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Effect of Position-dependent Mass on Dynamical Breaking of Type B and Type X_2 $\mathcal{N}$-fold Supersymmetry

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Abstract

We investigate effect of position-dependent mass profiles on dynamical breaking of $\mathcal{N}$-fold supersymmetry in several type B and type X_2 models. We find that $\mathcal{N}$-fold supersymmetry in rational potentials in the constant-mass background are steady against the variation of mass profiles. On the other hand, some physically relevant mass profiles can change the pattern of dynamical $\mathcal{N}$-fold supersymmetry breaking in trigonometric, hyperbolic, and exponential potentials of both type B and type X_2. The latter results open the possibility of detecting experimentally phase transition of $\mathcal{N}$-fold as well as ordinary supersymmetry at a realistic energy scale.

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I. INTRODUCTION

In recent years, the study of quantum mechanical systems with a position-dependent mass (PDM) have attracted a lot of interest due to their relevance in describing the physics of many microstructures of current interests, such as compositionally graded crystals [1], semiconductor heterostructure [2], quantum dots [3], ^{3}He clusters [4], metal clusters [5] etc. The concept of PDM comes from the effective-mass approximation [6, 7] which is a useful tool for studying the motion of carrier electrons in pure crystals and also for the virtual-crystal approximation in the treatment of homogeneous alloys (where the actual potential is approximated by a periodic potential) as well as in graded mixed semiconductors (where the potential is not periodic). Recent interest in this field stems from extraordinary development in crystal-growth techniques like molecular beam epitaxy, which allow the production of nonuniform semiconductor specimen with abrupt heterojunctions [8]. In these mesoscopic materials, the effective mass of the charge carrier are position dependent. Consequently, the study of the position-dependent mass Schrödinger equation (PDMSE) becomes relevant for deeper understanding of the non-trivial quantum effects observed on these nanostructures. It has also been found that such equations appear in many different areas. For example, it has been shown that constant mass Schrödinger equations in curved space and those based on deformed commutation relations can be interpreted in terms of PDMSE [9]. The PDM also appear in nonlinear oscillator [10, 11] and $\mathcal{PT}$-symmetric cubic anharmonic oscillator [12]. The most general form of the PDM Hamiltonian proposed by von Roos [13] is defined by

$$H = -\frac{1}{4}\left(m(q)^{\alpha}\frac{\mathrm{d}}{\mathrm{d}q}m(q)^{\beta}\frac{\mathrm{d}}{\mathrm{d}q}m(q)^{\gamma} + m(q)^{\gamma}\frac{\mathrm{d}}{\mathrm{d}q}m(q)^{\beta}\frac{\mathrm{d}}{\mathrm{d}q}m(q)^{\alpha}\right) + V(q), \tag{1.1}$$

where the ambiguity parameters α, β, γ are related by $\alpha + \beta + \gamma = -1$. The above Hamiltonian always has the following form:

$$H = -\frac{1}{2m(q)}\frac{\mathrm{d}^2}{\mathrm{d}q^2} + \frac{m'(q)}{2m(q)^2}\frac{\mathrm{d}}{\mathrm{d}q} + U(q), \tag{1.2}$$

where the effective potential $U(q)$ is given by

$$U(q) = V(q) - (\alpha + \gamma)\frac{m''(q)}{4m(q)^2} + (\alpha\gamma + \alpha + \gamma)\frac{m'(q)^2}{2m(q)^3}. \tag{1.3}$$

It is quite natural that physical interests just described above have also enhanced the studies on exact solutions to PDMSE [14–30] by employing various methods e.g. supersymmetric (SUSY) quantum mechanics [31] and point canonical transformation [32] to mention a few. Later, PDM quantum systems were successfully formulated in the framework of $\mathcal{N}$-fold SUSY in Ref. [33], which has provided until now the most general tool for constructing a PDM system which admits exact solutions because of its equivalence to weak quasi-solvability. To avoid confusion, we here note that $\mathcal{N}$-fold SUSY is different from *nonlinear SUSY* which has been long employed since the work by Samuel and Wess [34] in 1983 to indicate the nonlinearly realized SUSY originated from the work by Akulov and Volkov [35] in 1972. For a review of $\mathcal{N}$-fold SUSY see Ref. [36], while for recent works on nonlinear SUSY see, e.g., Ref. [37] and references cited therein.

Very recently, new classes of exactly solvable PDM quantum systems whose eigenfunctions are expressible in terms of so-called X_1 polynomials were constructed in Ref. [28]. The new findings of X_n polynomials ($n \geq 1$) were associated with the more fundamental mathematical concept of exceptional polynomial subspaces of codimension n introduced in Refs. [38–40], whose origin can be traced back to the pioneering work on the classification of monomial spaces preserved by second-order linear differential operators [41].

The purpose of the present paper is two-fold. The first one is to bring the purely mathematical concept of exceptional polynomial subspaces into more physical settings by allowing the position dependence of mass (in a spirit similar to Ref. [28]) in the framework of $\mathcal{N}$-fold SUSY. In the constant-mass case, form of potentials related to exceptional polynomial systems is very limited. Thus, we can enlarge the physical applicability of the mathematical concept by introducing PDM to quantum systems. On the other hand, the framework of $\mathcal{N}$-fold SUSY enables us to talk about the physical phenomenon of dynamical $\mathcal{N}$-fold SUSY breaking. The second purpose is actually to examine effect of PDM profiles on dynamical breaking of $\mathcal{N}$-fold SUSY. In this respect, it is rather surprising that there have been few papers, like Ref. [11], where broken as well as unbroken SUSY is described in PDM backgrounds depending on the mass profiles. One of the main reasons would be that SUSY has been mostly used just as a technique to obtain exact solutions. The true significance of the Witten's SUSY quantum mechanics [31], however, rather resides in the nonperturbative aspects of dynamical SUSY breaking. Hence, one of our main purposes is, in other words, to examine change of nonperturbative nature of quantum systems caused by variations of mass profiles in view of dynamical $\mathcal{N}$-fold SUSY breaking.

The paper is organized as follows. In Section II, we provide a self-contained review of $\mathcal{N}$-fold SUSY in a PDM background, especially for those who are not familiar with the subject. We also summarize mathematical structure of type B and type X_2 $\mathcal{N}$-fold SUSY. In Section III, we construct several $\mathcal{N}$-fold SUSY PDM quantum systems and examine dynamical $\mathcal{N}$-fold SUSY breaking in different PDM backgrounds. The first three models of type B $\mathcal{N}$-fold SUSY have rational, trigonometric, and exponential potentials in the constant mass case. We show in particular that the models whose bound state eigenfunctions were shown to be expressed in terms of X_1 polynomials in Ref. [42] for the constant mass case and in Ref. [28] for the PDM cases can be obtained as type B systems. The last three models of type X_2 $\mathcal{N}$-fold SUSY have rational, hyperbolic, and exponential potentials in the constant mass case. For both types of $\mathcal{N}$-fold SUSY, we find that the rational potentials have steady $\mathcal{N}$-fold SUSY against variation of mass profile while all the other types of potentials can receive effect of PDM on their dynamical breaking of $\mathcal{N}$-fold SUSY. Finally, we summarize the results and discuss their implications and prospects in Section IV.

II. REVIEW OF $\mathcal{N}$-FOLD SUPERSYMMETRY IN A PDM BACKGROUND

An $\mathcal{N}$-fold SUSY one-body quantum mechanical system with PDM is composed of a pair of PDM Hamiltonians

$$H^{\pm} = -\frac{1}{2m(q)}\frac{\mathrm{d}^2}{\mathrm{d}q^2} + \frac{m'(q)}{2m(q)^2}\frac{\mathrm{d}}{\mathrm{d}q} + U^{\pm}(q), \tag{2.1}$$

and an $\mathcal{N}$th-order linear differential operator

$$P_{\mathcal{N}}^{-} = m(q)^{-\mathcal{N}/2}\frac{\mathrm{d}^{\mathcal{N}}}{\mathrm{d}q^{\mathcal{N}}} + \sum_{k=0}^{\mathcal{N}-1} w_k^{[\mathcal{N}]}(q)\frac{\mathrm{d}^k}{\mathrm{d}q^k}, \tag{2.2}$$

which satisfy the following intertwining relations

$$P_{\mathcal{N}}^{-}H^{-} = H^{+}P_{\mathcal{N}}^{-}, \qquad P_{\mathcal{N}}^{+}H^{+} = H^{-}P_{\mathcal{N}}^{+}. \tag{2.3}$$

In the above, $P_{\mathcal{N}}^{+}$ is the transposition [43] of $P_{\mathcal{N}}^{-}$ given by

$$P_{\mathcal{N}}^{+} = (P_{\mathcal{N}}^{-})^{\mathrm{T}} = \left(-\frac{\mathrm{d}}{\mathrm{d}q}\right)^{\mathcal{N}} m(q)^{-\mathcal{N}/2} + \sum_{k=0}^{\mathcal{N}-1}\left(-\frac{\mathrm{d}}{\mathrm{d}q}\right)^{k} w_{k}^{[\mathcal{N}]}(q). \tag{2.4}$$

Actually, the two relations in (2.3) are not independent; the first implies the second and vice versa, since the PDM Hamiltonians (2.1) are invariant under the transposition $(H^{\pm})^{\mathrm{T}} = H^{\pm}$.

One of the significant consequences of the intertwining relations (2.3) is *weak quasi-sovability*, that is, $H^{\pm}$ preserves a finite-dimensional linear space $\mathcal{V}_{\mathcal{N}}^{\pm}$ spanned by the kernel of the operator $P_{\mathcal{N}}^{\pm}$

$$H^{\pm}\mathcal{V}_{\mathcal{N}}^{\pm} \subset \mathcal{V}_{\mathcal{N}}^{\pm}, \qquad \mathcal{V}_{\mathcal{N}}^{\pm} = \ker P_{\mathcal{N}}^{\pm}. \tag{2.5}$$

Each space $\mathcal{V}_{\mathcal{N}}^{\pm}$ is called a solvable sector of $H^{\pm}$. Except for the $\mathcal{N} = 2$ case (cf., Refs. [36, 44]), virtually all the $\mathcal{N}$-fold SUSY systems so far found admit analytic expression of $\mathcal{V}_{\mathcal{N}}^{\pm}$ in closed form, and thus are *quasi-solvable*. In addition, it sometimes happens when either H^{-} or H^{+} does not depend essentially on $\mathcal{N}$ and preserves an infinite flag of the solvable sectors

$$\mathcal{V}_{1}^{-/+} \subset \mathcal{V}_{2}^{-/+} \subset \cdots \subset \mathcal{V}_{\mathcal{N}}^{-/+} \subset \cdots . \tag{2.6}$$

In this case, it is said to be *solvable*, which is a necessary condition for exact solvability. We note that H^{-} and H^{+} are usually simultaneously solvable due to the intertwining relations (2.3).

