Actions for an Hierarchy of Attractive Nonlinear Oscillators Including The Quartic Oscillator in 1+1 Dimensions

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Abstract

In this paper, we present an explicit form in terms of end-point data for the classical action S_{2n} evaluated on extremals satisfying the Hamilton-Jacobi equation for each member of a hierarchy of classical non-relativistic oscillators characterized by even power potentials (i.e., attractive potentials $V_{2n}(y_{2n}) = \frac{1}{2n}k_{2n}y_{2n}^{2n}(t)|_{n\geq 1}$). The nonlinear quartic oscillator corresponds to $n = 2$ while the harmonic oscillator corresponds to $n = 1$.

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Part I

INTRODUCTION

The linearization map in [1] gives the solution for all members of the hierarchy $V_{2n}(y_{2n}) = \frac{1}{2n} k_{2n} y_{2n}^{2n}(t)|_{n\geq 1}$ in terms of the linear (harmonic) oscillator ($n = 1$). It consists of an explicit nonlinear deformation of coordinates and a nonlinear deformation of time coordinates involving a quadrature:

$$
\begin{aligned}
x(\hat{t}) &= (k_{2n}/nk_2)^{1/2} y_{2n}(t)(y_{2n}^2(t))^{(n-1)/2}, \\
y_{2n}(t) &= (nk_2/k_{2n})^{1/2n} x(\hat{t})(x^2(\hat{t}))^{(1/2)(1-n)/n},
\end{aligned}
\tag{1.1}
$$

$$
\begin{aligned}
\frac{dt}{d\hat{t}} &= n^{-(2n-1)/2n}(k_2/k_{2n})^{1/2n}(x^2(\hat{t}))^{-(n-1)/2n}, \\
\frac{d\hat{t}}{dt} &= \sqrt{n}(k_{2n}/k_2)^{1/2}(y_{2n}^2(t))^{(n-1)/2},
\end{aligned}
\tag{1.2}
$$

So, it important to keep in mind that all the quantities in this paper are known in principle as a consequence of (1.1) and (1.2)!

However, it is a non-canonical map. Therefore, to find a form for the action evaluated on an extremal, we can only use the linearization map as a guide, albeit an extremely useful one.

Critical to the form stated below is the fact that $y_{2n_{\max}}$ and $t_{\max}$ are constants of the motion and they uniquely characterize every extremal of the periodic systems studied in this paper. This allows us to find a form in terms of end-point data for the classical action S_{2n} evaluated on extremals for each member of a hierarchy of classical non-relativistic

oscillators characterized by even power potentials (i.e.,attractive potentials $V_{2n}(y_{2n}) = \frac{1}{2n}k_{2n}y_{2n}^{2n}(t)|_{n\geq 1}$). (The quartic oscillator corresponds to $n = 2$ while the harmonic oscillator corresponds to $n = 1$. The form is new for the harmonic linear oscillator and using the material in Part II one can readily check that it is equal in value to the one given in [2]-[3].) (See Appendix B.)

In particular, we arrive at a form for

$$S_{(2n)}(t_a, y_{2n_a}; t_b, y_{2n_b}) = \int_{t_{\max}}^{t_b} L_{2n}(y_{2n}(t), \frac{d}{dt}y_{2n}(t))dt|_{extremal} \quad ,$$
$$+ \int_{t_a}^{t_{\max}} L_{2n}(y_{2n}(t), \frac{d}{dt}y_{2n}(t))dt|_{extremal} \quad , \tag{1.3}$$

where L_{2n} equals the Lagrangian for the $2n$ oscillator, which satisfies

$$\frac{\partial S_{2n}}{\partial y_{2n_b}} = p_{2n_b} \quad ,$$
$$\frac{\partial S_{2n}}{\partial t_b} = -E = -H_{2n}(\frac{\partial S_{2n}}{\partial y_{2n_b}}, V_{2n}(y_{2n_b})) \quad , \tag{1.4}$$

and the time-reversed motion

$$\frac{\partial S_{2n}}{\partial y_{2n_a}} = -p_{2n_a} \quad ,$$
$$\frac{\partial S_{2n}}{\partial t_a} = +E = H_{2n}(\frac{\partial S_{2n}}{\partial y_{2n_a}}, V_{2n}(y_{2n_a})) \quad , \tag{1.5}$$

where H_{2n} = Hamiltonian for the $2n$ oscillator.

