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Optimal maintenance policy for a system subject to damage in a discrete time process

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ABSTRACT

Consider a system operating over n discrete time periods ($n=1, 2, \ldots$). Each operation period causes a random amount of damage to the system which accumulates over time periods. The system fails when the cumulative damage exceeds a failure level ζ and a corrective maintenance (CM) action is immediately taken. To prevent such a failure, a preventive maintenance (PM) may be performed. In an operation period without a CM or PM, a regular maintenance (RM) is conducted at the end of that period to maintain the operation of the system. We propose a maintenance policy which prescribes a PM when the accumulated damage exceeds a pre-specified level δ ($<\zeta$), or when the number of operation periods reaches N, whichever comes first. With the long-term average cost rate as an optimality criterion, we optimize the maintenance policy parameters δ^* and N^* and discuss some useful properties about them. It has been shown that a δ-based PM outperforms a N-based PM in terms of cost minimization. Numerical examples are presented to demonstrate the optimization of this class of maintenance policies.

1. Introduction

Maintenance policies for systems subject to stochastic failures have been studied extensively in the literature. A comprehensive review on these policies can be found in Wang [1] and Nakagawa [2,3]. Maintenance refers to planned or unplanned actions carried out to retain a system in, or restore it to, an acceptable operating condition. There are two types of maintenance actions. A corrective maintenance (CM) is to restore a failed item to a working condition. A preventive maintenance (PM) represents an action taken to retain or improve an operating item's condition. When a CM is conducted, it is usually more costly; thus, a PM may be performed to prevent the item's failure and related high CM cost. However, an appropriate PM frequency must be determined as too many PMs also lead to high cost. The optimal maintenance policy is to minimize the long-run average operating cost of the system

Preventive maintenance (PM) itself can be classified into two categories: predetermined maintenance and condition-based maintenance (CBM). Predetermined maintenance is scheduled without any monitoring activities. The maintenance schedule can be based on the number of operating hours, the number of times of usage, or the specific dates. In contrast, CBM does not use a predetermined schedule. It monitors the condition of the system to determine if a PM should be performed. A classical assumption in CBM modeling is that a system failure is caused by a deterioration process. One way to model a continuous and gradual deterioration due to wear and tear, such as erosion (hydraulic structures, dikes), or cumulative wear (cutting tools), etc, is to utilize a single continuous-state stochastic process. Failure occurs when the state exceeds a threshold value. Due to the complexity of the practical system, it is difficult to represent the failure mechanism by using a single deterioration process. However, some important partial information on the system state can be obtained by monitoring the observable covariates (e.g., vibration, temperature, humidity, etc.).

Recently, many researchers in 'Reliability Engineering & System Safety' have considered CBM policies for continuously deteriorating systems subject to stress. Deloux et al. [4] studied a system with two types of failure mechanisms due to an excessive deterioration level and shock. To optimize the maintenance policy, they proposed an approach that combines SPC (statistical process control) and CBM. SPC is used to monitor the stress covariate, and CBM is used to inspect and replace the system

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based on the observed deterioration level. Niu et al. [5] presented a novel CBM system that employs data fusion strategy to improve the condition monitoring, health assessment, and prognostics. Zhao et al. [6] focused on optimizing CBM policies for a deteriorating system with covariates. Tinga [7] proposed two CBS type maintenance concepts: usage based and load based maintenance. Tian and Liao [8] proposed a CBM policy based on a proportional hazards model, which applies to multi-component systems. Bouvard et al. [9] also presented a method to optimize the CBM planning for a multi-components system. Fouladirad and Grall [10] considered a gradually deteriorating system with sudden mean deterioration rate increases due to external causes. They proposed an adaptive maintenance model for such a system. Weide and Pandey [11] presented a probabilistic analysis of a system subject to shocks where damage is modeled as a cumulative stochastic process. To model the damage process in a nonlinear nature, they utilized a non-homogeneous Poisson process for damage increments instead of a renewal process.

There are different degrees of improvement after a maintenance action. 'Perfect maintenance' means a maintenance action that restores system to an "as good as new" condition. A complete overhaul of an engine with a broken connecting rod is an example of perfect maintenance. The replacement of a failed system with a new one signifies a perfect maintenance (or called perfect repair). 'Minimal maintenance' means a maintenance action that brings the system back to operation with the same condition as that prior to this action. Changing a flat tire on a car, or changing a broken fan belt on an engine is an example of minimal maintenance. The minimal repair that was first studied by Barlow and Proschan [12]. 'Imperfect maintenance' is a maintenance action that brings a system to between "as bad as old" and "as good as new" condition. An engine tune-up is an example of imperfect maintenance, as a tune-up may not make an engine as good as new, but its performance would be somewhat improved. Pham and Wang [13] presented the classification of maintenances. Recently, Bartholomew-Biggs et al. [14] addressed the problem of scheduling imperfect PM that improves the equipment's condition but not as good as new. Soro et al. [15] developed a model for evaluating the production rate and reliability of multi-state systems subjected to minimal repairs and imperfect PM. You et al. [16] investigated two component-level control-limit PM policies for systems subject to variable operational conditions. Kallen [17] used a superimposed renewal process to model the effect of imperfect PM in contrast to the common use of a virtual age process. In our model, both CM and PM are considered "perfect maintenance" actions.

Most maintenance models are based on a continuous time process. In failure studies, however, the system's time to failure is often measured as the number of operational cycles. Therefore, a discrete time process can be more appropriate for the system operation. Cumulative damage models were proposed by Cox [18] to analyze the system degradation process due to a sequence of shocks which occur randomly in time and cause some damage to a system. The system fails when the total damage accumulated exceeds a threshold or failure level. Zhao and Nakagawa [19] considered a modified cumulative damage model where the unit fails when the total damage due to shocks reaches a failure level or the total number of shocks reaches a certain number. They obtained the expected cost rates and the related optimal policies. Zhao et al. [20] studied the imperfect maintenance problem for used systems that suffer damage due to shocks. Zhao et al. [21] considered age replacement policies for combining additive independent damages. Other maintenance models were also studied by Wortman et al. [22], Sheu [23], Sheu and Griffith [24,25], Chien and Sheu [26] and Chien et al. [27]. A variety of optimal maintenance policies for different damage models were summarized in Nakagawa [3]. Another application of the cumulative damage process is the repair-cost limit policy. Lai [28], Chien et al., [29–31], and Chang et al. discussed the cumulative repair-cost limit policy for a maintenance model.

In this paper, we consider a cumulative damage model for a system operating for an indefinite length of period. Applied is a maintenance policy which prescribes PM based on the number of operating periods and the accumulated damage level. We first derive the expected cost rate for the system as the optimality criterion. Then we find the optimal policy to minimize the long-run average cost. Finally, we present numerical analysis and conduct sensitivity analysis based on our model.

2. Model descriptions and formulation

Consider a system operating over n periods ($n=1, 2, \ldots$). Each period's operation causes a random amount of damage to the system. These damages are accumulated to the system. In each operation period, a system fails when the total accumulated damage exceeds a threshold level ζ, then a corrective maintenance (CM) is immediately conducted. To prevent such a failure, a preventive maintenance (PM) action may be performed. The maintenance policy considered prescribes a PM when either the accumulated damage exceeds a pre-specified level δ (but less than the failure level ζ), or the number of operating periods reaches N ($N=1, 2, \ldots$) since the system installation, whichever occurs first. For an operation period without the CM and PM, a regular maintenance (RM) action is performed at the end of that period. A practical example fitting this model is the maintenance schedule of a transit bus. For a public transit bus, every 2000 miles of use can be considered as an operation period. At the end of each period, the general condition of the bus will be recorded. If the condition reaches a pre-determined level or a certain number of operating periods is reached, a PM is performed; if the condition reaches a failure level, a CM is performed. Both PM and CM bring the bus to a perfect condition. If none of PM or CM is conducted, at the end of each operating period, an RM is performed to improve the reliability of the bus which in turn is translated into better safety for the public transit service.