A set of an $\mathcal{N}$-fold SUSY system $H^{\pm}$ and $P_{\mathcal{N}}^{\pm}$ provides a representation of $\mathcal{N}$-fold superalgebra defined by

$$\left[\boldsymbol{Q}_{\mathcal{N}}^{\pm}, \boldsymbol{H}\right] = \left\{\boldsymbol{Q}_{\mathcal{N}}^{\pm}, \boldsymbol{Q}_{\mathcal{N}}^{\pm}\right\} = 0, \qquad \left\{\boldsymbol{Q}_{\mathcal{N}}^{-}, \boldsymbol{Q}_{\mathcal{N}}^{+}\right\} = 2^{\mathcal{N}}\mathsf{P}_{\mathcal{N}}(\boldsymbol{H}), \tag{2.7}$$

where $\mathsf{P}_{\mathcal{N}}(x)$ is a monic polynomial of degree $\mathcal{N}$ in x. Indeed, it is realized by defining $\boldsymbol{H}$ and $\boldsymbol{Q}_{\mathcal{N}}^{\pm}$ as

$$\boldsymbol{H} = H^{-}\psi^{-}\psi^{+} + H^{+}\psi^{+}\psi^{-}, \quad \boldsymbol{Q}_{\mathcal{N}}^{+} = P_{\mathcal{N}}^{-}\psi^{+}, \quad \boldsymbol{Q}_{\mathcal{N}}^{-} = P_{\mathcal{N}}^{+}\psi^{-}, \tag{2.8}$$

where $\psi^{\pm}$ is a pair of fermionic variables satisfying $\{\psi^{\pm}, \psi^{\pm}\} = 0$ and $\{\psi^{-}, \psi^{+}\} = 1$. It is easy to check that the above $\boldsymbol{H}$ and $\boldsymbol{Q}_{\mathcal{N}}^{\pm}$ satisfy the first part of algebra (2.7). In particular, the intertwining relations in (2.3) guarantee the commutativity of $\boldsymbol{H}$ and $\boldsymbol{Q}_{\mathcal{N}}^{\pm}$. Regarding the second part of algebra, the monic polynomial $\mathsf{P}_{\mathcal{N}}$ is given, in the above representation, by [33, 43]

$$\mathsf{P}_{\mathcal{N}}(\boldsymbol{H}) = \det\left(\boldsymbol{H} - H^{\pm}\big|_{\mathcal{V}_{\mathcal{N}}^{\pm}}\right), \tag{2.9}$$

namely, the characteristic polynomial for $H^{\pm}$ restricted to the solvable sectors $\mathcal{V}_{\mathcal{N}}^{\pm}$.

Whether $\mathcal{N}$-fold SUSY of the system under consideration is dynamically broken is determined by a property of the solvable sectors $\mathcal{V}_{\mathcal{N}}^{\pm}$ since they characterize $\mathcal{N}$-fold SUSY states, namely, states annihilated by the pair of $\mathcal{N}$-fold supercharges $\boldsymbol{Q}_{\mathcal{N}}^{\pm}$. Let $|0\rangle$ and $|1\rangle$ be the fermionic vacuum and the one fermion state, respectively, which satisfy

$$\psi^{-}|0\rangle = 0, \qquad |1\rangle = \psi^{+}|0\rangle. \tag{2.10}$$

Then, superstates $|\Psi_0^-\rangle = \Psi_0^-(q)|0\rangle$ and $|\Psi_0^+\rangle = \Psi_0^+(q)|1\rangle$, respectively, are annihilated by both of $\boldsymbol{Q}_{\mathcal{N}}^{\pm}$

$$\boldsymbol{Q}_{\mathcal{N}}^{\pm}|\Psi_0^-\rangle = 0, \qquad \boldsymbol{Q}_{\mathcal{N}}^{\pm}|\Psi_0^+\rangle = 0, \tag{2.11}$$

if and only if $\Psi_0^-(q) \in \mathcal{V}_{\mathcal{N}}^-$ and $\Psi_0^+(q) \in \mathcal{V}_{\mathcal{N}}^+$, respectively. However, such states do not necessarily satisfy physical requirements. Suppose $S \subset \mathbb{C}$ is a domain where both of the Hamiltonians $H^{\pm}$ have no singularities and are thus naturally defined, and $\mathfrak{F}(S)$ is a linear space of complex functions in which both of $H^{\pm}$ act. In a usual physical application, the domain S is the real line $\mathbb{R}$ or a real half-line $\mathbb{R}_+ = (0, \infty)$, and the linear space $\mathfrak{F}$ is a Hilbert space L^2, so that $\mathfrak{F}(S) = L^2(\mathbb{R})$, or $L^2(\mathbb{R}_+)$. In the latter cases, the physical requirement is the normalizability (square integrability) on S. Then, there exists physical (normalizable) $\mathcal{N}$-fold SUSY states $|\Psi_0^-\rangle$ and/or $|\Psi_0^+\rangle$ which satisfies (2.11) if $\mathcal{V}_{\mathcal{N}}^-(S) \subset L^2(S)$ and/or $\mathcal{V}_{\mathcal{N}}^+(S) \subset L^2(S)$, in other words, if H^- and/or H^+ is *quasi-exactly solvable*. If there are no such physical $\mathcal{N}$-fold SUSY states in the Hilbert space $L^2(S)$ exists then $\mathcal{N}$-fold SUSY of the system is said to be dynamically broken. It was first shown correctly in Ref. [45] that the generalized Witten index characterizes $\mathcal{N}$-fold SUSY breaking, which corrected the wrong statement made earlier in Ref. [46].

For $\mathcal{N} > 1$, we can have an intriguing situation where not the whole of, but a subspace of the solvable sectors $\mathcal{V}_{\mathcal{N}}^-(S)$ and/or $\mathcal{V}_{\mathcal{N}}^+(S)$ belong to the Hilbert space $L^2(S)$. In this case, $\mathcal{N}$-fold SUSY of the system is said to be *partially broken*. Partial breaking of $\mathcal{N}$-fold SUSY was first discovered in Ref. [47]. We note that it is different in nature from the partial breaking of (nonlinear) SUSY [48, 49].

Construction of an $\mathcal{N}$-fold SUSY system is in general quite difficult, especially for a larger value of $\mathcal{N}$, since the intertwining relations (2.3) compose of coupled nonlinear differential equations for $U^{\pm}(q)$ and $w_k^{[\mathcal{N}]}(q)$ $(k = 0, \dots, \mathcal{N}-1)$. For the direct calculations of intertwining relations in a PDM background in the cases of $\mathcal{N} = 1$ and 2, see Ref. [24]. To circumvent the difficulty, a systematic algorithm for constructing an $\mathcal{N}$-fold SUSY system was developed in Ref. [47] for constant-mass quantum mechanics and was later generalized to PDM systems in Ref. [33]. The significant feature which is common in both constant-mass and PDM systems is that an $\mathcal{N}$-dimensional linear space of functions

$$\tilde{\mathcal{V}}_{\mathcal{N}}^- = \langle \tilde{\varphi}_1(z), \dots, \tilde{\varphi}_{\mathcal{N}}(z) \rangle, \tag{2.12}$$

preserved by a second-order linear differential operator $\tilde{H}^-$ can determine whole of an $\mathcal{N}$-fold SUSY system. Indeed, we can construct a pair of $\mathcal{N}$th-order linear differential operators $\bar{\tilde{P}}_{\mathcal{N}}^{\pm}$ and another $\mathcal{N}$-dimensional vector space $\bar{\mathcal{V}}_{\mathcal{N}}^+$ such that $\bar{\tilde{\mathcal{V}}}_{\mathcal{N}}^{\pm} = \ker \bar{\tilde{P}}_{\mathcal{N}}^{\pm}$. Then, we can show that a pair of second-order linear differential operators given by

$$\begin{aligned}
\bar{\tilde{H}}^{\pm} = &- A(z)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \left[\frac{\mathcal{N}-2}{2}A'(z) \pm Q(z)\right]\frac{\mathrm{d}}{\mathrm{d}z} - C(z) \\
&- (1 \pm 1)\left[\frac{\mathcal{N}-1}{2}Q'(z) - \frac{1}{2}A'(z)\tilde{w}_{\mathcal{N}-1}^{[\mathcal{N}]}(z) - A(z)\tilde{w}_{\mathcal{N}-1}^{[\mathcal{N}]\prime}(z)\right],
\end{aligned} \tag{2.13}$$

is weakly quasi-solvable with respect to the spaces $\bar{\tilde{\mathcal{V}}}^{\pm}_{\mathcal{N}}$, namely, $\bar{\tilde{H}}^{\pm}\bar{\tilde{\mathcal{V}}}^{\pm}_{\mathcal{N}} \subset \bar{\tilde{\mathcal{V}}}^{\pm}_{\mathcal{N}}$.