We shall use the notation $y = y_{2n}$ where it does not cause confusion.

$$S_{2n}(t_a,y_a;t_b,y_b) = \frac{m\omega}{(n+1)}(\frac{k_{2n}}{nk_2})^{1/2}$$

$$\{[(y_b^2)^{(n+1)/2}\cos\omega\int_{t_{\max}}^{t_b}\gamma(t)dt - (n+1)y_b\,y_{\max}(y_{\max^2})^{(n-1)/2} + n(y_{\max}^2)^{(n+1)/2}(\cos^2\omega\int_{t_{\max}}^{t_b}\gamma(t)dt)^{(n+1)/2n}/\cos(\omega\int_{t_{\max}}^{t_b}\gamma(t)dt)]$$

$$/\sin(\omega\int_{t_{\max}}^{t_b}\gamma(t)dt)$$

$$+$$

$$[(y_a^2)^{(n+1)/2}\cos\omega\int_{t_a}^{t_{\max}}\gamma(t)dt - (n+1)y_a y_{\max}(y_{\max^2})^{(n-1)/2} + n(y_{\max}^2)^{(n+1)/2}(\cos^2\omega\int_{t_a}^{t_{\max}}\gamma(t)dt)^{(n+1)/2n}/\cos(\omega\int_{t_a}^{t_{\max}}\gamma(t)dt)]$$

$$/\sin(\omega\int_{t_a}^{t_{\max}}\gamma(t)dt)\}$$

$$+$$

$$\frac{(n-1)}{(n+1)}\frac{k_{2n}}{2n}y^{2n}(t_b-t_a). \qquad (1.6)$$

where $\gamma(t) = (\frac{nk_{2n}}{k_2})^{1/2}\ (y^2(t))^{n-1/2}$.

In Part II, we discuss the description of the hierarchy extremals using the known results for the harmonic (linear) oscillator ($n=1$) that are implied by the linearization map presented in [1]. We start with the quartic oscillator and then present the general case.

In Part III, we set $n=2$ in (1.6) and show that the classical action evaluated on an extremal S_{qo} satisfies the Hamilton-Jacobi equation equations (1.4)-(1.5)$|_{n=2}$ for the quartic oscillator.

In Part IV, we take n arbitrary in (1.6) and show that the classical action evaluated on an extremal S_{2n} satisfies the Hamilton-Jacobi equations (1.4)-(1.5) for each member of the hierarchy. This reproduces the quartic oscillator result in Part III and as well as yielding a new form for the harmonic oscillator.

Part II

Extremals

The set of extremals for the harmonic (linear) oscillator is described by the endpoint solution of Newton's equation of motion

$$x(\hat{t}) = (x_b \sin\omega(\hat{t}-\hat{t}_a) + x_a \sin\omega(\hat{t}_b-\hat{t})/\sin\omega(\hat{t}_b-\hat{t}_a), \tag{2.1}$$

where the spring constant $k_2 = m\omega^2$, $m = mass$, $\hat{t}$ denotes the harmonic oscillator time and x denotes the space coordinate of the linear oscillator (ref [3] and eq. (1.1) in [1]). It is important to note that each extremal is uniquely characterized by $x_{\max}$ and a $\hat{t}_{\max}$. Specifically, we take $0 \leq \hat{t}_{\max} < \frac{2\pi}{\omega}$ and of course $x_{\max}$ is fixed by the energy. In practice the evaluation of quantities here we can take $\hat{t}_{\max}$ up to a multiple of the period.

