD can be adapted to noisy factoring, yielding new attacks whose running time is essentially the "square root" of exhaustive search, that is, $\tilde{O}(2^{k/2})$ or $\tilde{O}(\sqrt{\left(\begin{array}{c} n \\ k \end{array}\right)})$ polynomial-time operations, depending on the case.

4.1 Known positions

We assume that the prime number p has n bits, so that:

$$p = \sum_{i=0}^{n-1} p_i 2^i,$$

where $p_i \in \{0, 1\}$ for $0 \leqslant i \leqslant n-1$.

In this subsection, we assume that all the bits p_i are known, except possibly at k positions $b_1, \ldots, b_k$, which we sort, so that: $0 \leq b_1 \leqslant \cdots \leqslant b_k < n$. Denote by $p^{(1)}, \ldots, p^{(2^k)}$ the 2^k possibilities for p, when $(p_{b_1}, \ldots, p_{b_k})$ ranges over $\{0, 1\}^k$.

With high probability, all the $p^{(i)}$'s are coprime with N, except one, which would imply that:

$$p = \gcd\left(N, \prod_{i=1}^{2^k} p^{(i)} (\text{mod } N)\right) \tag{7}$$

A naive evaluation of (7) costs 2^k modular multiplications, and one single gcd. We now show that this evaluation can be performed more efficiently using $\tilde{O}(2^{k/2})$ arithmetic operations with numbers with the same size as N.

The unknown bits $p_{b_1}, \ldots, p_{b_k}$ can be regrouped into two sets $\{p_{b_1}, \ldots, p_{b_\ell}\}$, and $\{p_{b_{\ell+1}}, \ldots, p_k\}$ of roughly the same size $\ell = \lfloor k/2 \rfloor$, as illustrated in Figure 3:

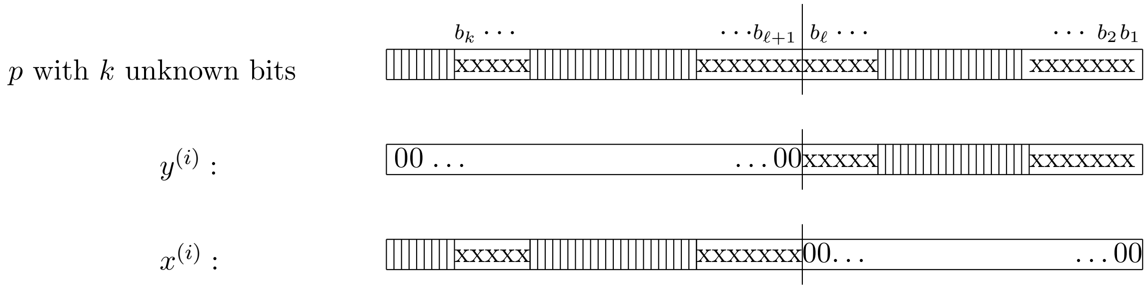
- For $1 \leqslant i \leq 2^\ell$, let $y^{(i)}$ be $\sum_{j=0}^{n-1} y_j^{(i)} 2^j$, where the j-th bit $y_j^{(i)}$ of $y^{(i)}$ is defined as

$$y_j^{(i)} = \begin{cases} 0 & \text{if } j > b_\ell \\ t\text{-th bit of } i & \text{if } \exists t \leqslant \ell, j = b_t \\ p_j & \text{otherwise} \end{cases},$$

- For $1 \leqslant i \leq 2^{k-\ell}$, let $x^{(i)}$ be $\sum_{j=0}^{n-1} x_j 2^j$, where $x_j^{(i)}$ is defined as

$$x_j^{(i)} = \begin{cases} 0 & \text{if } j \leqslant b_l \\ t\text{-th bit of } i & \text{if } \exists t > l, j = b_t \\ p_j & \text{otherwise} \end{cases},$$

Figure 3: Splitting the Unknown Bits in Two



Hence, by definition of $x^{(i)}$ and $y^{(i)}$, we have:

$$\prod_{i=1}^{2^k} p^{(i)} \equiv \prod_{i=1}^{2^\ell} \prod_{j=1}^{2^{k-\ell}} (x^{(j)} + y^{(i)}) \; (\text{mod } N) \tag{8}$$

which gives rise to a square-root algorithm (Alg. 6) to solve the noisy factorization problem with known positions.

Algorithm 6 Noisy Factorization With Known Positions

Input: An RSA modulus $N = pq$ and the bits $p_0, \ldots, p_{n-1}$ of p, except the k bits $p_{b_1}, \ldots, p_{b_k}$, where the bit positions $b_1 \leq b_2 \leq \cdots \leq b_k$ are known.