With the choice of the change of variable $z = z(q)$ and the gauge potential $\mathcal{W}^{\pm}_{\mathcal{N}}$ determined by

$$z'(q)^2 = 2m(q)A(z), \quad \mathcal{W}^{\pm}_{\mathcal{N}} = -\frac{1}{4}\ln|m(q)| + \frac{\mathcal{N}-1}{4}\ln|2A(z)| \pm \int \mathrm{d}z\, \frac{m(q)Q(z)}{2A(z)}, \quad (2.14)$$

we can obtain an $\mathcal{N}$-fold SUSY system by

$$H^{\pm} = \mathrm{e}^{-\mathcal{W}^{\pm}_{\mathcal{N}}}\bar{\tilde{H}}^{\pm}\mathrm{e}^{\mathcal{W}^{\pm}_{\mathcal{N}}}\Big|_{z=z(q)}, \qquad P^{\pm}_{\mathcal{N}} = \mathrm{e}^{-\mathcal{W}^{\pm}_{\mathcal{N}}}\bar{\tilde{P}}^{\pm}_{\mathcal{N}}\mathrm{e}^{\mathcal{W}^{\pm}_{\mathcal{N}}}\Big|_{z=z(q)}. \quad (2.15)$$

With the change of variable and the gauge transformation, both of $H^{\pm}$ get the form of PDM Hamiltonian (2.1) and their effective potentials $U^{\pm}(q)$ are given by

$$\begin{aligned} U^{\pm}(q) = {} & \frac{1}{2m(q)}\left[\left(\frac{\mathrm{d}\mathcal{W}^{-}_{\mathcal{N}}}{\mathrm{d}q}\right)^2 - \frac{\mathrm{d}^2\mathcal{W}^{-}_{\mathcal{N}}}{\mathrm{d}q^2} + \frac{m'(q)}{m(q)}\frac{\mathrm{d}\mathcal{W}^{-}_{\mathcal{N}}}{\mathrm{d}q}\right] - C(z(q)) \\ & - (1\pm 1)\left[\frac{\mathcal{N}-1}{2}Q'(z) - \frac{1}{2}A'(z)\tilde{w}^{[\mathcal{N}]}_{\mathcal{N}-1}(z) - A(z)\tilde{w}^{[\mathcal{N}]\prime}_{\mathcal{N}-1}(z)\right]_{z=z(q)}. \end{aligned} \quad (2.16)$$

The solvable sectors $\mathcal{V}^{\pm}_{\mathcal{N}}$ of $H^{\pm}$ are evidently given by

$$\mathcal{V}^{\pm}_{\mathcal{N}} = \ker P^{\pm}_{\mathcal{N}} = \mathrm{e}^{-\mathcal{W}^{\pm}_{\mathcal{N}}}\bar{\tilde{\mathcal{V}}}^{\pm}_{\mathcal{N}}\Big|_{z=z(q)}. \quad (2.17)$$

In principle, we can construct a pair of $\mathcal{N}$-fold SUSY PDM Hamiltonians $H^{\pm}$ and its solvable sectors $\mathcal{V}^{\pm}_{\mathcal{N}}$ by using the formulas (2.16) and (2.17). However, there is an easier way to obtain such a system when we have already had an ordinary $\mathcal{N}$-fold SUSY constant-mass quantum system at hand. Suppose the latter system is such that its pair of potentials $V^{(0)\pm}(q)$, its gauge potentials $\mathcal{W}^{(0)\pm}_{\mathcal{N}}(q)$, its solvable sectors $\mathcal{V}^{(0)\pm}_{\mathcal{N}}[q]$ are all known. Then, an $\mathcal{N}$-fold SUSY PDM system having a pair of effective potentials $U^{\pm}(q)$, gauge potentials $\mathcal{W}^{\pm}_{\mathcal{N}}(q)$, and solvable sectors $\mathcal{V}^{\pm}_{\mathcal{N}}[q]$ can be constructed immediately via the following prescription:

$$U^{\pm}(q) = V^{(0)\pm}(u(q)) + \frac{m''(q)}{8m(q)^2} - \frac{7m'(q)^2}{32m(q)^3}, \quad (2.18a)$$

$$\mathcal{W}^{\pm}_{\mathcal{N}}(q) = -\frac{1}{4}\ln|m(q)| + \mathcal{W}^{(0)\pm}_{\mathcal{N}}(u(q)), \quad (2.18b)$$

$$\mathcal{V}^{\pm}_{\mathcal{N}}[q] = m(q)^{1/4}\mathcal{V}^{(0)\pm}_{\mathcal{N}}[u(q)], \quad (2.18c)$$

where the function $u(q)$ is given by

$$u(q) = \int \mathrm{d}q\sqrt{m(q)}. \quad (2.19)$$

Actually, the above relations are consistent with the formulas obtained by the point canonical transformation, see, e.g., equations (2.7) and (2.8) in Ref. [22], equation (7) of [21] and equations (10), (13), and (14) in Ref. [19]. The above relations (2.18) have also been verified in Ref. [33] where type A $\mathcal{N}$-fold SUSY has been constructed in PDM background. One of the most salient features unveiled by the algorithmic construction is that both constant-mass and PDM quantum systems with $\mathcal{N}$-fold SUSY have totally the same structure in the gauged z-space. That is, the functional forms of the gauged operators such as $\bar{\tilde{P}}_{\mathcal{N}}^{\pm}$ and $\bar{\tilde{H}}^{\pm}$ given by (2.13) are identical in both the cases. It means in particular that the starting vector space $\tilde{\mathcal{V}}_{\mathcal{N}}^{-}$ determines all in the algorithm regardless of whether mass is constant or not. Hence, different types of $\mathcal{N}$-fold SUSY are characterized by different types of vector spaces $\tilde{\mathcal{V}}_{\mathcal{N}}^{-}$ and vice versa. Until now, four different types have been discovered, namely, type A [44, 50], type B [51], type C [47], and type X_2 [52]. We note that almost all the models having essentially the same symmetry as $\mathcal{N}$-fold SUSY but called with other terminologies in the literature, such as Pöschl–Teller and Lamé potentials, are actually particular cases of type A $\mathcal{N}$-fold SUSY. In this article, we focus on constructing PDM quantum systems with type B and type X_2 $\mathcal{N}$-fold SUSY since the other types (type A and type C) are not related to exceptional polynomial subspaces. In what follows, we shall review the general structure of these two types of $\mathcal{N}$-fold SUSY.

A. Type B $\mathcal{N}$-fold Supersymmetry

Type B $\mathcal{N}$-fold SUSY was first discovered in Ref. [51] and was found to be associated with the following monomial space

$$\tilde{\mathcal{V}}_{\mathcal{N}}^{-} = \tilde{\mathcal{V}}_{\mathcal{N}}^{(\mathrm{B})} := \langle 1, z, \dots, z^{\mathcal{N}-2}, z^{\mathcal{N}} \rangle, \tag{2.20}$$

called type B, which was considered in Ref. [41] in the context of the classification of monomial spaces preserved by second-order linear ordinary differential operators. Applying the algorithm to the type B monomial space, we obtain [36] the gauged $\mathcal{N}$-fold supercharge components

$$\tilde{P}_{\mathcal{N}}^{-} = z'(q)^{\mathcal{N}} \left(\frac{\mathrm{d}}{\mathrm{d}z} - \frac{1}{z} \right) \frac{\mathrm{d}^{\mathcal{N}-1}}{\mathrm{d}z^{\mathcal{N}-1}}, \qquad \bar{P}_{\mathcal{N}}^{+} = z'(q)^{\mathcal{N}} \frac{\mathrm{d}^{\mathcal{N}-1}}{\mathrm{d}z^{\mathcal{N}-1}} \left(\frac{\mathrm{d}}{\mathrm{d}z} + \frac{1}{z} \right), \tag{2.21}$$

and the functions which characterize the gauged Hamiltonians (2.13) are given by

$$A(z) = a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0, \tag{2.22}$$

$$2Q(z) = -\mathcal{N} a_3 z^2 + 2b_1 z - \mathcal{N} a_1, \tag{2.23}$$

$$C(z) = \mathcal{N}(\mathcal{N}-3) a_4 z^2 + \mathcal{N}(\mathcal{N}-2) a_3 z + c_0, \tag{2.24}$$

and $\tilde{w}_{\mathcal{N}-1}^{[\mathcal{N}]}(z) = -z^{-1}$. The other linear space $\bar{\mathcal{V}}_{\mathcal{N}}^{+}$ preserved by $\bar{H}^{+}$ is given by

$$\bar{\mathcal{V}}_{\mathcal{N}}^{+} = z^{-1} \langle 1, z^2, \dots, z^{\mathcal{N}} \rangle. \tag{2.25}$$

We note that both the monomial spaces (2.20) and (2.25) are actually exceptional polynomial subspaces of codimension 1, see Ref. [40]. We can easily check that the type B Hamiltonian H^{+} preserves an infinite flag of the following spaces

$$\bar{\mathcal{V}}_{1}^{+} \mathrm{e}^{-\mathcal{W}_{\mathcal{N}}^{+}} \subset \bar{\mathcal{V}}_{2}^{+} \mathrm{e}^{-\mathcal{W}_{\mathcal{N}}^{+}} \subset \cdots \subset \bar{\mathcal{V}}_{\mathcal{N}}^{+} \mathrm{e}^{-\mathcal{W}_{\mathcal{N}}^{+}} \subset \cdots, \tag{2.26}$$

where $\bar{\mathcal{V}}_{\mathcal{N}}^{+}$ and $\mathcal{W}_{\mathcal{N}}^{+}$ are given by (2.25) and (2.14), respectively, and thus H^{+} is solvable if and only if $a_3 = a_4 = 0$. On the other hand, the partner type B Hamiltonian H^{-} does not appear to be solvable for any parameter value since the type B monomial space (2.20) does not constitutes an infinite flag due to the fact that $\tilde{\mathcal{V}}_{\mathcal{N}}^{(\mathrm{B})} \not\subset \tilde{\mathcal{V}}_{\mathcal{N}+1}^{(\mathrm{B})}$ for all $\mathcal{N} = 1, 2, \ldots$. However, it turns out [36] that, when $a_3 = a_4 = 0$ and H^+ gets solvable, the partner Hamiltonian H^- does preserve an infinite flag of linear spaces given by

$$\tilde{\mathcal{V}}_1^{(\mathrm{A})} \mathrm{e}^{-\mathcal{W}_{\mathcal{N}}^-} \subset \tilde{\mathcal{V}}_2^{(\mathrm{A})} \mathrm{e}^{-\mathcal{W}_{\mathcal{N}}^-} \subset \cdots \subset \tilde{\mathcal{V}}_{\mathcal{N}}^{(\mathrm{A})} \mathrm{e}^{-\mathcal{W}_{\mathcal{N}}^-} \subset \cdots, \tag{2.27}$$

where $\mathcal{W}_{\mathcal{N}}^-$ is given by (2.14) and $\tilde{\mathcal{V}}_{\mathcal{N}}^{(\mathrm{A})}$ is the type A monomial space defined by

$$\tilde{\mathcal{V}}_{\mathcal{N}}^{(\mathrm{A})} = \langle 1, z, \ldots, z^{\mathcal{N}-1} \rangle. \tag{2.28}$$

That is, H^- and H^+ can be solvable simultaneously. In this paper, all the type B models we will consider later satisfy the solvability condition $a_3 = a_4 = 0$. Thus, all the pairs of type B Hamiltonians $H^\pm$ preserve the infinite-dimensional solvable sectors $\mathcal{V}^\pm$ given by

$$\begin{aligned} \mathcal{V}^- &= \langle 1, z(q), z(q)^2, \ldots \rangle \, \mathrm{e}^{-\mathcal{W}_{\mathcal{N}}^-(q)}, \\ \mathcal{V}^+ &= \langle 1, z(q)^2, z(q)^3, \ldots \rangle z(q)^{-1} \mathrm{e}^{-\mathcal{W}_{\mathcal{N}}^+(q)}. \end{aligned} \tag{2.29}$$

An interesting consequence of the fact that H^- and H^+ preserve different types of infinite flag of spaces in the solvable case is that the eigenfunctions of H^- are expressed in terms of a classical polynomial system while those of H^+ are in terms of an X_1 polynomial system. It is exactly the underlying reason why some of the Hamiltonians whose eigenfunctions are expressed in terms of the X_1 Laguerre or Jacobi polynomials were obtained by those whose bound state eigenfunctions are expressed in terms of the classical Laguerre or Jacobi polynomials using an intertwining or SUSY techniques in Refs. [27, 53, 54].