Now the extremals are also described by the equation

$$x(\hat{t}) = x_{\max} \cos\omega(\hat{t}-\hat{t}_{\max}) \tag{2.2}$$

[(see Appendix A for a demonstation)].

The set of extremals for a quartic oscillator with mass m is described by the equivalent endpoint solution of Newton's equation of motion

$$(y^2(t))^{1/2}\, y(t) =$$

$$\left[\frac{y_b(y_b^2)^{(1/2)}\sin\omega\int_{t_a}^{t}(\frac{2k_4}{k_2})^{1/2}(y^2(t'))^{1/2}dt' \;+\; y_a(y_a^2)^{1/2}\sin\omega\int_{t}^{t_b}(\frac{2k_4}{k_2})^{1/2}(y^2(t'))^{1/2}dt'}{\sin\omega\int_{a}^{b}(\frac{2k_4}{k_2})^{1/2}(y^2(t'))^{1/2}dt'}\right] \tag{2.3}$$

and the equivalent integral equation

$$(y(t)(y^2(t))^{1/2}) - (y_{\max}(y_{\max}^2)^{1/2})\cos(\omega\int_{t_{max}}^{t}\gamma(t)dt') = 0 \tag{2.4}$$

where $\gamma = (2k_4/k_2)^{1/2}(y^2)^{1/2}$, $y_{max} = (4E/k_4)^{1/4}$, k_4 = denotes the quartic spring constant, t denotes quartic oscillator time and $y = y_{qo} = y_4$ denotes the space coordinate of the quartic oscillator.

(We remind the reader that because of the linearization map given in [1], the integral equation (2.4) is solved. However, as an aside, we would like to point out that $\int_{t_{max}}^{t}\gamma(t')dt'$ can also be determined from (2.4) since we know the pairs $(t,y(t))$ and $(t_{\max},y_{\max})$.)

The argument in Appendix A generalizes to this case for the equivalence of (2.3) and (2.4). Further, that the integral equation (2.4) satisfies Newtons equation of motion $\frac{d^2}{dt^2}my(t) = -k_4y^3(t)$ follows by direct differentiation twice.

The pairs (2.1)-(2.2) and (2.3)-(2.4) are connected by the linearization map (1.1)-1.2) given in [1] between the linear oscillator and the quartic oscillator.

Using the preceding arguments, (2.3)-(2.4) generalize to

$$(y_{2n}^2(t))^{(n-1)/2}\, y_{2n}(t) =$$

$$\left[\frac{y_{2n_b}(y_{2n_b}^2)^{(n-1)/2}\sin\omega\int_{t_a}^{t}\gamma(t')dt' \;+\; y_{2n_a}(y_{2n_a}^2)^{(n-1)/2}\sin\omega\int_{t}^{t_b}\gamma(t')dt'}{\sin\omega\int_{t_a}^{t_b}\gamma(t')dt'}\right] \tag{2.5}$$

which is equivalent to the integral equation

$$(y_{2n}(t)(y_{2n}^2(t))^{(n-1)/2}) - (y_{2n_{\max}}(y_{2n_{\max}}^2)^{(n-1)/2})\cos(\omega\int_{t_{\max}}^{t}\gamma(t')dt') = 0, \tag{2.6}$$

where $\gamma(t') = (\frac{nk_{2n}}{k_2})^{1/2}(y_{2n}(t'))^{(n-1)/2}$.