To develop the expected cost rate for the maintenance policy, the following cost structure is imposed. Let c_0 be the fixed operating cost for each operation, c_{rm} be the fixed cost for each RM, C_{PM} be the fixed PM cost, and C_{CM} be the fixed CM cost. Without loss of generality, $C_{CM}>C_{PM}>c_{rm}$ is assumed. Furthermore, let random variable Y_j ($j=1, 2, \ldots$) be the amount of damage due to the jth operation with a distribution function $G(y)\equiv P_r(Y_j \le y)$. Then, the total damage $Z_j \equiv \sum_{j=0}^{j} Y_i$ at the jth operation has a distribution function $P_r(Z_j \le w)=P_r(Y_1+Y_2+\cdots+Y_j \le w)=G^{(j)}(w)$ where $G^{(j)}(w)$ is the jth-fold convolution of $G(w)$ with itself with $G^{(0)}(w)\equiv 1$ for $w \ge 0$. Clearly, the probability of a PM performed at the completion of Nth operation is $G^{(N)}(\delta)$.

For $N=\infty$, the probability that a PM is performed at the end of the jth operation (denoted by $P_j^{(PM)}$, $j=1, 2, \ldots$) is given by

$$P_j^{(PM)} = P_r(Y_1+Y_2+\cdots+Y_{j-1} < \delta \le Y_1+Y_2+\cdots+Y_j < \zeta)$$
$$= P_r(Z_{j-1} < \delta \le Z_j < \zeta) = \int_0^{\delta} [G(\zeta-y)-G(\delta-y)]dG^{(j-1)}(y)$$
$$= \int_0^{\delta} G(\zeta-y)dG^{(j-1)}(y) - G^{(j)}(\delta), \tag{1}$$

and the probability that a CM is performed at the end of the jth operation (denoted by $P_j^{(CM)}$, $j=1, 2, \ldots$) is given by

$$P_j^{(CM)} = P_r(Y_1+Y_2+\cdots+Y_{j-1} < \delta < \zeta \le Y_1+Y_2+\cdots+Y_j)$$

$$=P_r(Z_{j-1}<\delta<\zeta\le Z_j)=\int_0^{\delta}\overline{G}(\zeta-y)dG^{(j-1)}(y)$$

$$=G^{(j-1)}(\delta)-\int_0^{\delta}G(\zeta-y)dG^{(j-1)}(y). \quad (2)$$

Combining (1) and (2) (i.e., $P_j^{(PM)}+P_j^{(CM)}$) yields

$$G^{(j-1)}(\delta)-G^{(j)}(\delta)=P(Z_{j-1}<\delta\le Z_j), \quad (3)$$

which is the probability that a CM or a PM is performed at the end of the jth operation ($j=1, 2, \ldots$) when $N=\infty$.

In this study, CM and PM are assumed to be 'perfect maintenance' type, while RM is assumed to be 'minimal maintenance' type. That is, a system becomes as good as new after a CM or PM, and as bad as old after an RM. Therefore, the system becomes brand new after a scheduled PM at the end of period N, or an unscheduled PM at any period with damage δ, or an unscheduled CM at any period with damage ζ, whichever occurs first. The time between two successive perfect maintenance actions (i.e., CM or PM) can be regarded as a renewal cycle. From the renewal reward theorem (see Ross [32, p. 52]), the long-run expected cost rate is the expected total cost per renewal cycle divided by expected renewal cycle length.

Three types of the maintenance actions as well as the associated costs are illustrated in Figs. 1–3, respectively. Based on these diagrams, the expected total cost in a renewal cycle and the renewal cycle length are given by

$$\sum_{j=1}^{N}\{[(j-1)c_{rm}+jc_0+C_{PM}]\times P_j^{(PM)}+[(j-1)c_{rm}+jc_0+C_{CM}]\times P_j^{(CM)}\}$$

$$+[(N-1)c_{rm}+Nc_0+C_{PM}]G^{(N)}(\delta)=c_0\sum_{j=0}^{N-1}G^{(j)}(\delta)$$

$$+c_{rm}\sum_{j=1}^{N-1}G^{(j)}(\delta)+C_{PM}\left\{\sum_{j=1}^{N}P_j^{(PM)}+G^{(N)}(\delta)\right\}+C_{CM}\sum_{j=1}^{N}P_j^{(CM)}, \quad (4)$$

and

$$\sum_{j=1}^{N}j[G^{(j-1)}(\delta)-G^{(j)}(\delta)]+NG^{(N)}(\delta)=\sum_{j=0}^{N-1}G^{(j)}(\zeta). \quad (5)$$

Using (4) and (5), the expected cost rate is obtained as

$$CR(\delta,N)=\frac{c_0\sum_{j=0}^{N-1}G^{(j)}(\delta)+c_{rm}\sum_{j=1}^{N-1}G^{(j)}(\delta)+C_{PM}\{\sum_{j=1}^{N}P_j^{(PM)}+G^{(N)}(\delta)\}+C_{CM}\sum_{j=1}^{N}P_j^{(CM)}}{\sum_{j=0}^{N-1}G^{(j)}(\zeta)}. \quad (6)$$

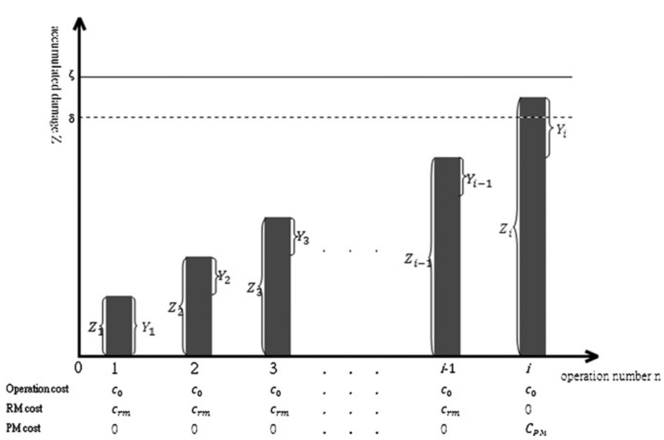

Fig. 1. PM due to damage level δ at i th (i=1, 2,..., N) operation.

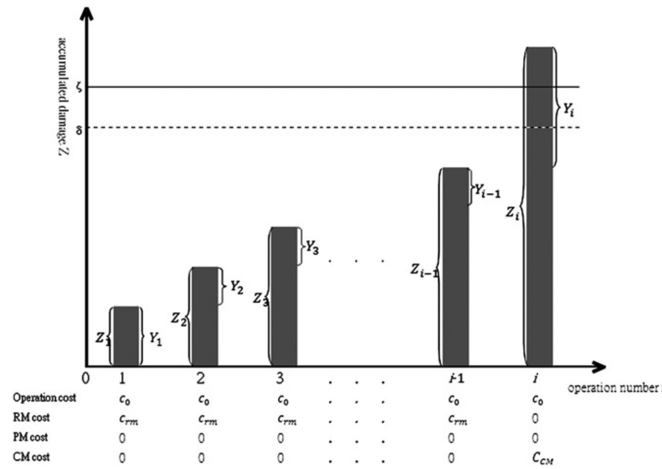

Fig. 2. CM due to damage level ζ at i th (i=1, 2,..., N) operation.

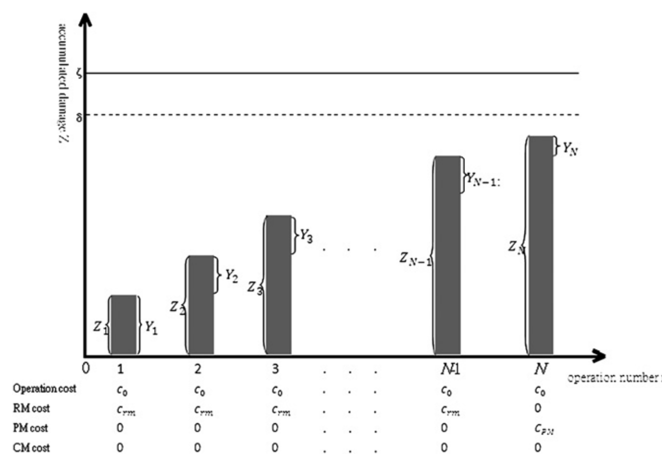

Fig. 3. PM due to the completion of N th operation without exceeding damage level δ.