B. Type X_2 $\mathcal{N}$-fold Supersymmetry

Type X_2 $\mathcal{N}$-fold SUSY constructed in Ref. [52] is associated with the following exceptional polynomial subspace of codimension 2

$$\tilde{\mathcal{V}}_{\mathcal{N}}^- = \langle \tilde{\varphi}_1(z;\alpha), \ldots, \tilde{\varphi}_{\mathcal{N}}(z;\alpha) \rangle, \tag{2.30}$$

where $\tilde{\varphi}_n(z;\alpha)$ is a polynomial of degree $n+1$ in z with a parameter $\alpha (\neq 0, 1)$ defined by

$$\tilde{\varphi}_n(z;\alpha) = (\alpha + n - 2) z^{n+1} + 2(\alpha + n - 1)(\alpha - 1) z^n + (\alpha + n)(\alpha - 1)\alpha z^{n-1}. \tag{2.31}$$

Applying the algorithm to the X_2 space (2.30), we obtain [52] the gauged $\mathcal{N}$-fold supercharge components

$$\begin{aligned} \tilde{P}_{\mathcal{N}}^- &= z'(q)^{\mathcal{N}} \frac{f(z;\alpha)}{f(z;\alpha+\mathcal{N})} \prod_{k=0}^{\mathcal{N}-1} \frac{f(z;\alpha+k+1)}{f(z;\alpha+k)} \left(\frac{\mathrm{d}}{\mathrm{d}z} - \frac{f'(z;\alpha+k+1)}{f(z;\alpha+k+1)} \right), \\ \bar{P}_{\mathcal{N}}^+ &= z'(q)^{\mathcal{N}} \left[\prod_{k=0}^{\mathcal{N}-1} \left(\frac{\mathrm{d}}{\mathrm{d}z} + \frac{f'(z;\alpha+\mathcal{N}-k)}{f(z;\alpha+\mathcal{N}-k)} \right) \frac{f(z;\alpha+\mathcal{N}-k)}{f(z;\alpha+\mathcal{N}-k-1)} \right] \frac{f(z;\alpha)}{f(z;\alpha+\mathcal{N})}, \end{aligned} \tag{2.32}$$

where $\prod_{k=0}^{\mathcal{N}-1} A_k := A_{\mathcal{N}-1} \dots A_1 A_0$, and the functions $f(z;\alpha)$ and $\tilde{w}_{\mathcal{N}-1}^{[\mathcal{N}]}(z)$ are given by

$$f(z;\alpha) = z^2 + 2(\alpha-1)z + (\alpha-1)\alpha, \tag{2.33}$$

$$\tilde{w}_{\mathcal{N}-1}^{[\mathcal{N}]}(z) = -(\mathcal{N}-1)\frac{f'(z;\alpha)}{f(z;\alpha)} - \frac{f'(z;\alpha+\mathcal{N})}{f(z;\alpha+\mathcal{N})}. \tag{2.34}$$

The most general forms of the functions $A(z)$, $Q(z)$, and $C(z)$ appeared in $\bar{\bar{H}}^{\pm}$ depend on four parameters a_i ($i = 1,\dots,4$), but in this paper we only consider models with $a_4 = a_3 = 0$. In the latter case, they read as

$$A(z) = a_2 z^2 + a_1 z + (\alpha-1)(\alpha+\mathcal{N}-1)a_2, \tag{2.35}$$

$$\begin{aligned} Q(z) = &- a_2 z^2 - (3a_2 + a_1)z - (\alpha-1)(3\alpha+3\mathcal{N}-7)a_2 \\ &+ \frac{2\alpha+\mathcal{N}-8}{2}a_1 + \frac{4(\alpha-1)D(z)}{f(z;\alpha)}, \end{aligned} \tag{2.36}$$

$$C(z) = a_2 z + c_0 - \frac{4(\alpha-1)D(z)}{f(z;\alpha)}, \tag{2.37}$$

where $D(z)$ is given by

$$D(z) = -[(2\alpha+\mathcal{N}-3)a_2 - a_1]z - (\alpha-1)(2\alpha+\mathcal{N}-1)a_2 + \alpha a_1. \tag{2.38}$$

For their most general forms, please refer to Ref. [52]. The other linear space $\bar{\mathcal{V}}_{\mathcal{N}}^{+}$ preserved by $\bar{H}^{+}$ is given by

$$\bar{\mathcal{V}}_{\mathcal{N}}^{+} = \langle \bar{\chi}_1(z;\alpha+\mathcal{N}), \dots, \bar{\chi}_{\mathcal{N}}(z;\alpha+\mathcal{N}) \rangle f(z;\alpha)^{-1} f(z;\alpha+\mathcal{N})^{-1}, \tag{2.39}$$

where $\bar{\chi}_n(z;\alpha)$ is a polynomial of degree $n+1$ in z defined by

$$\begin{aligned} \bar{\chi}_n(z;\alpha) = &(\alpha-n)(\alpha-n+1)z^{n+1} + 2(\alpha-n-1)(\alpha-n+1)(\alpha-1)z^n \\ &+ (\alpha-n-1)(\alpha-n)(\alpha-1)\alpha z^{n-1}. \end{aligned} \tag{2.40}$$

The solvable sectors $\mathcal{V}_{\mathcal{N}}^{\pm}$ of the constant-mass Hamiltonians $H^{\pm}$ are

$$\mathcal{V}_{\mathcal{N}}^{-} = \langle \tilde{\varphi}_1(z(q);\alpha), \dots, \tilde{\varphi}_{\mathcal{N}}(z(q);\alpha) \rangle \, \mathrm{e}^{-\mathcal{W}_{\mathcal{N}}^{-}(q)}, \tag{2.41}$$

$$\mathcal{V}_{\mathcal{N}}^{+} = \frac{\langle \bar{\chi}_1(z(q);\alpha+\mathcal{N}), \dots, \bar{\chi}_{\mathcal{N}}(z(q);\alpha+\mathcal{N}) \rangle}{f(z(q);\alpha) f(z(q);\alpha+\mathcal{N})} \, \mathrm{e}^{-\mathcal{W}_{\mathcal{N}}^{+}(q)}. \tag{2.42}$$

Finally, the type X_2 Hamiltonians $H^{\pm}$ preserve the infinite flag of the spaces $\mathcal{V}_{\mathcal{N}}^{\pm}$ ($\mathcal{N} = 1, 2, \dots$) and are simultaneously solvable if and only if $a_2 = (a_3 = a_4 =)0$.

III. TYPE B AND TYPE X_2 $\mathcal{N}$-FOLD SUPERSYMMETRY FOR POSITION-DEPENDENT MASS

In this section, we shall consider some models which belong to type B and type X_2 $\mathcal{N}$-fold supersymmetry. In order to study effect of PDM in these models, we need to consider simultaneously the corresponding constant-mass type B and type X_2 models as well. In particular, we shall address ourselves to the following question: Does position dependent mass have any effect on dynamical breaking of type B and type X_2 $\mathcal{N}$-fold SUSY? By comparing the solvable sectors of both constant and position-dependent mass cases, we shall see below that the answer is in the affirmative in some cases for particular choices of physically interesting mass functions. In order to explore in detail the impact of mass functions on symmetry breaking or restoration, it will be appropriate to consider more than one mass function in a few examples. Also it will be shown that the bound state wavefunctions of one of the partner potentials obtained in type B $\mathcal{N}$-fold SUSY are associated with exceptional X_1 Laguerre and Jacobi polynomials while those of the other partner are associated with classical Laguerre and Jacobi polynomials.

A. Effects of PDM on Dynamical Symmetry Breaking of Type B $\mathcal{N}$-fold SUSY

Here we shall consider three examples of type B $\mathcal{N}$-fold SUSY corresponding to three different choices of $A(z)$. In each of the examples, we first show the results in the constant mass case, followed by the corresponding results in the PDM case. As we referred to before, all the type B models constructed below satisfy the solvability condition $a_3 = a_4 = 0$ and thus their solvable sectors in the constant-mass case are given by (2.29).

Example 3.1. $A(z) = k(z - z_0) \quad (k \neq 0)$
Potentials:

$$V^{(0)-}(q) = \frac{b_1^2}{8}q^2 + \frac{4(z_0 b_1 - \mathcal{N}k)^2 - k^2}{8k^2 q^2} + \frac{\mathcal{N}b_1}{2} + V_0, \tag{3.1}$$

$$V^{(0)+}(q) = \frac{b_1^2}{8}q^2 + \frac{4z_0^2 b_1^2 - k^2}{8k^2 q^2} + \frac{2k}{kq^2 + 2z_0} - \frac{8kz_0}{(kq^2 + 2z_0)^2} + V_0, \tag{3.2}$$

where V_0 is an irrelevant constant given by

$$V_0 = \frac{(z_0 b_1 - \mathcal{N}k)b_1}{2k} + \frac{b_1}{\mathcal{N}} - R.$$

Solvable sectors:

$$\mathcal{V}^{(0)-} = \langle 1, z(q), z(q)^2, \dots \rangle q^{(2z_0 b_1 - 2\mathcal{N}k + k)/(2k)} \mathrm{e}^{b_1 q^2/4}, \tag{3.3}$$

$$\mathcal{V}^{(0)+} = \langle 1, z(q)^2, z(q)^3, \dots \rangle z(q)^{-1} q^{-(2z_0 b_1 - k)/(2k)} \mathrm{e}^{-b_1 q^2/4}. \tag{3.4}$$

We assume $k > 0$ and $z_0 > 0$ so that the pair of potentials $V^{\pm}(q)$ has no singularities except for at $q = 0$. Thus, the system is naturally defined in $L^2(\mathbb{R}_+)$, $\mathbb{R}_+ = (0, \infty)$. In the latter Hilbert space, $\mathcal{V}^{(0)-}(\mathbb{R}_+) \subset L^2(\mathbb{R}_+)$ if and only if

$$b_1 < 0 \quad \text{and} \quad z_0 b_1 > (\mathcal{N} - 1)k, \tag{3.5}$$

which cannot be satisfied by any $b_1 \in \mathbb{R}$. On the other hand, $\mathcal{V}^{(0)+}(\mathbb{R}_+) \subset L^2(\mathbb{R}_+)$ if and only if

$$b_1 > 0 \quad \text{and} \quad k > z_0 b_1. \tag{3.6}$$

Hence, $\mathcal{N}$-fold SUSY of the system is unbroken if and only if $0 < b_1 < k/z_0$ on the constant-mass background.