The above imply the following momenta since all systems have the same mass m:

$$p_{2n_b} = \frac{m\omega}{}(\frac{k_{2n}}{nk_2})^{1/2}\left[(y_{2n_b}^2)^{(n-1)/2}y_{2n_b}\cos(\omega\int_{t_{\max}}^{t_b}\gamma(t)dt) - (y_{2n_{\max}}^2)^{(n-1)/2}y_{2n_{\max}}\right]/\sin(\omega\int_{t_{\max}}^{t_b}\gamma(t)dt), \tag{2.7a}$$

$$p_{2n_a} = \frac{m\omega}{}(\frac{k_{2n}}{nk_2})^{1/2}\left[(y_{2n_a}^2)^{(n-1)/2}y_{2n_a}\cos(\omega\int_{t_a}^{t_{\max}}\gamma(t)dt) - (y_{2n_{\max}}^2)^{(n-1)/2}y_{2n_{\max}}\right]/\sin(\omega\int_{t_a}^{t_{\max}}\gamma(t)dt), \tag{2.7b}$$

These are the equations we have to reproduce with our S_{2n}.

Note (2.6) implies

$$(y_{2n}^2(t_b))^{(n-1)/2}) = (y_{2n_{\max}}^2)^{(n-1)/2})(\cos^2(\omega\int_{t_{\max}}^{t_b}\gamma(t)dt))^{(n-1)/2n}, \tag{2.8a}$$

and

$$(y_{2n}^2(t_a))^{(n-1)/2}) = (y_{2n_{\max}}^2)^{(n-1)/2})(\cos^2(\omega\int_{t_a}^{t_{\max}}\gamma(t)dt))^{(n-1)/2n}. \tag{2.8b}$$

These latter relations are needed to evaluate γ when we differentiate w.r.t time for $n = 2$ in Part III and arbitrary n in Part IV.

Part III

Nonlinear action for the quartic oscillator

As mentioned above, because the linearization map given in [1] is a non-canonical one, we do not have a derivation of the actions for the nonlinear quartic oscillator in terms of y_b, y_a, t_b and t_a, rather we have constructed it, namely (1.6) with $n = 2$, using the linearization map as a guide.

$$
\begin{aligned}
&S_{qo}(t_a,y_a;t_b,y_b) = S_4(t_a,y_a;t_b,y_b,)\\
&= \frac{m\omega}{3}(\frac{k_4}{2k_2})^{1/2}\\
&\{\left[(y_b^2)^{3/2}\cos(\omega\int_{t_{\max}}^{t_b}\gamma(t)dt) - 3y_b\ y_{\max}(y_{\max}^2)^{1/2}\ +\ 2(y_{\max}^2)^{3/2}(\cos^2(\omega\int_{t_{\max}}^{t_b}\gamma(t)dt)^{3/4}\ /\cos(\omega\int_{t_{\max}}^{t_b}\gamma(t)dt))\right]\\
&/\sin(\omega\int_{t_{\max}}^{t_b}\gamma(t)dt)\\
&+\\
&\left[(y_a^2)^{3/2}\cos(\omega\int_{t_a}^{t_{\max}}\gamma(t)dt) - 3y_a\ y_{\max}(y_{\max}^2)^{1/2}\ +\ 2(y_{\max}^2)^{3/2}(\cos^2(\omega\int_{t_a}^{t_{\max}}\gamma(t)dt)^{3/4}\ /\cos(\omega\int_{t_a}^{t_{\max}}\gamma(t)dt))\right]\\
&/\sin(\omega\int_{t_a}^{t_{\max}}\gamma(t)dt)\}\\
&+\frac{1}{12}\,k_4\,y_{\max}^4(t_b-t_a)
\end{aligned}
\tag{3.1}
$$