Output: The secret factor $p = \sum_{i=0}^{n-1} p_i 2^i$ of N.

```
1: Compute the polynomial $f(X) = \prod_{i=1}^{2^\ell} \left(X + y^{(i)}\right) \bmod N$ of degree $2^\ell$, with coefficients modulo $N$, using Alg. 2.
2: Compute the evaluation of $f(X)$ at the points $\{x^{(1)}, \ldots, x^{(2^{k-\ell})}\}$, using $1 + (k \bmod 2)$ times Alg. 3 with $2^\ell$ points.
3: return $p \leftarrow \gcd\left(N, \prod_{i=1}^{2^{k-\ell}} \left(f(x^{(i)})\right) \bmod N\right)$
```

Similary to Section 2, the cost of Alg. 6 is $\tilde{O}(2^{k/2})$ polynomial-time operations. This is an exponential improvement over naive exhaustive search, but Alg. 6 requires exponential space.

4.2 Unknown positions

In this subsection, we assume that p' differs from p by exactly k bits at unknown positions, and that p' has bit-length n. Our attack is somewhat reminiscent of Coppersmith's baby-step/giant-step attack on low-Hamming-weight discrete logarithm [23], but that attack uses sorting, not multipoint evaluation. To simplify the description, we assume that both k and n are even, but the attack can easily be adapted to the general case.

Pick a random subset S of $\{0, \ldots, n-1\}$ containing exactly $n/2$ elements. The probability that S contains the indices of exactly $k/2$ flipped bits is:

$$\frac{\binom{n/2}{k/2}^2}{\binom{n}{k}} \approx \frac{1}{\sqrt{k}}.$$

We now assume that this event holds, and let $\ell = \binom{n/2}{k/2}$. Similarly to the previous subsection, we define:

- Let $x^{(i)}$ for $1 \leqslant i \leq \ell$ be the numbers obtained by copying the bits of p' at all the positions inside S, and flipping exactly $k/2$ bits: all the other bits are set to zero.
- Let $y^{(i)}$ for $1 \leqslant i \leq \ell$ be the numbers obtained by copying the bits of p' at all the positions outside S, and flipping exactly $k/2$ bits: all the other bits are set to zero.

Now, with high probability over the choice of (p, q), we may write:

$$p = \gcd(N, \prod_{i=1}^{\ell} \prod_{j=1}^{\ell} (x^{(j)} + y^{(i)}) \pmod N) \tag{9}$$

which gives rise to a square-root algorithm (Alg. 7) to solve the noisy factorization problem with unknown positions.

Algorithm 7 Noisy Factorization With Unknown Positions

Input: An RSA modulus $N = pq$ and a number p' differing from p by exactly k bits of unknown position.
Output: The secret factor p.

```
repeat
    Pick a random subset $S$ of $\{0, \ldots, n-1\}$ containing exactly $n/2$ elements.
    Compute the integers $x^{(i)}$ and $y^{(i)}$ for $1 \leqslant i \leq \ell = \binom{n/2}{k/2}$.
    Compute the polynomial $f(X) = \prod_{j=1}^{\ell}(X + y^{(j)}) \mod N$.
    Compute the evaluation of $f(X)$ at the $\ell$ points $\{x^{(1)}, \ldots, x^{(\ell)}\}$.
    $p'' \leftarrow \gcd\left(N, \prod_{i=1}^{\ell}\left(f(x^{(i)})\right) \mod N\right)$
until $p'' > 1$
return $p''$
```

Similary to Section 2, the expected cost of Alg. 7 is $\tilde{O}(\ell\sqrt{k})$ polynomial-time operations, where $\ell = \binom{n/2}{k/2}$ is roughly $\sqrt{\binom{n}{k}}$. This is an exponential improvement over naive exhaustive search, but Alg. 7 requires exponential space.

Alg. 7 is randomized, but like Coppersmith's baby-step/giant-step attack on low-Hamming-weight discrete logarithm [23], it can easily be derandomized using splitting systems. Deterministic versions are slightly less efficient, by a small polynomial factor: see [23].

5 Applications to Low-Exponent RSA

In this section, we show that our previous algorithms for noisy factoring can be adapted to attacks on low-exponent RSA encryption. Consider an RSA ciphertext $c = m^e \bmod N$, where the public exponent e is very small. Assume that one knows a noisy version m' of the plaintext m, which differs from m by at most k bits, not necessarily consecutive, under either of the following two cases:

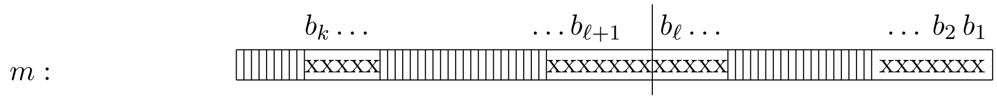
- If the k positions of the noisy bits are known, we can recover m by exhaustive search using at most 2^k polynomial-time operations: we stress that in this case, we assume that we do not know if each of the k bits has been flipped, otherwise no search would be necessary.
- If instead, none of the positions is known, but we know that exactly k bits have been modified, we can recover m by exhaustive search using at most $\binom{n}{k}$ polynomial-time operations, where n is the bit-length of m. If we only know an upper bound on the number of modified bits, we can simply repeat the attack with decreasing values of k.

This setting is usually called stereotyped RSA encryption [3]: there are well-known lattice attacks [3, 15] against stereotyped RSA, but they require that the unknown bits are consecutive, or split across extremely few blocks.