Assumed that the amount of damage due to each period of operation has an exponential distribution with a mean of μ; that is, $G(y)=1-\exp[-(y/\mu)]$, and $G^{(j)}(y)=\sum_{i=j}^{\infty}[(y/\mu)^i/i!]\exp[-(y/\mu)]$. Thus, the probability $P_j^{(PM)}$ given in (1) can be expressed as $G(\zeta-\delta)G^{(j-1)}(\delta)-(1/\mu)\times\int_0^{\delta}G^{(j-1)}(y)[\overline{G}(\delta-y)-\overline{G}(\zeta-y)]dy$, and then the term $\sum_{j=1}^{N}P_j^{(PM)}$ in (6) can be further reduced to $G(\zeta-\delta)[1-G^{(N)}(\delta)]$. On the other hand, the probability $P_j^{(CM)}$ in (2) can be expressed as $\overline{G}(\zeta-\delta)G^{(j-1)}(\delta)-(1/\mu)\times\int_0^{\delta}G^{(j-1)}(y)\overline{G}(\zeta-y)dy$, and then the term $\sum_{j=1}^{N}P_j^{(CM)}$ in (6) can be further reduced to $\overline{G}(\zeta-\delta)[1-G^{(N)}(\delta)]$. Therefore, the expected cost rate in (6) reduce to

$$CR(\delta,N)=\frac{(c_0+c_{rm})\sum_{j=0}^{N-1}G^{(j)}(\delta)+(C_{CM}-C_{PM})\overline{G}(\zeta-\delta)[1-G^{(N)}(\delta)]+(C_{PM}-c_{rm})}{\sum_{j=0}^{N-1}G^{(j)}(\delta)}. \quad (7)$$

Remark 1. Note that $\sum_{j=0}^{\infty}G^{(j)}(y)=1+\sum_{j=1}^{\infty}G^{(j)}(y)=1+M(y)$ where $M(y)$ is the renewal function of the distribution $G(y)$; and if $G(y)=1-\exp[-(y/\mu)]$, then $M(y)=(y/\mu)$.

3. Optimal PM policies

For the infinite-horizon case, the optimal PM policy is to minimize $CR(\delta,N)$ with respect to the pair (δ,N), where $0\le\delta\le\zeta$ and $N=1, 2, \ldots$. For a fixed δ, $CR(\delta,N)$ in (7) has the following properties: (i) the inequalities $CR(\delta,N+1)\ge CR(\delta,N)$ and $CR(\delta,N)<CR(\delta,N-1)$ holds if, and only if

$$\Phi(\delta,N)\ge(C_{PM}-c_{rm}) \text{ and } \Phi(\delta,N-1)<(C_{PM}-c_{rm}), \tag{8}$$

where

$$\Phi(\delta,N)=(C_{CM}-C_{PM})\overline{G}(\zeta-\delta)\left\{\left[1-\frac{G^{(N+1)}(\delta)}{G^{(N)}(\delta)}\right]\sum_{j=0}^{N-1}G^{(j)}(\delta)+G^{(N)}(\delta)-1\right\}. \tag{9}$$

(ii) Since

$$\Phi(\delta,N+1)-\Phi(\delta,N)=-(C_{CM}-C_{PM})\overline{G}(\zeta-\delta)\left[\frac{G^{(N+2)}(\delta)}{G^{(N+1)}(\delta)}-\frac{G^{(N+1)}(\delta)}{G^{(N)}(\delta)}\right]\sum_{j=0}^{N}G^{(j)}(\delta), \tag{10}$$

and we can show that $G^{(N+1)}(\delta)/G^{(N)}(\delta)$ decreases strictly with N when $G^{(j)}(y)=\sum_{i=j}^{\infty}[(y/\mu)^i/i!]\exp[-(y/\mu)]$, $j=0, 1, 2, \ldots$(see Nakagawa [3, p. 24]), thus $\Phi(\delta,N)$ increases strictly with N.

On the other hand, differentiating $CR(\delta,N)$ in (7) with respect to δ and setting it equal to zero yields

$$(C_{CM}-C_{PM})\overline{G}(\zeta-\delta)\sum_{j=1}^{N}G^{(j)}(\delta)=(C_{PM}-c_{rm}). \tag{11}$$

Let $\Omega(\delta,N)$ be the left-hand-side of (11); thus, it is easy to see that $\Omega(\delta,N)$ is strictly increasing in δ for a fixed N.

Based on (8)–(11), we obtain the following two theorems regarding the optimal δ^* that minimizes $CR(\delta,N)$ for a fixed N, and the optimal N^* that minimizes $CR(\delta,N)$ for a fixed δ.

Theorem 1. Given a fixed δ $(0<\delta<\zeta)$.

(i) If $(C_{CM}-C_{PM})\overline{G}(\zeta-\delta)(\delta/\mu)>(C_{PM}-c_{rm})$, then there exists a N^* (where $1\le N^*<\infty$) that minimizes $CR(\delta,N)$ in (7), and the resulting cost rate satisfies

$$(c_0+c_{rm})+(C_{CM}-C_{PM})\overline{G}(\zeta-\delta)\left[1-\frac{G^{(N^*)}(\delta)}{G^{(N^*-1)}(\delta)}\right]<CR(\delta,N^*)\le(c_0+c_{rm})+(C_{CM}-C_{PM})\overline{G}(\zeta-\delta)\left(1-\frac{G^{(N^*+1)}(\delta)}{G^{(N^*)}(\delta)}\right). \tag{12}$$

(ii) If $(C_{CM}-C_{PM})\overline{G}(\zeta-\delta)(\delta/\mu)\le(C_{PM}-c_{rm})$, then $N^*=\infty$, and

$$CR(\delta,\infty)=(c_0+c_{rm})+\frac{\mu}{\mu+\delta}\left[(C_{CM}-C_{PM})\overline{G}(\zeta-\delta)+(C_{PM}-c_{rm})\right]. \tag{13}$$

Proof. Since $\Phi(\delta,N)$ is strictly increasing in N for a given fixed δ, and

$$\lim_{N\to\infty}\Phi(\delta,N)=(C_{CM}-C_{PM})\overline{G}(\zeta-\delta)\left\{[1-0]\sum_{j=0}^{\infty}G^{(j)}(\delta)+G^{(\infty)}(\delta)-1\right\}$$
$$=(C_{CM}-C_{PM})\overline{G}(\zeta-\delta)\left\{\sum_{j=1}^{\infty}G^{(j)}(\delta)\right\}$$
$$=(C_{CM}-C_{PM})\overline{G}(\zeta-\delta)M(\delta)=(C_{CM}-C_{PM})\overline{G}(\zeta-\delta)\frac{\delta}{\mu}.$$

Thus, if $\lim_{N\to\infty}\Phi(\delta,N)=(C_{CM}-C_{PM})\overline{G}(\zeta-\delta)(\delta,\mu)>(C_{PM}-c_{rm})$, then there exists a finite N^* satisfies (8), which minimizes $CR(\delta,N)$ in (7) with respect to N. That is, the optimal N^* satisfies (8) or

$$(C_{CM}-C_{PM})\overline{G}(\zeta-\delta)\left\{\left[1-\frac{G^{(N^*)}(\delta)}{G^{(N^*-1)}(\delta)}\right]\sum_{j=0}^{N^*-2}G^{(j)}(\delta)+G^{(N^*-1)}(\delta)-1\right\}$$
$$<(C_{PM}-c_{rm})\le(C_{CM}-C_{PM})\overline{G}(\zeta-\delta)\times\left\{\left[1-\frac{G^{(N^*+1)}(\delta)}{G^{(N^*)}(\delta)}\right]\sum_{j=0}^{N^*-1}G^{(j)}(\delta)+G^{(N^*)}(\delta)-1\right\}.$$

Algebraic manipulation of the above inequality yields (12).

Otherwise, if $\lim_{N\to\infty}\Phi(\delta,N)=(C_{CM}-C_{PM})\overline{G}(\zeta-\delta)(\delta/\mu)\le(C_{PM}-c_{rm})$, then $N^*=\infty$; and by (7), the resulting cost rate yields (13). □

Theorem 2. Given a fixed N $(N=1, 2, \ldots)$.