This implies

$$
\begin{aligned}
&\frac{\partial}{\partial y_b}S_{qo}(t_a,y_a;t_b,y_b)\\
&= \frac{m\omega}{3}(\frac{k_4}{2k_2})^{1/2}\\
&\left[(y_b^2)^{1/2}\ y_b\ \cos(\omega\int_{t_{\max}}^{t_b}\gamma(t)dt) - (y_{\max}^2)^{1/2}y_{max}\right]/\sin(\omega\int_{t_{\max}}^{t_b}\gamma(t)dt) = p_b
\end{aligned}
\tag{3.2}
$$

and

$$
\begin{aligned}
&\frac{\partial}{\partial t_b} S_{qo}(t_a, y_a; t_b, y_b) \\
&= \frac{m\omega^2}{3} (\frac{k_4}{2k_2})^{1/2} \gamma_b \{ \\
&\left[(y_b^2)^{3/2} \cos(\omega \int_{t_{\max}}^{t_b} \gamma dt) - 3 y_b \; y_{\max} (y_{\max}^2)^{1/2} \;+\; 2(y_{\max}^2)^{3/2} (\cos^2(\omega \int_{t_{\max}}^{t_b} \gamma(t) dt)^{3/4} \; / \cos(\omega \int_{t_{\max}}^{t_b} \gamma(t) dt)) \right] \\
&(-\cos(\omega \int_{t_{\max}}^{t_b} \gamma(t) dt) / \sin^2(\omega \int_{t_{\max}}^{t_b} \gamma(t) dt)) \\
&+ [(y_b^2)^{3/2} (-\sin(\omega \int_{t_{\max}}^{t_b} \gamma(t) dt) \\
&+ 2(y_{\max}^2)^{3/2} (3/4) \langle -2\cos(\omega \int_{t_{\max}}^{t_b} \gamma(t) dt) \sin(\omega \int_{t_{\max}}^{t_b} \gamma(t) dt) \rangle / ((\cos^2(\omega \int_{t_{\max}}^{t_b} \gamma(t) dt))^{1/4} \cos(\omega \int_{t_{\max}}^{t_b} \gamma(t) dt)) \\
&+ 2(y_{\max}^2)^{3/2} (\cos^2(\omega \int_{t_{\max}}^{t_b} \gamma(t) dt))^{3/4} \sin(\omega \int_{t_{\max}}^{t_b} \gamma(t) dt) / \cos^2(\omega \int_{t_{\max}}^{t_b} \gamma(t) dt)] \\
&/ \sin(\omega \int_{t_{\max}}^{t_b} \gamma(t) dt) \} \\
&+ \frac{1}{12} k_4 y_{\max}^4 \\
&= -\frac{k_4}{3} y_{\max}^4 + \frac{1}{12} k_4 y_{\max}^4 = -\frac{k_4 y_{\max}^4}{4}
\end{aligned}
\tag{3.3}
$$

(We have used (2.4) $|_{t=t_b}$ and (2.8a) $|_{n=2}$ after differentiating wrt t_b to obtain (3.3).) The first four terms sum to zero.

$$
\begin{aligned}
&\frac{\partial S_{qo}}{\partial y_b} = p_{qo_b} \quad , \\
&\frac{\partial S_{qo}}{\partial t_b} = -E = H_{qo}(\frac{\partial S_{qo}}{\partial y_b}, V_{qo}(y_b)) \quad .
\end{aligned}
\tag{3.4}
$$

This agrees with (2.7a) and (1.4) for $n = 2$.

The a-differentiations parallel the b-differentiations and yield

$$\frac{\partial S_{qo}}{\partial y_a} = -p_{qo_a} \quad ,$$
$$\frac{\partial S_{qo}}{\partial t_a} = +E = H_{qo}(\frac{\partial S_{qo}}{\partial y_a}, V_{qo}(y_a)) \quad . \tag{3.5}$$

This agrees with (2.7b) and (1.5) for $n = 2$.

It is important to note that the value of S_{qo} is not changed if the substitution of $(2.4)|_{t=t_b}$ is made in (3.1). However, this substitution cannot be made before all differentiations are made because the choice of the form in terms of space-time endpoints for S_{qo} is critical.

Part IV

Actions for the nonlinear hierarchy

$$V_2(y_{2n}) = \frac{1}{2n} k_{2n} y_{2n}^{2n}(t) \,|_{n \geq 1}.$$

Here we generalize the approach from Part III.