Now, the relevant expressions for partner potentials, gauge potentials and corresponding solvable sectors of type B PDM systems can be obtained using Eqs. (3.1)–(3.4) and relations (2.18). Since our main objective in this section is to study effect of mass function on dynamical breaking of $\mathcal{N}$-fold SUSY, we give below only the solvable sectors $\mathcal{V}^{\pm}$ for an arbitrary mass function $m(q)$:

$$\mathcal{V}^{-} = \left\langle 1, z(u(q)), z(u(q))^2, \dots \right\rangle m(q)^{1/4} u(q)^{(2z_0 b_1 - 2\mathcal{N}k + k)/(2k)} \mathrm{e}^{b_1 u(q)^2/4}, \tag{3.7}$$

$$\mathcal{V}^{+} = \left\langle 1, z(u(q))^2, z(u(q))^3, \dots \right\rangle z(u(q))^{-1} m(q)^{1/4} u(q)^{-(2z_0 b_1 - k)/(2k)} \mathrm{e}^{-b_1 u(q)^2/4}, \tag{3.8}$$

where $u(q)$ is given by (2.19). At this point, we are in a position to choose a particular mass function. Let the mass function be

$$m(q) = \mathrm{e}^{-bq}, \quad b > 0, \quad q \in (-\infty, \infty), \tag{3.9}$$

which was considered in Ref. [28] where the PDM potentials were associated with X_1-Laguerre polynomials. This exponentially behaved mass function has been often used in the study of confined energy states for carriers in semiconductor quantum well [19, 28]. It has also been used to compute transmission probabilities for scattering in abrupt heterostructures [25] which may be useful in the design of semiconductor devices [55]. For the mass function, the change of variable is given by

$$u(q) = -\frac{2}{b}\mathrm{e}^{-bq/2}, \tag{3.10}$$

and the pair of potentials $U^{\pm}(q)$ reads from (2.18a) as

$$U^{-}(q) = \frac{b_1^2}{2b^2}\mathrm{e}^{-bq} + \frac{b^2[(z_0 b_1 - \mathcal{N}k)^2 - k^2]}{8k^2}\mathrm{e}^{bq} + \frac{\mathcal{N}b_1}{2} + V_0, \tag{3.11}$$

$$U^{+}(q) = \frac{b_1^2}{2b^2}\mathrm{e}^{-bq} + \frac{b^2(z_0^2 b_1^2 - k^2)}{8k^2}\mathrm{e}^{bq} + \frac{kb^2}{2k\mathrm{e}^{-bq} + z_0 b^2} - \frac{2kz_0 b^4}{(2k\mathrm{e}^{-bq} + z_0 b^2)^2} + V_0, \tag{3.12}$$

respectively. It is worth mentioning here that the potential $U^{+}(q)$ given in (3.11) is identical with the potential $V_{eff}(q)$ associated with exceptional X_1 Laguerre polynomials [e.g., Eq. (12) of Ref. [28]], if one takes $k = 1/2$, $b_1 = b^2/2$, and $z_0 = \alpha/b^2$. On the other hand, for the same choices of parameters the other potential $U^{-}(q)$ coincides with the potential [after making a translation $\alpha \to \alpha - \mathcal{N}$] previously obtained in Ref. [27] corresponding to classical Laguerre polynomials.

The solvable sectors of the potentials (3.11) and (3.12) are respectively given by

$$\begin{aligned}\mathcal{V}^{-} &= \left\langle 1, \mathrm{e}^{-bq} + \bar{z}_0, (\mathrm{e}^{-bq} + \bar{z}_0)^2, \dots \right\rangle \\ &\quad \times \exp\left[-\left(\frac{z_0 b_1}{k} - \mathcal{N} + 1\right)\frac{b}{2}q + \frac{b_1}{b^2}\mathrm{e}^{-bq}\right],\end{aligned} \tag{3.13}$$

$$\begin{aligned}\mathcal{V}^{+} &= \left\langle 1, (\mathrm{e}^{-bq} + \bar{z}_0)^2, (\mathrm{e}^{-bq} + \bar{z}_0)^3, \dots \right\rangle \\ &\quad \times (\mathrm{e}^{-bq} + \bar{z}_0)^{-1} \exp\left[\left(\frac{z_0 b_1}{k} - 1\right)\frac{b}{2}q - \frac{b_1}{b^2}\mathrm{e}^{-bq}\right],\end{aligned} \tag{3.14}$$

where $\bar{z}_0 = z_0 b^2/(2k)$. Here the potentials have no singularities in the finite part of the real line, so the domain is $\mathbb{R}$. Since $b > 0$, so $\mathcal{V}^-(\mathbb{R}) \subset L^2(\mathbb{R})$ if and only if $b_1 < 0$. On the other hand, $\mathcal{V}^+(\mathbb{R}) \subset L^2(\mathbb{R})$ if and only if $b_1 > 0$. Hence, the $\mathcal{N}$-fold SUSY of the PDM system is unbroken unless $b_1 = 0$. Comparing the solvable sectors of both the constant and position-dependent mass scenarios, it can be observed that it is not possible to break $\mathcal{N}$-fold SUSY dynamically for the particular choice of mass function $m(q) = \mathrm{e}^{-bq}$. In addition, we have checked that many physically interesting mass functions also have no effect on symmetry breaking.

Example 3.2. $A(z) = a^2[1-(z-z_0)^2]/2 \quad (a>0)$
Potentials:

$$V^{(0)-}(q) = \frac{(4b_1^2 - \mathcal{N}^2 a^4) z_0}{4a^2}\frac{\sin aq}{\cos^2 aq} + \frac{(2b_1 - \mathcal{N}a^2)^2 z_0^2 + (2b_1 + \mathcal{N}a^2)^2 - a^4}{8a^2}\tan^2 aq + \frac{b_1\mathcal{N}}{2} + V_0, \tag{3.15}$$

$$V^{(0)+}(q) = \frac{(2b_1 - \mathcal{N}a^2)^2 z_0}{4a^2}\frac{\sin aq}{\cos^2 aq} + \frac{(2b_1 - \mathcal{N}a^2)^2 (z_0^2+1) - a^4}{8a^2}\tan^2 aq + \frac{a^2 z_0}{\sin aq + z_0} - \frac{a^2(z_0^2-1)}{(\sin aq + z_0)^2} - \frac{b_1\mathcal{N}}{2} + V_0, \tag{3.16}$$

where V_0 is an irrelevant constant given by

$$V_0 = \frac{b_1}{\mathcal{N}} + \frac{a^2(\mathcal{N}^2-7)}{12} + \frac{(2b_1 z_0 - \mathcal{N} z_0 a^2)^2}{8a^2} - R.$$

Solvable sectors:

$$\mathcal{V}^{(0)-} = \langle 1, z(q), z(q)^2, \dots\rangle |\cos aq|^{\frac{b_1}{a^2}+\frac{\mathcal{N}-1}{2}} \left(\frac{1+\sin aq}{1-\sin aq}\right)^{-\frac{(2b_1-\mathcal{N}a^2)z_0}{4a^2}}, \tag{3.17}$$

$$\mathcal{V}^{(0)+} = \langle 1, z(q)^2, z(q)^3, \dots\rangle z(q)^{-1} |\cos aq|^{-\frac{b_1}{a^2}+\frac{\mathcal{N}-1}{2}} \left(\frac{1+\sin aq}{1-\sin aq}\right)^{\frac{(2b_1-\mathcal{N}a^2)z_0}{4a^2}}. \tag{3.18}$$

It is worth mentioning here that the potential $V^{(0)+}(q)$ coincides with the potential whose bound state wave functions are given in terms of exceptional X_1 Jacobi polynomial [42] for $a = 1$, $b_1 = B + \mathcal{N}/2$, $z_0 = -(2A-1)/(2B)$ whereas potential $V^{(0)-}(q)$ coincides with the Scarf I potential [54] [after making an change $B \to B + \mathcal{N}$] whose bound state wave functions are given in terms of classical Jacobi polynomials.