(i) If $(C_{CM}-C_{PM})\sum_{j=1}^{N}G^{(j)}(\zeta)>(C_{PM}-c_{rm})$, then there exists an unique δ^* (where $0<\delta^*<\zeta$) that minimizes $CR(\delta,N)$ in (7), and

$$CR(\delta^*,N)=(c_0+c_{rm})+(C_{CM}-C_{PM})\overline{G}(\zeta-\delta^*). \tag{14}$$

(ii) If $(C_{CM}-C_{PM})\sum_{j=1}^{N}G^{(j)}(\zeta)\le(C_{PM}-c_{rm})$, then $\delta^*=\zeta$, and

$$CR(\zeta,N)=(c_0+c_{rm})+\frac{(C_{CM}-C_{PM})[1-G^{(N)}(\zeta)]+(C_{PM}-c_{rm})}{\sum_{j=0}^{N-1}G^{(j)}(\zeta)}. \tag{15}$$

Proof. From (11), since $\Omega(0,N)=0<(C_{PM}-c_{rm})$ and $\Omega(\delta,N)$ is strictly increasing in δ $(0\le\delta\le\zeta)$, thus if $\lim_{\delta\to\zeta}\Omega(\delta,N)=\Omega(\zeta,N)=(C_{CM}-C_{PM})\sum_{j=1}^{N}G^{(N)}(\zeta)>(C_{PM}-c_{rm})$, there exists a unique δ^* $(0<\delta^*<\zeta)$ that satisfies (11), which minimizes the cost rate $CR(\delta,N)$ in (7) with respect to δ. Applying the condition $\Omega(\delta^*,N)=(C_{PM}-c_{rm})$ into (7) yields (14).

On the other hand, if $\lim_{\delta\to\zeta}\Omega(\delta,N)=\Omega(\zeta,N)=(C_{CM}-C_{PM})\sum_{j=1}^{N}G^{(N)}(\zeta)\le(C_{PM}-c_{rm})$, then $\delta^*=\zeta$; and substituting $\delta=\zeta$ into (7) gives (15). □

Remark 2. Theorems 1 and 2, respectively, reveal the relation between the optimal N^* and the given δ, as well as the relation between the optimal δ^* and the given N. They show a common characteristic: when the ratio $(C_{CM}-C_{PM})/(C_{PM}-c_{rm})$ exceeds a threshold, a PM needs to be implemented (i.e. $1\le N^*<\infty$, $0<\delta^*<\zeta$). Specifically, the higher the C_{CM}, or the higher the c_{rm}, the earlier the PM will be implemented (i.e. the N^* and δ^* values will be smaller); on the other hand, the higher the C_{PM}, the later the PM will be implemented (i.e. the N^* and δ^* values will be larger).

Remark 3. It is observed from the cost rate function in (7) and Theorems 1 and 2 that the fixed cost c_0 of each operation period of the system does not influence the optimal maintenance policy; however, a larger c_0 indicates a higher long-run average cost rate. On the other hand, the cost of each RM c_{rm} influences the optimal maintenance policy. According to (8) and Theorem 1, a higher c_{rm} leads to a smaller optimum PM N^*; Similarly, according to (11) and Theorem 2, a higher c_{rm} results in a smaller optimal PM δ^*.

4. Two special cases

There are two special cases of the maintenance model.

- Case 1: $\delta=\zeta$. This is the maintenance policy that PM only depends on N.

In this case, a CM is performed at a failure (i.e., the total damage exceeds failure level ζ) or a PM is performed at at the end of period N ($N=1, 2, \ldots$), whichever occurs first. Substituting $\delta=\zeta$ into (7), we have

$$CR(\zeta,N)=\frac{(c_0+c_{rm})\cdot\sum_{j=0}^{N-1}G^{(j)}(\zeta)+(C_{CM}-C_{PM})[1-G^{(N)}(\zeta)]+(C_{PM}-c_{rm})}{\sum_{j=0}^{N-1}G^{(j)}(\zeta)}=CR_1(N). \tag{16}$$

Then, the following properties regarding the optimal N^* can be obtained.

Corollary 1. The optimal N^* that minimizes the cost rate $CR_1(N)$ in (16) has the following properties:

(i) If $(C_{CM}-C_{PM})(\zeta/\mu)>(C_{PM}-c_{rm})$, then there exists a finite and unique N^* (where $1\le N^*<\infty$) which satisfies $CR_1(N^*+1)\ge CR_1(N^*)$ and $CR_1(N^*)<CR_1(N^*-1)$, and the associated cost rate is

$$(c_0+c_{rm})+(C_{CM}-C_{PM})\left[1-\frac{G^{(N^*)}(\zeta)}{G^{(N^*-1)}(\zeta)}\right] < CR_1(N^*)\le(c_0+c_{rm})+(C_{CM}-C_{PM})\left[1-\frac{G^{(N^*+1)}(\zeta)}{G^{(N^*)}(\zeta)}\right]. \tag{17}$$

(ii) If $(C_{CM}-C_{PM})(\zeta/\mu)\le(C_{PM}-c_{rm})$, then $N^*=\infty$, and the cost rate is given as

$$CR_1(\infty)=(c_0+c_{rm})+\frac{\mu}{\mu+\delta}(C_{CM}-c_{rm}). \tag{18}$$

- Case 2: $N=\infty$. This is the maintenance policy that PM only depends on the damage level δ.

In this case, a CM is performed at a failure (i.e., the total damage exceeds failure level ζ) or a PM is performed when the accumulated damage exceeds a pre-specified PM level δ (but less than the failure level ζ), whichever occurs first. Substituting $N=\infty$ into (7), we have

$$CR(\delta,\infty)=(c_0+c_{rm})+\frac{\mu\left[(C_{PM}-c_{rm})+(C_{CM}-C_{PM})\overline{G}(\zeta-\delta)\right]}{\mu+\delta}=CR_2(\delta) \tag{19}$$

Then, the following properties regarding the optimal δ^* can be found.

Corollary 2. The optimal δ^* that minimizes the cost rate $CR_2(\delta)$ in (19) has the following properties:

i) If $(C_{CM}-C_{PM})(\zeta/\mu)>(C_{PM}-c_{rm})$, then there exists a unique δ^* $(0<\delta^*<\zeta)$, which satisfies $(C_{CM}-C_{PM})(\delta^*/\mu)\overline{G}(\zeta-\delta^*)=(C_\delta-c_m)$, and the cost rate is the same as (12).
ii) If $(C_{CM}-C_{PM})(\zeta/\mu)\le(C_{PM}-c_{rm})$, then $\delta^*=\zeta$ and the cost rate is

$$CR_2(\delta^*)=CR_2(\zeta)=(c_0+c_{rm})+\frac{(C_{Cm}-c_{rm})}{\mu+\zeta}. \tag{20}$$

Remark 4. Corollaries 1 and 2 indicate the properties of the optimal policies for the two special cases, they are coincides with Theorems 1 and 2, respectively. It is worth noting that if $\delta=\zeta$ and $N=\infty$, a CM will be implemented only in case of system failure, and a PM will never be implemented. Substituting $\delta=\zeta$ and $N=\infty$ into (7), then the cost rate become $(c_0+c_{rm})+\mu((C_{CM}-c_{rm})/(\mu+\zeta))$. On the other hand, if $\delta\to0^+$, a PM will be implemented immediately once the first period of operation is finished after the system installation. At this point, the cost rate will be $\lim_{\delta\to0^+}CR(\delta,N)=(c_0+C_{PM})+(C_{CM}-C_{PM})\exp(-(\zeta/\mu))$.

5. Numerical example and discussion

In this section, a numerical example is used to demonstrate the optimal preventive maintenance (PM) policies, where parameters $\zeta=20$, $C_{PM}=10$ and $c_{rm}=c_0=1$ are fixed, and C_{CM} and μ are varied to observe their impacts on the optimal policies. Tables 1-1–1-4 show the optimal N^* and optimal $CR(\delta,N^*)$ when δ is chosen to be at 20, 18, 16 and 14. Note that $\delta=20$ is the first special case discussed in Section 4. On the other hand, Tables 2-1–2-5 show the optimal δ^* and optimal $CR(\delta^*,N)$ when the N is chosen to be at ∞, 15, 10 and 5. And $N=\infty$ is the second special case discussed in Section 4.