The action on an extremal $S_{2n}(t_a, y_a; t_b, y_b)$ is given (1.4) (Here, we shall use the notation $y = y_{2n}$.). We now proceed to verify that (1.6) satisfies the Hamilton-Jacobi equations (1.4)-(1.5) by verifying that the following equations are satisfied (We shall use the notation $y = y_{2n}$.):

$$
\begin{aligned}
&\frac{\partial}{\partial y_b} S_{2n}(t_a, y_a; t_b, y_b) \\
&= \; m\omega (\frac{k_{2n}}{nk_2})^{1/2} \\
&\left[(y_b^2)^{(n-1)/2} \; y_b \; \cos(\omega \int_{t_{\max}}^{t_b} \gamma(t)dt) - y_{\max} (y_{\max}^2)^{(n-1)/2} \right] \\
&/\sin(\omega \int_{t_{\max}}^{t_b} \gamma(t)dt) = p_{2n_b}, \qquad (4.1)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial}{\partial t_b} S_{2n}(t_a, y_a; t_b, y_b) = \frac{m\omega^2}{(n+1)} (\frac{k_{2n}}{nk_2})^{1/2} \gamma_b \\
&\{ \left[(y_b^2)^{(n+1)/2} \cos(\omega \int_{t_{\max}}^{t_b} \gamma(t)dt - (n+1) y_b \; y_{\max} (y_{\max}^2)^{(n-1)/2} \; + \; n (y_{\max}^2)^{(n+1)/2} (\cos^2(\omega \int_{t_{\max}}^{t_b} \gamma(t)dt))^{(n+1)/2n} / \cos(\omega \int_{t_{\max}}^{t_b} \gamma(t)dt) \right] \\
&(-\cos(\omega \int_{t_{\max}}^{t_b} \gamma(t)dt) / \sin^2(\omega \int_{t_{\max}}^{t_b} \gamma(t)dt) \\
&+ \left[(y_a^2)^{(n+1)/2} - \sin(\omega \int_{t_{\max}}^{t_b} \gamma(t)dt)) + n (y_{\max}^2)^{(n+1)/2n} ((n+1)/2) \cos^2(\omega \int_{t_{\max}}^{t_b} \gamma(t)dt)^{((n+1)/2n-1} \langle -2\cos(\omega \int_{t_{\max}}^{t_b} \gamma(t)dt) \right. \\
&\left. \sin(\omega \int_{t_{\max}}^{t_b} \gamma(t)dt) \rangle / \cos(\omega \int_{t_{\max}}^{t_b} \gamma(t)dt) + n (y_{\max}^2)^{(n+1)/2} \cos^2 \omega \int_{t_{\max}}^{t_b} \gamma(t)dt)^{(n+1)/2n} \; \sin(\omega \int_{t_{\max}}^{t_b} \gamma(t)dt) / (\cos^2(\omega \int_{t_{\max}}^{t_b} \gamma(t)dt.) \right] \\
&/\sin(\omega \int_{t_{\max}}^{t_b} (t)dt) \} \\
&+ \frac{n-1}{n+1} \frac{k_{2n}}{2n} y_{\max}^{2n} \\
&= -\frac{k_{2n}}{n+1} y_{\max}^{2n} + \frac{(n-1)}{(n+1)} \frac{k_{2n}}{2n} y_{\max}^{2n}, \qquad (4.2)
\end{aligned}
$$

where $\gamma_b = (\frac{nk_{2n}}{k_2})^{1/2} (y_b^2)^{(n-1)/2}$.

(We have used (2.4) $|_{t=t_b}$ and (2.8a) after differentiating t_b to obtain (4.2).) The first four terms sum to zero.

Thus,

$$\frac{\partial S_{2n}}{\partial y_b} = p_{2n_b} \ ,$$
$$\frac{\partial S_{2n}}{\partial t_b} = -E, \tag{4.3}$$

which agrees with (2.7a) and (1.4).