We choose here a domain of the system as $S = (-\frac{\pi}{2a}, \frac{\pi}{2a})$ and assume $z_0 > 1$ so that the pair of potentials $V^{(0)\pm}(q)$ has no singularities except for at the boundary $\partial S = \{-\frac{\pi}{2a}, \frac{\pi}{2a}\}$. Thus, the Hilbert space for the system is $L^2(S)$. Then, $\mathcal{V}^{(0)-}(S) \subset L^2(S)$ if and only if

$$\frac{b_1}{a^2} + \frac{\mathcal{N}-1}{2} \pm \frac{(2b_1 - \mathcal{N}a^2)z_0}{2a^2} > -\frac{1}{2},$$

that is,

$$\frac{\mathcal{N}a^2}{2}\frac{z_0-1}{z_0+1} < b_1 < \frac{\mathcal{N}a^2}{2}\frac{z_0+1}{z_0-1} \quad \text{for} \quad z_0 > 1. \tag{3.19}$$

Similarly, $\mathcal{V}^{(0)+}(S) \subset L^2(S)$ if and only if

$$-\frac{b_1}{a^2} + \frac{\mathcal{N}-1}{2} \pm \frac{(2b_1 - \mathcal{N}a^2)z_0}{2a^2} > -\frac{1}{2},$$

that is,

$$b_1 > \frac{\mathcal{N}a^2}{2} \quad \text{and} \quad z_0 > 1. \tag{3.20}$$

Hence, $\mathcal{N}$-fold SUSY of the system is broken for the constant mass case if and only if $z_0 > 1$ and

$$b_1 \leq \frac{\mathcal{N}a^2}{2}\frac{z_0 - 1}{z_0 + 1} \quad \text{or} \quad b_1 \geq \frac{\mathcal{N}a^2}{2}\frac{z_0 + 1}{z_0 - 1}. \tag{3.21}$$

In a PDM case, the solvable sectors $\mathcal{V}^{\pm}$ of the type B PDM $\mathcal{N}$-fold SUSY partner Hamiltonians $H^{\pm}$ for an arbitrary mass function $m(q)$ are deformed according to (2.18c) as

$$\begin{aligned}\mathcal{V}^{-} = &\langle 1, z(u(q)), z(u(q))^2, \dots \rangle m(q)^{\frac{1}{4}} \\ &\times |\cos au(q)|^{\frac{b_1}{a^2} + \frac{\mathcal{N}-1}{2}} \left(\frac{1 + \sin au(q)}{1 - \sin au(q)}\right)^{-\frac{(2b_1 - \mathcal{N}a^2)z_0}{4a^2}},\end{aligned} \tag{3.22}$$

$$\begin{aligned}\mathcal{V}^{+} = &\langle 1, z(u(q))^2, z(u(q))^3, \dots \rangle m(q)^{\frac{1}{4}} \\ &\times \frac{|\cos au(q)|^{-\frac{b_1}{a^2} + \frac{\mathcal{N}-1}{2}}}{\sin au(q) + z_0} \left(\frac{1 + \sin au(q)}{1 - \sin au(q)}\right)^{\frac{(2b_1 - \mathcal{N}a^2)z_0}{4a^2}}.\end{aligned} \tag{3.23}$$

where $u(q)$ is given by (2.19). In this case, the choice of mass function and the corresponding change of variable are given by

$$m(q) = \frac{2}{\pi}\mathrm{e}^{-2q^2}, \qquad u(q) = \operatorname{Erf} q, \qquad q \in (-\infty, \infty). \tag{3.24}$$

Consequently, the partner potentials $U^{\pm}(q)$ read as

$$\begin{aligned}U^{-}(q) = &\frac{(4b_1^2 - \mathcal{N}^2 a^4)z_0}{4a^2}\frac{\sin(a \operatorname{Erf} q)}{\cos^2(a \operatorname{Erf} q)} - \frac{(3q^2+1)\pi \mathrm{e}^{2q^2}}{4} + \frac{b_1\mathcal{N}}{2} + V_0 \\ &+ \frac{(2b_1 - \mathcal{N}a^2)^2 z_0^2 + (2b_1 + \mathcal{N}a^2)^2 - a^4}{8a^2}\tan^2(a \operatorname{Erf} q),\end{aligned} \tag{3.25}$$

$$\begin{aligned}U^{+}(q) = &\frac{(2b_1 - \mathcal{N}a^2)^2 z_0}{4a^2}\frac{\sin(a \operatorname{Erf} q)}{\cos^2(a \operatorname{Erf} q)} - \frac{(3q^2+1)\pi \mathrm{e}^{2q^2}}{4} - \frac{b_1\mathcal{N}}{2} + V_0 \\ &+ \frac{a^2 z_0}{\sin(a \operatorname{Erf} q) + z_0} - \frac{a^2(z_0^2 - 1)}{[\sin(a \operatorname{Erf} q) + z_0]^2} \\ &+ \frac{(2b_1 - \mathcal{N}a^2)^2(z_0^2 + 1) - a^4}{8a^2}\tan^2(a \operatorname{Erf} q).\end{aligned} \tag{3.26}$$

The solvable sectors of the potentials (3.25) and (3.26) are given by

$$\mathcal{V}^- = \langle 1, z(u(q)), z(u(q))^2, \ldots \rangle \, \mathrm{e}^{-q^2/4} \times |\cos(a \,\mathrm{Erf}\, q)|^{\frac{b_1}{a^2}+\frac{\mathcal{N}-1}{2}} \left(\frac{1+\sin(a \,\mathrm{Erf}\, q)}{1-\sin(a \,\mathrm{Erf}\, q)} \right)^{-\frac{(2b_1-\mathcal{N}a^2)z_0}{4a^2}}, \tag{3.27}$$

$$\mathcal{V}^+ = \langle 1, z(u(q))^2, z(u(q))^3, \ldots \rangle \, \mathrm{e}^{-q^2/4} \times \frac{|\cos(a \,\mathrm{Erf}\, q)|^{-\frac{b_1}{a^2}+\frac{\mathcal{N}-1}{2}}}{\sin(a \,\mathrm{Erf}\, q)+z_0} \left(\frac{1+\sin(a \,\mathrm{Erf}\, q)}{1-\sin(a \,\mathrm{Erf}\, q)} \right)^{\frac{(2b_1-\mathcal{N}a^2)z_0}{4a^2}}. \tag{3.28}$$

The potentials $U^{\pm}(q)$ as well as the mass function are well behaved in $q \in (-\infty, \infty)$. So, we can take the domain as the whole real line $\mathbb{R}$. Since $\mathrm{Erf}\, q \to \pm 1$ as $q \to \pm\infty$, so both the solvable sectors $\mathcal{V}^{\pm}(\mathbb{R})$ belong to $L^2(\mathbb{R})$, irrespective of the parameter values of b_1 and z_0. Hence, it manifests unbroken SUSY. So, in this case position-dependent mass affects the symmetry breaking scenario. But the mass profile $m(q) = \mathrm{sech}^2\, aq$, $q \in (-\infty, \infty)$ has no effect on dynamical breaking of $\mathcal{N}$-fold SUSY which can be observed by considering the leading behavior of the solvable sectors (3.22) and (3.23). We have found that same is true for many other mass functions.

Also associated to this mass profile, one of the partner potentials given in equation (3.29) is identical with the $V_{eff}(q)$ whose bound state wave functions are given by exceptional X_1 Jacobi polynomials [e.g., Eq. (18) of Ref. [28]], for the choice of parameters $b_1 = (\alpha - \beta + \mathcal{N})a^2/2$, $z_0 = (\alpha+\beta)/(\alpha-\beta)$. The simplified form of the other partner potential $U^-(q)$ matches with the potential previously obtained in [27] corresponding to classical Jacobi polynomials. It is worth mentioning that this mass profile $m(q) = \mathrm{sech}^2\, aq$ has been previously used in PDM Hamiltonians of BenDaniel–Duke [56] and Zhu–Kroemer [57] type and interesting connection was shown [58] between the discrete eigenvalues of such Hamiltonians and the stationary 1-soliton and 2-soliton solutions of the Korteweg-de Vries (KdV) equation.

For the latter choice of the mass function, the change of variable is given by $u(q) = \tan^{-1}(\sinh aq)/a$ and corresponding pair of potentials $U^{\pm}(q)$ read as

$$\begin{aligned} U^{\pm}(q) = & \frac{[2b_1(z_0+1) - \mathcal{N}a^2 z_0 \mp (\mathcal{N}-2)a^2][2b_1(z_0+1) - \mathcal{N}a^2 z_0 \mp (\mathcal{N}+2)a^2]}{32a^2} \mathrm{e}^{2aq} \\ & + \frac{[2b_1(z_0-1) - \mathcal{N}a^2 z_0 \pm (\mathcal{N}-2)a^2][2b_1(z_0-1) - \mathcal{N}a^2 z_0 \pm (\mathcal{N}+2)a^2]}{32a^2} \mathrm{e}^{-2aq} \\ & + \frac{1 \pm 1}{2} \frac{a^2}{z_0+1} \left[1 - \frac{2(z_0-2)}{z_0 - 1 + (z_0+1)\mathrm{e}^{2aq}} - \frac{4(z_0-1)}{(z_0-1+(z_0+1)\mathrm{e}^{2aq})^2} \right] \mp \frac{\mathcal{N} b_1}{4} + V_0. \end{aligned} \tag{3.29}$$

Example 3.3. $A(z) = (z - z_0)^2/2$

Potentials:

$$V^{(0)-}(q) = \frac{(2b_1+\mathcal{N})^2 z_0^2}{8} \mathrm{e}^{-2q} + \frac{(4b_1^2 - \mathcal{N}^2) z_0}{4} \mathrm{e}^{-q} + V_0, \tag{3.30a}$$

$$V^{(0)+}(q) = \frac{(2b_1+\mathcal{N})^2 z_0^2}{8} \mathrm{e}^{-2q} + \frac{(2b_1+\mathcal{N})^2 z_0}{4} \mathrm{e}^{-q} - \frac{z_0 \mathrm{e}^{-q}}{(1+z_0\mathrm{e}^{-q})^2} + V_0, \tag{3.30b}$$

where V_0 is an irrelevant constant given by

$$V_0 = \frac{b_1^2}{2} + \frac{b_1}{\mathcal{N}} + \frac{\mathcal{N}^2 + 11}{24} - R.$$

Solvable sectors:

$$\mathcal{V}^{(0)-} = \langle 1, z(q), z(q)^2, \dots \rangle \exp\left[-\frac{(2b_1 + \mathcal{N})z_0}{2}\mathrm{e}^{-q} - \frac{\mathcal{N} - 1 - 2b_1}{2}q\right], \tag{3.31a}$$

$$\mathcal{V}^{(0)+} = \langle 1, z(q)^2, z(q)^3, \dots \rangle z(q)^{-1} \exp\left[\frac{(2b_1 + \mathcal{N})z_0}{2}\mathrm{e}^{-q} - \frac{\mathcal{N} - 1 + 2b_1}{2}q\right]. \tag{3.31b}$$

We assume $z_0 > 0$ so that the pair of potentials $V^{(0)\pm}(q)$ has no singularities in $(-\infty, \infty)$. As we will show in what follows, the $\mathcal{N}$-fold SUSY in this case can be partially broken. To see this, we first introduce a pair of k-dimensional subspaces $\mathcal{V}_k^{(0)\pm}$ of the solvable sectors $\mathcal{V}^{(0)\pm}$ as

$$\mathcal{V}_k^{(0)-} = \langle 1, z(q), \dots, z(q)^{k-1} \rangle \exp\left[-\frac{(2b_1 + \mathcal{N})z_0}{2}\mathrm{e}^{-q} - \frac{\mathcal{N} - 1 - 2b_1}{2}q\right], \tag{3.32}$$

$$\mathcal{V}_k^{(0)+} = \langle 1, z(q)^2, \dots, z(q)^k \rangle z(q)^{-1} \exp\left[\frac{(2b_1 + \mathcal{N})z_0}{2}\mathrm{e}^{-q} - \frac{\mathcal{N} - 1 + 2b_1}{2}q\right]. \tag{3.33}$$