Based on these numerical results, the following observations are made:

1. As shown in Tables 1-1–1-4, N^* decreases as C_{CM} or μ increases, whereas $CR(\delta,N^*)$ increases with C_{CM} or μ. These behaviors are intuitive. For a higher CM cost C_{CM}, or a larger

Table 1-1
N^* and $CR(\delta,N^*)$ for a given $\delta=20$ and under $\zeta=20$, $C_{PM}=10$, and $c_{rm}=c_0=1$.

	$\mu=1$	$\mu=2$	$\mu=3$	$\mu=4$	$\mu=5$
$C_{CM}=30$					
$N^*=$	14	7	5	4	3
$CR(\delta,N^*)=$	2.741	3.684	4.703	5.735	6.761
$C_{CM}=40$					
$N^*=$	13	6	4	3	3
$CR(\delta,N^*)=$	2.784	3.848	5.043	6.314	7.585
$C_{CM}=50$					
$N^*=$	13	6	4	3	2
$CR(\delta,N^*)=$	2.814	3.961	5.299	6.736	8.390
$C_{CM}=60$					
$N^*=$	12	6	4	3	2
$CR(\delta,N^*)=$	2.840	4.073	5.554	7.158	8.852
$C_{CM}=70$					
$N^*=$	12	5	3	3	2
$CR(\delta,N^*)=$	2.858	4.156	5.774	7.580	9.314
$C_{CM}=80$					
$N^*=$	12	5	3	2	2
$CR(\delta,N^*)=$	2.876	4.215	5.901	7.934	9.776
$C_{CM}=90$					
$N^*=$	12	5	3	2	2
$CR(\delta,N^*)=$	2.893	4.274	6.029	8.137	10.238
$C_{CM}=100$					
$N^*=$	11	5	3	2	2
$CR(\delta,N^*)=$	2.907	4.332	6.156	8.340	10.700
$C_{CM}=120$					
$N^*=$	11	5	3	2	2
$CR(\delta,N^*)=$	2.926	4.450	6.410	8.746	11.624
$C_{CM}=150$					
$N^*=$	11	4	3	2	2
$CR(\delta,N^*)=$	2.956	4.613	6.792	9.354	13.011
$C_{CM}=180$					
$N^*=$	10	4	3	2	1
$CR(\delta,N^*)=$	2.985	4.691	7.174	9.963	14.113
$C_{CM}=200$					
$N^*=$	10	4	3	2	1
$CR(\delta,N^*)=$	2.995	4.743	7.429	10.368	14.479
$C_{CM}=250$					
$N^*=$	10	4	2	2	1
$CR(\delta,N^*)=$	3.020	4.872	7.674	11.382	15.395
$C_{CM}=300$					
$N^*=$	10	4	2	2	1
$CR(\delta,N^*)=$	3.045	5.001	7.918	12.397	16.311
$C_{CM}=400$					
$N^*=$	9	4	2	1	1
$CR(\delta,N^*)=$	3.090	5.260	8.406	13.627	18.143
$C_{CM}=500$					
$N^*=$	9	3	2	1	1
$CR(\delta,N^*)=$	3.113	5.452	8.894	14.301	19.974
$C_{CM}=1000$					
$N^*=$	8	3	2	1	1
$CR(\delta,N^*)=$	3.221	5.914	11.335	17.670	29.132

Table 1-2
N^* and $CR(\delta,N^*)$ for a given $\delta=18$ and under $\zeta=20$, $C_{PM}=10$, and $c_{rm}=c_0=1$.

	$\mu=1$	$\mu=2$	$\mu=3$	$\mu=4$	$\mu=5$
$C_{CM}=30$					
$N^*=$	20	9	6	5	4
$CR(\delta,N^*)=$	2.606	3.493	4.476	5.502	6.488
$C_{CM}=40$					
$N^*=$	18	8	5	4	3
$CR(\delta,N^*)=$	2.649	3.655	4.809	6.056	7.299
$C_{CM}=50$					
$N^*=$	17	7	4	3	3
$CR(\delta,N^*)=$	2.682	3.770	5.089	6.512	8.012
$C_{CM}=60$					
$N^*=$	16	6	4	3	2
$CR(\delta,N^*)=$	2.708	3.880	5.287	6.871	8.697
$C_{CM}=70$					
$N^*=$	15	6	4	3	2
$CR(\delta,N^*)=$	2.729	3.952	5.485	7.231	9.124
$C_{CM}=80$					
$N^*=$	15	6	4	3	2
$CR(\delta,N^*)=$	2.748	4.024	5.683	7.591	9.551
$C_{CM}=90$					
$N^*=$	14	6	3	3	2
$CR(\delta,N^*)=$	2.764	4.096	5.874	7.950	9.979
$C_{CM}=100$					
$N^*=$	14	6	3	2	2
$CR(\delta,N^*)=$	2.778	4.168	5.980	8.202	10.406
$C_{CM}=120$					
$N^*=$	13	5	3	2	2
$CR(\delta,N^*)=$	2.804	4.257	6.194	8.574	11.260
$C_{CM}=150$					
$N^*=$	13	5	3	2	2
$CR(\delta,N^*)=$	2.833	4.379	6.514	9.133	12.541
$C_{CM}=180$					
$N^*=$	12	5	3	2	2
$CR(\delta,N^*)=$	2.859	4.501	6.834	9.692	13.822
$C_{CM}=200$					
$N^*=$	12	5	3	2	1
$CR(\delta,N^*)=$	2.871	4.583	7.048	10.065	14.479
$C_{CM}=250$					
$N^*=$	12	4	2	2	1
$CR(\delta,N^*)=$	2.902	4.723	7.575	10.997	15.395
$C_{CM}=300$					
$N^*=$	11	4	2	2	1
$CR(\delta,N^*)=$	2.928	4.821	7.798	11.928	16.311
$C_{CM}=400$					
$N^*=$	11	4	2	1	1
$CR(\delta,N^*)=$	2.966	5.017	8.244	13.627	18.143
$C_{CM}=500$					
$N^*=$	10	4	2	1	1
$CR(\delta,N^*)=$	3.003	5.212	8.690	14.301	19.974
$C_{CM}=1000$					
$N^*=$	9	3	2	1	1
$CR(\delta,N^*)=$	3.105	5.758	10.920	17.670	29.132

Table 1-3
N^* and $CR(\delta,N^*)$ for a given $\delta=16$ and under $\zeta=20$, $C_{PM}=10$, and $c_{rm}=c_0=1$.

	$\mu=1$	$\mu=2$	$\mu=3$	$\mu=4$	$\mu=5$
$C_{CM}=30$					
$N^*=$	∞	14	8	6	5
$CR(\delta,N^*)=$	2.550	3.300	4.232	5.219	6.208
$C_{CM}=40$					
$N^*=$	∞	11	6	4	4
$CR(\delta,N^*)=$	2.561	3.436	4.549	5.774	7.023
$C_{CM}=50$					
$N^*=$	54	9	5	4	3
$CR(\delta,N^*)=$	2.572	3.548	4.809	6.211	7.677
$C_{CM}=60$					
$N^*=$	43	8	5	3	3
$CR(\delta,N^*)=$	2.583	3.644	5.027	6.629	8.289
$C_{CM}=70$					
$N^*=$	34	8	4	3	3
$CR(\delta,N^*)=$	2.594	3.728	5.233	6.932	8.901
$C_{CM}=80$					
$N^*=$	29	7	4	3	2
$CR(\delta,N^*)=$	2.604	3.799	5.384	7.235	9.342
$C_{CM}=90$					
$N^*=$	26	7	4	3	2
$CR(\delta,N^*)=$	2.615	3.863	5.535	7.538	9.734
$C_{CM}=100$					
$N^*=$	24	7	4	3	2
$CR(\delta,N^*)=$	2.626	3.927	5.686	7.841	10.127
$C_{CM}=120$					
$N^*=$	21	6	4	2	2
$CR(\delta,N^*)=$	2.646	4.028	5.988	8.411	10.912
$C_{CM}=150$					
$N^*=$	19	6	3	2	2
$CR(\delta,N^*)=$	2.673	4.161	6.271	8.921	12.909
$C_{CM}=180$					
$N^*=$	18	5	3	2	2
$CR(\delta,N^*)=$	2.697	4.285	6.535	9.431	13.268
$C_{CM}=200$					
$N^*=$	17	5	3	2	2
$CR(\delta,N^*)=$	2.711	4.340	6.712	9.771	14.053
$C_{CM}=250$					
$N^*=$	16	5	3	2	1
$CR(\delta,N^*)=$	2.742	4.476	7.153	10.621	15.395
$C_{CM}=300$					
$N^*=$	15	5	3	2	1
$CR(\delta,N^*)=$	2.768	4.613	7.594	11.471	16.311
$C_{CM}=400$					
$N^*=$	14	4	2	2	1
$CR(\delta,N^*)=$	2.810	4.821	8.086	13.171	18.143
$C_{CM}=500$					
$N^*=$	13	4	2	1	1
$CR(\delta,N^*)=$	2.844	4.965	8.490	14.301	19.974
$C_{CM}=1000$					
$N^*=$	11	3	2	1	1
$CR(\delta,N^*)=$	2.952	5.618	10.510	17.670	29.132