The *a*-differentiations parallel the *b*-differentiations and yield.

$$\frac{\partial S_{2n}}{\partial y_a} = p_{2n_a} \ ,$$
$$\frac{\partial S_{2n}}{\partial t_a} = E, \tag{4.4}$$

which agrees with (2.7b) and (1.5).

Part V

Concluding Remarks

To the best of the author's knowledge of the existing literature on classical mechanics, he can not find any literature on general transformation theory devoted to transforming one system at a given time to another at a distinctly different time, hence the absence of references to the classical literature on this point in this paper.

Part VI

Acknowledgements

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References

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[2] Feynmans Thesis-A New Approach To Quantum Theory, Laurie M. Brown-editor. Singapore: World Scientific, 2005. See equation (27), p 18 and set . We are grateful to Justin Manning for bringing this result to our attention.

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Appendix A

As stated in Part II, the set of extremals for the harmonic (linear) oscillator is given by

$$x(\hat{t}) = (x_b \sin\omega(\hat{t}-\hat{t}_a) + x_a \sin\omega(\hat{t}_b-\hat{t})/\sin\omega(\hat{t}_b-\hat{t}_a), \qquad (2.1)$$

where the spring constant $k_2 = m\omega^2$, $m = mass$, $\hat{t}$ denotes the harmonic oscillator time and x denotes the space coordinate of the linear oscillator.

We now demonstrate that equation (A.1) is equivalent to the relationship

$$x(\hat{t}) = x_{\max} \cos\omega(\hat{t}-\hat{t}_{\max}). \qquad (2.2)$$

Rewriting (2.1) as

$$x(\hat{t}) = (x_b \sin\omega(\hat{t}-\hat{t}_{\max}+\hat{t}_{\max}-\hat{t}_a) + x_a \sin\omega(\hat{t}_b-\hat{t}_{\max}+\hat{t}_{\max}-\hat{t}))/\sin\omega(\hat{t}_b-\hat{t}_{\max}+\hat{t}_{\max}-\hat{t}_a), \qquad (A.1)$$

cross-multiplying and expanding, we have

$$x(\hat{t})\sin\omega(\hat{t}_b-\hat{t}_{\max}+\hat{t}_{\max}-\hat{t}_a) = (x_b \sin\omega(\hat{t}-\hat{t}_{\max}+\hat{t}_{\max}-\hat{t}_a) + x_a \sin\omega(\hat{t}_b-\hat{t}_{\max}+\hat{t}_{\max}-\hat{t}))$$

or

$$\begin{aligned} & x(\hat{t})\{\sin\omega(\hat{t}_b-\hat{t}_{\max})\cos\omega(\hat{t}_{\max}-\hat{t}_a) + \cos\omega(\hat{t}_b-\hat{t}_{\max})\sin\omega(\hat{t}_{\max}-\hat{t}_a)\} \\ & = x_b\{\sin\omega(\hat{t}-\hat{t}_{\max})\cos\omega(\hat{t}_{\max}-\hat{t}_a) + \cos\omega(\hat{t}-\hat{t}_{\max})\sin\omega(\hat{t}_{\max}-\hat{t}_a)\} \\ & + x_a\{\sin\omega(\hat{t}_b-\hat{t}_{\max})\cos\omega(\hat{t}_{\max}-\hat{t}) + \cos\omega(\hat{t}_b-\hat{t}_{\max})\sin\omega(\hat{t}_{\max}-\hat{t})\} \end{aligned} \qquad (A.2)$$

Substituting (2.2), we obtain the identity

$$\begin{aligned} & x(\hat{t})\,\sin\omega(\hat{t}_b-\hat{t}_{\max})\frac{x_a}{x_{\max}}+x(\hat{t})\frac{x_b}{x_{\max}}\sin\omega(\hat{t}_{\max}-\hat{t}_a) \\ & =x_b\sin\omega(\hat{t}-\hat{t}_{\max})\frac{x_a}{x_{\max}}+x_b\frac{x(\hat{t})}{x_{\max}}\sin\omega(\hat{t}_{\max}-\hat{t}_a) \\ & +x_a\sin\omega(\hat{t}_b-\hat{t}_{\max})\frac{x(\hat{t})}{x_{\max}}+x_a\frac{x_b}{x_{\max}}\sin\omega(\hat{t}_{\max}-\hat{t}), \end{aligned} \tag{A.3}$$

where the 1st and the 4th terms on the r.h.s. of (A.3) cancel.