Then, for a fixed $k \in \mathbb{N}$, we have

$$\mathcal{V}_k^{(0)-}(\mathbb{R}) \subset L^2(\mathbb{R}) \quad \Longleftrightarrow \quad -\mathcal{N} < 2b_1 < \mathcal{N} + 1 - 2k, \tag{3.34}$$

$$\mathcal{V}_k^{(0)+}(\mathbb{R}) \subset L^2(\mathbb{R}) \quad \Longleftrightarrow \quad 2k - \mathcal{N} - 1 < 2b_1 < -\mathcal{N}. \tag{3.35}$$

From these conditions, it is easy to observe that $\mathcal{V}_k^{(0)-}(\mathbb{R}) \subset L^2(\mathbb{R})$ if and only if $-\mathcal{N}/2 < b_1 < (\mathcal{N} + 1 - 2k)/2$ for a $k \in \mathbb{N}$ satisfying $k < \mathcal{N} + 1/2$, while there is no $k \in \mathbb{N}$ which satisfy the condition (3.35) and thus $\mathcal{V}^{(0)+}(\mathbb{R}) \not\subset L^2(\mathbb{R})$ $\forall b_1 \in \mathbb{R}$. Hence, the $\mathcal{N}$-fold SUSY in the constant-mass background is partially broken if there is a positive integer $k \leq \mathcal{N}$ for which the parameter b_1 satisfies

$$-\frac{\mathcal{N}}{2} < b_1 < \frac{\mathcal{N} + 1 - 2k}{2},$$

and fully broken otherwise.

The solvable sectors $\mathcal{V}^{\pm}$ of the corresponding PDM Hamiltonians $H^{\pm}$ are written as

$$\mathcal{V}^- = \langle 1, z(u(q)), z(u(q))^2, \dots \rangle m(q)^{1/4} \times \exp\left[-\frac{(2b_1 + \mathcal{N})z_0}{2}\mathrm{e}^{-u(q)} - \frac{\mathcal{N} - 1 - 2b_1}{2}u(q)\right], \tag{3.36a}$$

$$\mathcal{V}^+ = \langle 1, z(u(q))^2, z(u(q))^3, \dots \rangle z(u(q))^{-1} m(q)^{1/4} \times \exp\left[\frac{(2b_1 + \mathcal{N})z_0}{2}\mathrm{e}^{-u(q)} - \frac{\mathcal{N} - 1 + 2b_1}{2}u(q)\right]. \tag{3.36b}$$

and the potentials $U^{\pm}(q)$ can be obtained using Eqs. (2.18a), (3.30a) and (3.30b). We have checked the normalizability of the solvable sectors (3.36) with the following two mass functions.

(i) $m(q) = (1-q^2)^{-1}$, $q \in (-1,1)$ for which the change of variable is $u(q) = \sin^{-1} q$. This mass profile has been used in Refs. [10, 11] while considering the effective-mass quantum nonlinear oscillator. This mass function has effect on dynamical symmetry breaking because it manifests broken SUSY [i.e., neither $\mathcal{V}^-$ nor $\mathcal{V}^+$ belongs to $L^2(-1,1)$], which is clear from the following expressions of $\mathcal{V}^-$ and $\mathcal{V}^+$:

$$\begin{aligned}\mathcal{V}^- &= \big\langle 1, z(u(q)), z(u(q))^2, \dots \big\rangle \frac{1}{(1-q^2)^{1/4}} \\ &\quad\times \exp\left[-\frac{(2b_1+\mathcal{N})z_0}{2} \mathrm{e}^{-\sin^{-1} q} - \frac{\mathcal{N}-1-2b_1}{2} \sin^{-1} q \right], \end{aligned} \tag{3.37}$$

$$\begin{aligned}\mathcal{V}^+ &= \big\langle 1, z(u(q))^2, z(u(q))^3, \dots \big\rangle \frac{1}{(1-q^2)^{1/4}(\mathrm{e}^{\sin^{-1} q} + z_0)} \\ &\quad\times \exp\left[\frac{(2b_1+\mathcal{N})z_0}{2} \mathrm{e}^{-\sin^{-1} q} - \frac{\mathcal{N}-1+2b_1}{2} \sin^{-1} q \right]. \end{aligned} \tag{3.38}$$

(ii) $m(q) = 2\mathrm{e}^{-2q^2}/\pi$ for which the solvable sectors (3.36) reduce to

$$\begin{aligned}\mathcal{V}^- &= \big\langle 1, z(u(q)), z(u(q))^2, \dots \big\rangle \\ &\quad\times \exp\left[-\frac{q^2}{4} - \frac{(2b_1+\mathcal{N})z_0}{2} \mathrm{e}^{-\operatorname{Erf} q} - \frac{\mathcal{N}-1-2b_1}{2} \operatorname{Erf} q \right], \end{aligned} \tag{3.39}$$

$$\begin{aligned}\mathcal{V}^+ &= \big\langle 1, z(u(q))^2, \dots, z(u(q))^{\mathcal{N}} \big\rangle z(u(q))^{-1} \\ &\quad\times \exp\left[-\frac{q^2}{4} + \frac{(2b_1+\mathcal{N})z_0}{2} \mathrm{e}^{-\operatorname{Erf} q} - \frac{\mathcal{N}-1+2b_1}{2} \operatorname{Erf} q \right]. \end{aligned} \tag{3.40}$$

From the above solvable sectors, we observe that both $\mathcal{V}^-(\mathbb{R})$ and $\mathcal{V}^+(\mathbb{R})$ belong to $L^2(\mathbb{R})$, irrespective of the parameter value b_1, which means unbroken $\mathcal{N}$-fold SUSY. Hence, the mass function $m(q) = 2\mathrm{e}^{-2q^2}/\pi$ affects dynamical breaking of the $\mathcal{N}$-fold SUSY.

Hence, comparing the normalizability conditions in both the constant and position dependent mass cases, we conclude that both the mass functions change the behaviours of symmetry breaking.

B. Effects of PDM on Dynamical Symmetry Breaking of Type X_2 $\mathcal{N}$-fold SUSY

In this section, we examine three different models of type X_2 $\mathcal{N}$-fold SUSY characterized by different choices of the two parameters a_1 and a_2; $a_1 \neq 0$ and $a_2 = 0$ for the first model, $a_1 = 0$ and $a_2 \neq 0$ for the second, and $a_1 a_2 \neq 0$ for the third. The first two choices lead to the rational- and hyperbolic-type potential pairs already shown in Ref. [52], while the last choice to an exponential-type potential pair which is new and has not been investigated in the literature.

Example 3.4. $A(z) = 2z$ $[a_1 = 2]$.
Potentials:

$$V^{(0)-}(q) = \frac{q^2}{2} + \frac{4\alpha^2-1}{8q^2} + 4\left[\frac{q^2-\alpha+1}{f(q^2;\alpha)} - \frac{4(\alpha-1)q^2}{f(q^2;\alpha)^2} \right] - \mathcal{N} + V_0, \tag{3.41a}$$

$$V^{(0)+}(q) = \frac{q^2}{2} + \frac{4(\alpha+\mathcal{N})^2-1}{8q^2} + 4\left[\frac{q^2-\alpha-\mathcal{N}+1}{f(q^2;\alpha+\mathcal{N})} - \frac{4(\alpha+\mathcal{N}-1)q^2}{f(q^2;\alpha+\mathcal{N})^2} \right] + V_0, \tag{3.41b}$$

where $V_0 = \mathcal{N} - \alpha + 3 - c_0$ is an irrelevant constant.
Solvable sectors:

$$\mathcal{V}_{\mathcal{N}}^{(0)-} = \left\langle \tilde{\varphi}_1(q^2;\alpha), \dots, \tilde{\varphi}_{\mathcal{N}}(q^2;\alpha) \right\rangle \frac{q^{\alpha+1/2}\mathrm{e}^{-q^2/2}}{f(q^2;\alpha)}, \tag{3.42a}$$

$$\mathcal{V}_{\mathcal{N}}^{(0)+} = \left\langle \bar{\chi}_1(q^2;\alpha+\mathcal{N}), \dots, \bar{\chi}_{\mathcal{N}}(q^2;\alpha+\mathcal{N}) \right\rangle \frac{q^{-\alpha-\mathcal{N}+1/2}\mathrm{e}^{q^2/2}}{f(q^2;\alpha+\mathcal{N})}. \tag{3.42b}$$

In this case, the solvability condition $a_2 (= a_3 = a_4) = 0$ for type X_2 is satisfied and thus the corresponding constant-mass Hamiltonians $H^{(0)\pm}$ are simultaneously solvable.

For $\alpha > 1$, a natural choice for the domain of these potentials is a real half-line $S = \mathbb{R}_+$. On this domain $\mathbb{R}_+$, it is evident from (3.42) that $\mathcal{V}_{\mathcal{N}}^{(0)-}(\mathbb{R}_+) \subset L^2(\mathbb{R}_+)$ and $\mathcal{V}_{\mathcal{N}}^{(0)-}(\mathbb{R}_+) \not\subset L^2(\mathbb{R}_+)$. Therefore, it manifests unbroken $\mathcal{N}$-fold SUSY of the system in the constant-mass background.