Table 1-4
N^* and $CR(\delta,N^*)$ for a given $\delta=14$ and under $\zeta=20$, $C_{PM}=10$, and $c_{rm}=c_0=1$.

	$\mu=1$	$\mu=2$	$\mu=3$	$\mu=4$	$\mu=5$
$C_{CM}=30$					
$N^*=$	∞	∞	19	10	7
$CR(\delta,N^*)=$	2.603	3.249	4.065	4.991	5.952
$C_{CM}=40$					
$N^*=$	∞	35	10	6	5
$CR(\delta,N^*)=$	2.604	3.311	4.303	5.467	6.697
$C_{CM}=50$					
$N^*=$	∞	22	7	5	4
$CR(\delta,N^*)=$	2.606	3.373	4.528	5.891	7.352
$C_{CM}=60$					
$N^*=$	∞	15	6	4	3
$CR(\delta,N^*)=$	2.608	3.463	4.731	6.266	7.933
$C_{CM}=70$					
$N^*=$	∞	13	6	4	3
$CR(\delta,N^*)=$	2.609	3.497	4.914	6.606	8.456
$C_{CM}=80$					
$N^*=$	∞	11	5	3	3
$CR(\delta,N^*)=$	2.611	3.556	5.077	6.944	8.978
$C_{CM}=90$					
$N^*=$	∞	10	5	3	3
$CR(\delta,N^*)=$	2.613	3.613	5.228	7.196	9.500
$C_{CM}=100$					
$N^*=$	∞	9	5	3	2
$CR(\delta,N^*)=$	2.614	3.667	5.380	7.449	9.871
$C_{CM}=120$					
$N^*=$	∞	8	4	3	2
$CR(\delta,N^*)=$	2.618	3.764	5.619	7.954	10.589
$C_{CM}=150$					
$N^*=$	∞	8	4	3	2
$CR(\delta,N^*)=$	2.623	3.893	5.957	8.712	11.665

Table 1-4 (*continued*)

	$\mu=1$	$\mu=2$	$\mu=3$	$\mu=4$	$\mu=5$
$C_{CM}=180$					
$N^*=$	∞	7	3	2	2
$CR(\delta,N^*)=$	2.628	3.998	6.283	9.185	12.742
$C_{CM}=200$					
$N^*=$	∞	7	3	2	2
$CR(\delta,N^*)=$	2.631	4.068	6.426	9.493	13.460
$C_{CM}=250$					
$N^*=$	∞	6	3	2	2
$CR(\delta,N^*)=$	2.639	4.206	6.785	10.263	15.254
$C_{CM}=300$					
$N^*=$	50	6	3	2	1
$CR(\delta,N^*)=$	2.647	4.337	7.144	11.032	16.311
$C_{CM}=400$					
$N^*=$	44	5	3	2	1
$CR(\delta,N^*)=$	2.664	4.532	7.861	12.572	18.143
$C_{CM}=500$					
$N^*=$	31	5	2	2	1
$CR(\delta,N^*)=$	2.680	4.708	8.296	14.111	19.974
$C_{CM}=1000$					
$N^*=$	18	4	2	1	1
$CR(\delta,N^*)=$	2.759	5.288	10.107	17.670	29.132

Table 2-1
δ^* and $CR(\delta^*,N)$ for a given $N=\infty$ and under $\zeta=20$, $C_{PM}=10$, and $c_{rm}=c_0=1$.

	$\mu=1$	$\mu=2$	$\mu=3$	$\mu=4$	$\mu=5$
$C_{CM}=30$					
$\delta^*=$	16.41	14.45	13.17	12.31	11.74
$CR(\delta^*,N)=$	2.549	3.246	4.050	4.924	5.833
$C_{CM}=40$					
$\delta^*=$	16.02	13.74	12.18	11.10	10.34
$CR(\delta^*,N)=$	2.562	3.310	4.216	5.243	6.350
$C_{CM}=50$					
$\delta^*=$	15.75	13.24	11.49	10.26	9.39
$CR(\delta^*,N)=$	2.571	3.360	4.349	5.507	6.792
$C_{CM}=60$					
$\delta^*=$	15.54	12.85	10.97	9.63	8.67
$CR(\delta^*,N)=$	2.579	3.401	4.462	5.739	7.189
$C_{CM}=70$					
$\delta^*=$	15.37	12.53	10.54	9.12	8.10
$CR(\delta^*,N)=$	2.586	3.436	4.562	5.949	7.555
$C_{CM}=80$					
$\delta^*=$	15.22	12.27	10.18	8.69	7.63
$CR(\delta^*,N)=$	2.591	3.467	4.652	6.142	7.898
$C_{CM}=90$					
$\delta^*=$	15.10	12.04	9.87	8.33	7.23
$CR(\delta^*,N)=$	2.596	3.495	4.735	6.323	8.223
$C_{CM}=100$					
$\delta^*=$	14.99	11.84	9.61	8.01	6.89
$CR(\delta^*,N)=$	2.600	3.520	4.812	6.494	8.535
$C_{CM}=120$					
$\delta^*=$	14.80	11.49	9.15	7.48	6.32
$CR(\delta^*,N)=$	2.608	3.566	4.952	6.811	9.125
$C_{CM}=150$					
$\delta^*=$	14.58	11.09	8.60	6.86	5.66
$CR(\delta^*,N)=$	2.617	3.624	5.138	7.246	9.952
$C_{CM}=180$					
$\delta^*=$	14.39	10.76	8.18	6.38	5.15
$CR(\delta^*,N)=$	2.625	3.673	5.302	7.644	10.730
$C_{CM}=200$					
$\delta^*=$	14.29	10.57	7.93	6.11	4.87
$CR(\delta^*,N)=$	2.630	3.703	5.403	7.894	11.228
$C_{CM}=250$					
$\delta^*=$	14.07	10.18	7.43	5.55	4.32
$CR(\delta^*,N)=$	2.639	3.768	5.634	8.482	12.424
$C_{CM}=300$					
$\delta^*=$	13.89	9.86	7.03	5.12	3.89
$CR(\delta^*,N)=$	2.648	3.825	5.842	9.030	13.566
$C_{CM}=400$					
$\delta^*=$	13.62	9.37	6.41	4.47	3.27
$CR(\delta^*,N)=$	2.661	3.920	6.210	10.044	15.747

Table 2-1 (*continued*)

	$\mu=1$	$\mu=2$	$\mu=3$	$\mu=4$	$\mu=5$
$C_{CM}=500$					
$\delta^*=$	13.41	9.00	5.95	4.01	2.84
$CR(\delta^*,N)=$	2.671	4.000	6.536	10.987	17.840
$C_{CM}=1000$					
$\delta^*=$	12.75	7.86	4.61	2.73	1.75
$CR(\delta^*,N)=$	2.706	4.290	7.857	15.195	27.727

Table 2-2
δ^* and $CR(\delta^*,N)$ for a given $N=15$ and under $\zeta=20$, $C_{PM}=10$, and $c_{rm}=c_0=1$.