You can now run the above argument backwards. Thus we have shown that (2.1) and (2.2) are equivalent.

Appendix B

It follows from differentiating (2.1) and (2.2) and setting $\hat{t}=\hat{t}_b$ that

$$-x_{\max}\,\sin\omega(\hat{t}_b-\hat{t}_{\max})=\frac{x_b\cos\omega(\hat{t}_b-\hat{t}_a)-x_a}{\sin\omega(\hat{t}_b-\hat{t}_a)}. \tag{B.1}$$

Similarly, it follows that

$$x_{\max}\,\sin\omega(\hat{t}_{\max}-\hat{t}_a)=\frac{x_b-x_a\cos\omega(\hat{t}_b-\hat{t}_a)}{\sin\omega(\hat{t}_b-\hat{t}_a)}. \tag{B.2}$$

Hence

$$m\omega[\frac{(x_b^2+x_a^2)\cos\omega(\hat{t}_b-\hat{t}_a)-2x_bx_a}{2\sin\omega(\hat{t}_b-\hat{t}_a)}]$$
$$=\frac{m\omega}{2}[-x_{\max}x_b\sin\omega(\hat{t}_b-\hat{t}_{\max})-x_{\max}x_a\sin\omega(\hat{t}_{\max}-\hat{t}_a)]. \quad \text{(B.3)}$$

where the l.h.s. is the S_{ho} of [2] - [3].

The action $S_{2n}|_{n=1}=S_{ho}$ with $x=y_2$ given by (1.6) is

$$S_{ho}=\frac{m\omega}{2}\{[x_b^2\cos\omega(\hat{t}_b-\hat{t}_{\max})-2x_bx_{\max}+\frac{x_{\max}^2\cos^2\omega(\hat{t}_b-\hat{t}_{\max})}{\cos\omega(\hat{t}_b-\hat{t}_{\max})}]/\sin\omega(\hat{t}_b-\hat{t}_{\max})$$
$$+[x_a^2\cos\omega(\hat{t}_{\max}-\hat{t}_a)-2x_ax_{\max}+\frac{x_{\max}^2\cos^2\omega(\hat{t}_{\max}-\hat{t}_a)}{\cos\omega(\hat{t}_{\max}-\hat{t}_a)}]/\sin\omega(\hat{t}_{\max}-\hat{t}_a) \quad \text{(B.4)}$$

Now $\text{(B.4)}|_{(2.2)}$ in value is given by

$$\frac{m\omega}{2}\{x_b^2[\cos\omega(\hat{t}_b-\hat{t}_{\max})-\frac{1}{\cos\omega(\hat{t}_b-\hat{t}_{\max})}]/\sin\omega(\hat{t}_b-\hat{t}_{\max})$$
$$+x_a^2[\cos\omega(\hat{t}_{\max}-\hat{t}_a)-\frac{1}{\cos\omega(\hat{t}_{\max}-\hat{t}_a)}]/\sin\omega(\hat{t}_{\max}-\hat{t}_a)\}$$
$$=\frac{m\omega}{2}[-x_bx_{\max}\sin\omega(\hat{t}_b-\hat{t}_{\max})-x_ax_{\max}\sin\omega(\hat{t}_{\max}-\hat{t}_a)]. \quad \text{(B.5)}$$

The r.h.s. of (B.5) equals the l.h.s. of (B.3).