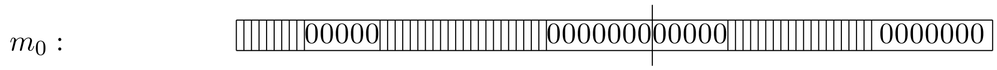
5.1 Known Positions

Assume that m is a plaintext of n bits, among which only k bits are unknown, whose (arbitrary) positions are $b_1, \ldots, b_k$. Let $c = m^e \mod N$ be the raw RSA ciphertext of m. If e is small (say, constant), we can "square root" the time of exhaustive search, using multipoint polynomial evaluation.

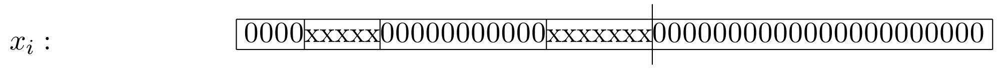
Let $\ell = \lceil (k - \log_2 e)/2 \rceil$, and assume that $k > 0$.



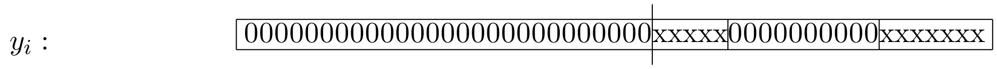
Let m_0 be derived from m by keeping all the known $n-k$ bits, and setting all the k unknown bits to 0.



For $1 \leqslant i \leqslant 2^{k-\ell}$, let the x_i's enumerate all the integers when $(b_{\ell+1}, \ldots, b_k)$ ranges over $\{0,1\}^{k-\ell}$.



Similarly, for $1 \leqslant j \leqslant 2^\ell$, let the y_j's enumerate all the integers when $(b_1, \ldots, b_\ell)$ ranges over $\{0,1\}^\ell$.



Thus, by construction, there is a unique pair (i, j) such that:

$$c = (m_0 + x_i + y_j)^e \bmod N.$$

Now, we define the polynomial $f(X) = \prod_{i=1}^{2^\ell} ((m_0 + y_i + X)^e - c) \bmod N$, which is of degree $e2^\ell$. If x_t corresponds to the correct guess for the bits $b_{\ell+1}, \ldots, b_k$, then $f(x_t) = 0$. Hence, if we evaluate $f(X)$ at $x_1, \ldots, x_{2^{k-\ell}}$, we would be able to derive the $k - \ell$ higher bits $b_{\ell+1}, \ldots b_k$, which gives rise to Alg. 8.

Algorithm 8 Decrypting Low-Exponent RSA With Known Positions

```
Input: An RSA modulus $N = pq$ and a ciphertext $c = m^e \bmod N$, where all the bits of $m$ are known, except
    at $k$ positions $b_1, \ldots, b_k$.
Output: The plaintext $m$.
1: Compute the polynomial $f(X) = \prod_{i=1}^{2^\ell} ((m_0 + y_i + X)^e - c) \bmod N$ of degree $e2^\ell$, with coefficients modulo
    $N$, using Alg. 2.
2: Compute the evaluation of $f(X)$ at the points $x^{(1)}, \ldots, x^{(2^{k-\ell})}$, using sufficiently many times Alg. 3.
3: Find the unique $i$ such that $f(x^{(i)}) = 0$.
4: Deduce from $x^{(i)}$ the bits $b_{\ell+1}, \ldots, b_k$.
5: Find the remaining bits $b_1, \ldots, b_\ell$ by exhaustive search.
```

By definition of ℓ, we have:

$$\sqrt{2^k/e} \le 2^\ell \le 2 \times \sqrt{2^k/e}$$
$$\sqrt{e2^k}/2 \le 2^{k-\ell} \le 2 \times \sqrt{e2^k}$$

It follows that the overall complexity of Alg. 8. is $\tilde{O}(\sqrt{e2^k})$ polynomial-time operations, which is the "square root" of exhaustive search if e is constant.

5.2 Unknown Positions

In the previous section, we showed how to adapt our noisy factoring algorithm with known positions (Alg. 6) to the RSA case. Similarly, our noisy factoring algorithm with unknown positions (Alg. 7) can also be adapted. If the plaintext m is known except for exactly k unknown bit positions, then one can recover m using on the average $\tilde{O}(\ell\sqrt{ke})$ polynomial-time operations, where $\ell = \begin{pmatrix} n/2 \\ k/2 \end{pmatrix}$ is roughly $\sqrt{\begin{pmatrix} n \\ k \end{pmatrix}}$.

5.3 Variants

Our technique was presented to decrypt stereotyped low-exponent RSA ciphertexts, but the same technique clearly applies to a slightly more general setting, where the RSA equation is replaced by an arbitrary univariate low-degree polynomial equation.

More precisely, instead of $c = m^e \bmod N$, we may assume that $P(m) \equiv 0 \pmod N$ where P is a univariate integer polynomial of degree e.

This allows to adapt various attacks [3] on low-exponent RSA, such as randomized padding across several blocks.

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