	$\mu=1$	$\mu=2$	$\mu=3$	$\mu=4$	$\mu=5$
$C_{CM}=30$					
$\delta^*=$	16.52	14.45	13.17	12.31	11.74
$CR(\delta^*,N)=$	2.639	3.246	4.051	4.924	5.833
$C_{CM}=40$					
$\delta^*=$	16.13	13.74	12.18	11.10	10.34
$CR(\delta^*,N)=$	2.645	3.310	4.216	5.243	6.350
$C_{CM}=50$					
$\delta^*=$	15.85	13.24	11.49	10.26	9.39
$CR(\delta^*,N)=$	2.650	3.360	4.349	5.507	6.792
$C_{CM}=60$					
$\delta^*=$	15.63	12.85	10.97	9.63	8.67
$CR(\delta^*,N)=$	2.654	3.401	4.462	5.739	7.189
$C_{CM}=70$					
$\delta^*=$	15.45	12.53	10.54	9.12	8.10
$CR(\delta^*,N)=$	2.657	3.436	4.562	5.949	7.555
$C_{CM}=80$					
$\delta^*=$	15.31	12.27	10.18	8.69	7.63
$CR(\delta^*,N)=$	2.661	3.467	4.652	6.142	7.898
$C_{CM}=90$					
$\delta^*=$	15.18	12.04	9.87	8.33	7.23
$CR(\delta^*,N)=$	2.663	3.495	4.735	6.323	8.223
$C_{CM}=100$					
$\delta^*=$	15.07	11.84	9.61	8.01	6.89
$CR(\delta^*,N)=$	2.666	3.520	4.812	6.494	8.535
$C_{CM}=120$					
$\delta^*=$	14.87	11.49	9.15	7.48	6.32
$CR(\delta^*,N)=$	2.670	3.566	4.952	6.811	9.125
$C_{CM}=150$					
$\delta^*=$	14.64	11.09	8.60	6.86	5.66
$CR(\delta^*,N)=$	2.676	3.624	5.138	7.246	9.952
$C_{CM}=180$					
$\delta^*=$	14.45	10.76	8.18	6.38	5.15
$CR(\delta^*,N)=$	2.681	3.673	5.302	7.644	10.730
$C_{CM}=200$					
$\delta^*=$	14.35	10.57	7.93	6.11	4.87
$CR(\delta^*,N)=$	2.684	3.703	5.403	7.894	11.228
$C_{CM}=250$					
$\delta^*=$	14.12	10.18	7.43	5.55	4.32
$CR(\delta^*,N)=$	2.690	3.768	5.634	8.482	12.424
$C_{CM}=300$					
$\delta^*=$	13.95	9.86	7.03	5.12	3.89
$CR(\delta^*,N)=$	2.696	3.825	5.842	9.030	13.566
$C_{CM}=400$					
$\delta^*=$	13.66	9.37	6.41	4.47	3.27
$CR(\delta^*,N)=$	2.705	3.920	6.210	10.044	15.747
$C_{CM}=500$					
$\delta^*=$	13.45	9.00	5.95	4.01	2.84
$CR(\delta^*,N)=$	2.712	4.000	6.536	10.987	17.840
$C_{CM}=1000$					
$\delta^*=$	12.78	7.86	4.61	2.73	1.75
$CR(\delta^*,N)=$	2.738	4.290	7.857	15.195	27.727

average damage μ, the PM will be implemented earlier to avoid system's failure; meanwhile, a higher CM cost C_{CM} or a larger damage μ, will lead to the higher cost rate of system. Tables 2-1–2-5 also show similar results.

2. As observed in Tables 1-1–1-4, N^* increases as δ decreases. This point is reasonable, because for smaller δ, the damage-based PM will be implemented more often. At the same time,

Table 2-3
δ^* and $CR(\delta^*,N)$ for a given $N=10$ and under $\zeta=20$, $C_{PM}=10$, and $c_{rm}=c_0=1$.

	$\mu=1$	$\mu=2$	$\mu=3$	$\mu=4$	$\mu=5$
$C_{CM}=30$					
$\delta^*=$	16.81	14.48	13.17	12.31	11.74
$CR(\delta^*,N)=$	2.904	3.284	4.053	4.924	5.833
$C_{CM}=40$					
$\delta^*=$	16.41	13.93	12.19	11.10	10.34
$CR(\delta^*,N)=$	2.905	3.341	4.218	5.243	6.350
$C_{CM}=50$					
$\delta^*=$	16.12	13.26	11.50	10.26	9.39
$CR(\delta^*,N)=$	2.906	3.386	4.350	5.507	6.792
$C_{CM}=60$					
$\delta^*=$	15.91	12.86	10.97	9.63	8.67
$CR(\delta^*,N)=$	2.907	3.424	4.463	5.739	7.189
$C_{CM}=70$					
$\delta^*=$	15.72	12.55	10.54	9.12	8.10
$CR(\delta^*,N)=$	2.908	3.456	4.562	5.949	7.555
$C_{CM}=80$					
$\delta^*=$	15.57	12.28	10.18	8.69	7.63
$CR(\delta^*,N)=$	2.908	3.485	4.652	6.142	7.898
$C_{CM}=90$					
$\delta^*=$	15.43	12.05	9.87	8.33	7.23
$CR(\delta^*,N)=$	2.909	3.512	4.735	6.323	8.223
$C_{CM}=100$					
$\delta^*=$	15.32	11.85	9.61	8.01	6.89
$CR(\delta^*,N)=$	2.910	3.536	4.812	6.494	8.535
$C_{CM}=120$					
$\delta^*=$	15.12	11.50	9.15	7.48	6.32
$CR(\delta^*,N)=$	2.911	3.579	4.952	6.811	9.125
$C_{CM}=150$					
$\delta^*=$	14.88	11.09	8.60	6.86	5.66
$CR(\delta^*,N)=$	2.912	3.635	5.138	7.246	9.952
$C_{CM}=180$					
$\delta^*=$	14.69	10.76	8.18	6.38	5.15
$CR(\delta^*,N)=$	2.914	3.683	5.302	7.644	10.730
$C_{CM}=200$					
$\delta^*=$	14.58	10.57	7.93	6.11	4.87
$CR(\delta^*,N)=$	2.914	3.711	5.403	7.894	11.228
$C_{CM}=250$					
$\delta^*=$	14.35	10.18	7.43	5.55	4.32
$CR(\delta^*,N)=$	2.916	3.775	5.634	8.482	12.424
$C_{CM}=300$					
$\delta^*=$	14.16	9.86	7.03	5.12	3.89
$CR(\delta^*,N)=$	2.918	3.831	5.842	9.030	13.566
$C_{CM}=400$					
$\delta^*=$	13.87	9.37	6.41	4.47	3.27
$CR(\delta^*,N)=$	2.921	3.925	6.210	10.044	15.747
$C_{CM}=500$					
$\delta^*=$	13.65	9.00	5.95	4.01	2.84
$CR(\delta^*,N)=$	2.923	4.004	6.536	10.987	17.840
$C_{CM}=1000$					
$\delta^*=$	12.96	7.86	4.61	2.73	1.75
$CR(\delta^*,N)=$	2.933	4.291	7.857	15.195	27.727

Table 2-4
δ^* and $CR(\delta^*,N)$ for a given $N=8$ and under $\zeta=20$, $C_{PM}=10$, and $c_{rm}=c_0=1$.

	$\mu=1$	$\mu=2$	$\mu=3$	$\mu=4$	$\mu=5$
$C_{CM}=30$					
$\delta^*=$	17.01	14.56	13.18	12.32	11.74
$CR(\delta^*,N)=$	3.126	3.375	4.073	4.929	5.834
$C_{CM}=40$					
$\delta^*=$	16.60	13.83	12.20	11.11	10.35
$CR(\delta^*,N)=$	3.126	3.422	4.232	5.245	6.350
$C_{CM}=50$					
$\delta^*=$	16.32	13.31	11.50	10.27	9.39
$CR(\delta^*,N)=$	3.126	3.459	4.361	5.509	6.792
$C_{CM}=60$					
$\delta^*=$	16.10	12.92	10.98	9.63	8.68
$CR(\delta^*,N)=$	3.126	3.492	4.472	5.740	7.189
$C_{CM}=70$					
$\delta^*=$	15.91	12.60	10.55	9.12	8.11
$CR(\delta^*,N)=$	3.127	3.520	4.570	5.949	7.554
$C_{CM}=80$					
$\delta^*=$	15.76	12.32	10.18	8.69	7.63
$CR(\delta^*,N)=$	3.127	3.545	4.659	6.142	7.898
$C_{CM}=90$					
$\delta^*=$	15.62	12.09	9.88	8.33	7.23
$CR(\delta^*,N)=$	3.127	3.568	4.741	6.323	8.223
$C_{CM}=100$					
$\delta^*=$	15.51	11.89	9.61	8.01	6.89
$CR(\delta^*,N)=$	3.127	3.590	4.817	6.494	8.535
$C_{CM}=120$					
$\delta^*=$	15.30	11.53	9.15	7.48	6.32
$CR(\delta^*,N)=$	3.128	3.629	4.956	6.811	9.125
$C_{CM}=150$					
$\delta^*=$	15.07	11.12	8.61	6.86	5.66
$CR(\delta^*,N)=$	3.128	3.679	5.140	7.246	9.952
$C_{CM}=180$					
$\delta^*=$	14.87	10.79	8.18	6.38	5.15
$CR(\delta^*,N)=$	3.129	3.723	5.304	7.644	10.730
$C_{CM}=200$					
$\delta^*=$	14.76	10.60	7.94	6.11	4.87
$CR(\delta^*,N)=$	3.129	3.749	5.405	7.894	11.228
$C_{CM}=250$					
$\delta^*=$	14.53	10.20	7.43	5.55	4.32
$CR(\delta^*,N)=$	3.130	3.809	5.635	8.482	12.424
$C_{CM}=300$					
$\delta^*=$	14.34	9.88	7.03	5.12	3.89
$CR(\delta^*,N)=$	3.130	3.861	5.842	9.030	13.566
$C_{CM}=400$					
$\delta^*=$	14.05	9.38	6.41	4.47	3.27
$CR(\delta^*,N)=$	3.131	3.950	6.210	10.044	15.747
$C_{CM}=500$					
$\delta^*=$	13.82	9.01	5.95	4.01	2.84
$CR(\delta^*,N)=$	3.132	4.025	6.536	10.987	17.840
$C_{CM}=1000$					
$\delta^*=$	13.12	7.86	4.61	2.73	1.75
$CR(\delta^*,N)=$	3.136	4.303	7.857	15.195	27.727

the scheduled PM should be delayed or N^* becomes larger. Therefore, N^* increases gradually as δ decreases. Note that some values of N^* are infinite (i.e. $N^*=\infty$), as shown in Tables 1-3 and 1-4. This is because the condition $(C_{CM}-C_{PM})\overline{G}(\zeta-\delta)(\delta/\mu)>(C_{PM}-c_{rm})$ cannot be satisfied as predicted in Theorem 1.

3. According to Tables 2-1–2-5, if N is large, there will be no significant difference in optimal policies $(\delta^*,CR(\delta^*,N))$ for different N's; This is because that if N is large, then the scheduled PM is less frequently implemented. At this point, if the damage μ is large, thus its effects on δ^* and $CR(\delta^*,N)$ will becomes less related to N. As a result, the values of the optimal $(\delta^*,CR(\delta^*,N))$ are almost the same for the case with large N and μ. Only if N is small, the impact of N on the optimal $(\delta^*,CR(\delta^*,N))$ is significant.
4. Tables 1-1 and 2-1 show that $CR_1(N^*)>CR_2(\delta^*)$, which means that the damage-based PM outperforms the operation number-based PM This is because that the damage amount contains more information about the system condition than the number of operation periods does.

All of these observations are consistent with the analytical results in Sections 3 and 4.

6. Concluding remarks

In this paper, a maintenance policy for a continuously operating system is studied. Each period of operation causes a random amount of damage to the system and these damages are accumulated to trigger a PM or CM action. The system fails when the total damage exceeds a pre-specified failure level, and then corrective maintenance (CM) is performed to bring the system to a new condition. To prevent such a costly failure, a preventive

Table 2-5
δ^* and $CR(\delta^*,N)$ for a given $N=5$ and under $\zeta=20$, $C_{PM}=10$, and $c_{rm}=c_0=1$.

	$\mu=1$	$\mu=2$	$\mu=3$	$\mu=4$	$\mu=5$
$C_{CM}=30$					
$\delta^*=$	17.41	14.98	13.34	12.37	11.76
$CR(\delta^*,N)=$	3.800	3.875	4.316	5.041	5.889
$C_{CM}=40$					
$\delta^*=$	17.01	14.21	12.32	11.14	10.36
$CR(\delta^*,N)=$	3.800	3.896	4.441	5.332	6.388
$C_{CM}=50$					
$\delta^*=$	16.72	13.66	11.60	10.29	9.40
$CR(\delta^*,N)=$	3.800	3.914	4.546	5.579	6.820
$C_{CM}=60$					
$\delta^*=$	16.49	13.25	11.06	9.65	8.68
$CR(\delta^*,N)=$	3.800	3.931	4.639	5.799	7.210
$C_{CM}=70$					
$\delta^*=$	16.31	12.90	10.62	9.14	8.11
$CR(\delta^*,N)=$	3.800	3.946	4.723	6.000	7.572
$C_{CM}=80$					
$\delta^*=$	16.16	12.61	10.25	8.70	7.62
$CR(\delta^*,N)=$	3.800	3.960	4.801	6.186	7.911
$C_{CM}=90$					
$\delta^*=$	16.02	12.37	9.93	8.34	7.24
$CR(\delta^*,N)=$	3.800	3.973	4.873	6.362	8.234
$C_{CM}=100$					
$\delta^*=$	15.91	12.15	9.66	8.02	6.89
$CR(\delta^*,N)=$	3.800	3.985	4.940	6.528	8.544
$C_{CM}=120$					
$\delta^*=$	15.71	11.78	9.19	7.49	6.32
$CR(\delta^*,N)=$	3.800	4.008	5.066	6.839	9.132
$C_{CM}=150$					
$\delta^*=$	15.46	11.34	8.64	6.87	5.66
$CR(\delta^*,N)=$	3.800	4.039	5.234	7.266	9.956
$C_{CM}=180$					
$\delta^*=$	15.27	10.99	8.20	6.38	5.16
$CR(\delta^*,N)=$	3.800	4.067	5.387	7.660	10.733
$C_{CM}=200$					
$\delta^*=$	15.16	10.79	7.96	6.11	4.88
$CR(\delta^*,N)=$	3.800	4.085	5.481	7.908	11.231
$C_{CM}=250$					
$\delta^*=$	14.92	10.37	7.45	5.56	4.32
$CR(\delta^*,N)=$	3.800	4.124	5.699	8.492	12.425
$C_{CM}=300$					
$\delta^*=$	14.74	10.04	7.04	5.12	3.89
$CR(\delta^*,N)=$	3.800	4.160	5.897	9.037	13.566
$C_{CM}=400$					
$\delta^*=$	14.44	9.52	6.42	4.47	3.27
$CR(\delta^*,N)=$	3.800	4.224	6.252	10.049	15.747
$C_{CM}=500$					
$\delta^*=$	14.21	9.12	5.96	4.01	2.84
$CR(\delta^*,N)=$	3.800	4.280	6.569	10.990	17.840
$C_{CM}=1000$					
$\delta^*=$	13.51	7.94	4.61	2.73	1.75
$CR(\delta^*,N)=$	3.801	4.499	7.871	15.195	27.726

maintenance (PM) action is carried out at suitable time based on both the cumulative damage level and the number of operation periods. Using the renewal reward cycle, we derive the long-run expected cost rate and determine the cost minimization optimal policy. Both analytical and numerical results are presented and some important observations are made.

The model and the maintenance policy studied in this paper have wide applications. Besides the public transit bus maintenance example mentioned in Section 2, some entertainment facilities in amusement parks, such as roller-coasters, are another example of our model. Each day of operating a roller-coaster can be regarded as one operation period and corresponding CM, PM, and RM may be performed at the end of each period according to a maintenance policy like the one considered in this paper. Analyzing the maintenance policy with imperfect CM or PM actions can be a good topic for future research.

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