According to (2.18c), the solvable sectors $\mathcal{V}_{\mathcal{N}}^{\pm}$ of the corresponding PDM Hamiltonians $H^{\pm}$ for an arbitrary mass function $m(q)$ read as

$$\mathcal{V}_{\mathcal{N}}^{-} = \left\langle \tilde{\varphi}_1(u(q)^2;\alpha), \dots, \tilde{\varphi}_{\mathcal{N}}(u(q)^2;\alpha) \right\rangle \frac{m(q)^{1/4} u(q)^{\alpha+1/2}\mathrm{e}^{-u(q)^2/2}}{f(u(q)^2;\alpha)}, \tag{3.43}$$

$$\mathcal{V}_{\mathcal{N}}^{+} = \left\langle \bar{\chi}_1(u(q)^2;\alpha+\mathcal{N}), \dots, \bar{\chi}_{\mathcal{N}}(u(q)^2;\alpha+\mathcal{N}) \right\rangle \frac{m(q)^{1/4} u(q)^{-\alpha-\mathcal{N}+1/2}\mathrm{e}^{u(q)^2/2}}{f(u(q)^2;\alpha+\mathcal{N})}, \tag{3.44}$$

where $u(q)$ is given by (2.19) and the PDM potentials $U^{\pm}(q)$ can be obtained using Eqs. (2.18a) and (3.41). In this case, we have not been able to find out any realistic mass function which could break the $\mathcal{N}$-fold SUSY. In other words, we can say that the $\mathcal{N}$-fold SUSY in this case is steady against many variations of mass functions [e.g., $m(q) = \mathrm{e}^{-q}$, $\operatorname{sech}^2 q$].

Example 3.5. $A(z) = (z^2 + \zeta^2)/2$, $[a_2 = 1/2, \zeta^2 = (\alpha-1)(\alpha+\mathcal{N}-1) > 0]$.
Potentials:

$$\begin{aligned} V^{(0)-}(q) = {} & \frac{\zeta^2}{8}\cosh^2 q + \frac{\mathcal{N}-1}{4}\zeta\sinh q + V_0 \\ & + \frac{1}{8\cosh^2 q}\left[4(\mathcal{N}-1)\zeta\sinh q + 4\alpha^2 + 4(\mathcal{N}-2)\alpha - \mathcal{N}^2 - 2\mathcal{N} + 4\right] \\ & - 2(\alpha-1)\left[\frac{\zeta\sinh q - \alpha - \mathcal{N} + 3}{f(\zeta\sinh q;\alpha)} - 2(\alpha-1)\frac{2\zeta\sinh q - \mathcal{N} + 1}{f(\zeta\sinh q;\alpha)^2}\right], \end{aligned} \tag{3.45}$$

$$\begin{aligned} V^{(0)+}(q) = {} & \frac{\zeta^2}{8}\cosh^2 q + \frac{3\mathcal{N}-1}{4}\zeta\sinh q + V_0 \\ & - \frac{1}{8\cosh^2 q}\left[4(\mathcal{N}+1)\zeta\sinh q - 4\alpha^2 - 4(\mathcal{N}-2)\alpha + \mathcal{N}^2 + 6\mathcal{N} - 4\right] \\ & - 2(\alpha+\mathcal{N}-1)\left[\frac{\zeta\sinh q - \alpha + 3}{f(\zeta\sinh q;\alpha+\mathcal{N})} - 2(\alpha+\mathcal{N}-1)\frac{2\zeta\sinh q + \mathcal{N} + 1}{f(\zeta\sinh q;\alpha+\mathcal{N})^2}\right], \end{aligned} \tag{3.46}$$

where V_0 is an irrelevant constant given by

$$V_0 = \frac{4\alpha^2 + 4(\mathcal{N}-4)\alpha + \mathcal{N}^2 + 16}{8} - c_0.$$

Solvable sectors:

$$\mathcal{V}_{\mathcal{N}}^{(0)-} = \big\langle \tilde{\varphi}_1(\zeta \sinh q; \alpha), \dots, \tilde{\varphi}_{\mathcal{N}}(\zeta \sinh q; \alpha) \big\rangle \frac{\mathrm{e}^{-\zeta (\sinh q)/2 - \zeta \operatorname{gd} q}}{(\cosh q)^{\mathcal{N}/2-1} f(\zeta \sinh q; \alpha)}, \tag{3.47a}$$

$$\mathcal{V}_{\mathcal{N}}^{(0)+} = \big\langle \bar{\chi}_1(\zeta \sinh q; \alpha + \mathcal{N}), \dots, \bar{\chi}_{\mathcal{N}}(\zeta \sinh q; \alpha + \mathcal{N}) \big\rangle \times \frac{\mathrm{e}^{\zeta (\sinh q)/2 + \zeta \operatorname{gd} q}}{(\cosh q)^{\mathcal{N}/2} f(\zeta \sinh q; \alpha + \mathcal{N})}, \tag{3.47b}$$

where $\operatorname{gd} q = \tan^{-1}(\sinh q)$ is the Gudermann function. The solvability condition is not satisfied in this case and both of the Hamiltonians are only quasi-solvable. For $\alpha > 1$, the potentials $V^{\pm}(q)$ given in (3.45) are defined on the whole real line $\mathbb{R}$. From the solvable sectors (3.47), it is clear that neither $\mathcal{V}_{\mathcal{N}}^{(0)-}(\mathbb{R})$ nor $\mathcal{V}_{\mathcal{N}}^{(0)+}(\mathbb{R})$ belongs to $L^2(\mathbb{R})$, so the $\mathcal{N}$-fold SUSY is dynamically broken in the constant-mass background.

Now, the PDM potentials $U^{\pm}(q)$ can be obtained with help of Eqs. (2.18a), (3.45), and (3.46), and the solvable sectors $\mathcal{V}_{\mathcal{N}}^{\pm}$ of the corresponding PDM Hamiltonians $H^{\pm}$ for an arbitrary mass function $m(q)$ read from (2.18c) as

$$\mathcal{V}_{\mathcal{N}}^{-} = \big\langle \tilde{\varphi}_1(\zeta \sinh u(q); \alpha), \dots, \tilde{\varphi}_{\mathcal{N}}(\zeta \sinh u(q); \alpha) \big\rangle \times \frac{m(q)^{1/4} \mathrm{e}^{-\zeta (\sinh u(q))/2 - \zeta \operatorname{gd} u(q)}}{(\cosh u(q))^{\mathcal{N}/2-1} f(\zeta \sinh u(q); \alpha)}, \tag{3.48a}$$

$$\mathcal{V}_{\mathcal{N}}^{+} = \big\langle \bar{\chi}_1(\zeta \sinh u(q); \alpha + \mathcal{N}), \dots, \bar{\chi}_{\mathcal{N}}(\zeta \sinh u(q); \alpha + \mathcal{N}) \big\rangle \times \frac{m(q)^{1/4} \mathrm{e}^{\zeta (\sinh u(q))/2 + \zeta \operatorname{gd} u(q)}}{(\cosh u(q))^{\mathcal{N}/2} f(\zeta \sinh u(q); \alpha + \mathcal{N})}, \tag{3.48b}$$

where $u(q)$ is given by (2.19). Let us now consider two cases:

(i) $m(q) = \operatorname{sech}^2 q$, $q \in (-\infty, \infty)$, for which the change of variable is $u(q) = \operatorname{gd} q$. Then, the solvable sectors of $U^{\pm}(q)$ are given by

$$\mathcal{V}_{\mathcal{N}}^{-} = \big\langle \tilde{\varphi}_1(\zeta \sinh u(q); \alpha), \dots, \tilde{\varphi}_{\mathcal{N}}(\zeta \sinh u(q); \alpha) \big\rangle \times \frac{\sqrt{\operatorname{sech} q}\, \mathrm{e}^{-\zeta \sinh(\operatorname{gd} q)/2 - \zeta \operatorname{gd}(\operatorname{gd} q)}}{[\cosh(\operatorname{gd} q)]^{\mathcal{N}/2-1} f(\zeta \sinh u(q); \alpha)}, \tag{3.49a}$$

$$\mathcal{V}_{\mathcal{N}}^{+} = \big\langle \bar{\chi}_1(\zeta \sinh u(q); \alpha + \mathcal{N}), \dots, \bar{\chi}_{\mathcal{N}}(\zeta \sinh u(q); \alpha + \mathcal{N}) \big\rangle \times \frac{\sqrt{\operatorname{sech} q}\, \mathrm{e}^{\zeta \sinh(\operatorname{gd} q)/2 + \zeta \operatorname{gd}(\operatorname{gd} q)}}{[\cosh(\operatorname{gd} q)]^{\mathcal{N}/2} f(\zeta \sinh u(q); \alpha + \mathcal{N})}. \tag{3.49b}$$

In this case, the mass function as well as the potentials $U^{\pm}(q)$ are well behaved on $(-\infty, \infty)$, so we can consider the whole real line $\mathbb{R}$ as a domain of the potentials. From the solvable sectors (3.49), it is clear that both $\mathcal{V}_{\mathcal{N}}^{\pm}(\mathbb{R})$ belong to $L^2(\mathbb{R})$, which means unbroken $\mathcal{N}$-fold SUSY, i.e., the mass profile affects symmetry restoration.

(ii) $m(q) = 2\mathrm{e}^{-2q^2}/\pi$. In this case, the solvable sectors $\mathcal{V}_{\mathcal{N}}^{\pm}$ reduce to

$$\mathcal{V}_{\mathcal{N}}^{-} = \big\langle \tilde{\varphi}_1(\zeta \sinh u(q); \alpha), \dots, \tilde{\varphi}_{\mathcal{N}}(\zeta \sinh u(q); \alpha) \big\rangle \times \frac{\exp[-q^2/4 - \zeta \sinh(\operatorname{Erf} q)/2 - \zeta \operatorname{gd}(\operatorname{Erf} q)]}{[\cosh(\operatorname{Erf} q)]^{\mathcal{N}/2-1} f(\zeta \sinh u(q); \alpha)}, \tag{3.50a}$$

$$\mathcal{V}_{\mathcal{N}}^{+} = \big\langle \bar{\chi}_1(\zeta \sinh u(q); \alpha + \mathcal{N}), \dots, \bar{\chi}_{\mathcal{N}}(\zeta \sinh u(q); \alpha + \mathcal{N}) \big\rangle \times \frac{\exp[-q^2/4 + \zeta \sinh(\operatorname{Erf} q)/2 + \zeta \operatorname{gd}(\operatorname{Erf} q)]}{[\cosh(\operatorname{Erf} q)]^{\mathcal{N}/2} f(\zeta \sinh u(q); \alpha + \mathcal{N})}. \tag{3.50b}$$

From the above solvable sectors (3.50), it is clear that both $\mathcal{V}_{\mathcal{N}}^{\pm}(\mathbb{R})$ belong to $L^2(\mathbb{R})$. That is, in this case we again have unbroken $\mathcal{N}$-fold SUSY.

We note that there are other mass functions, e.g., $m(q) = (\beta + q^2)^2/(1+q^2)^2$, which have no effect on the dynamical breaking of $\mathcal{N}$-fold SUSY, i.e., it is also possible to construct PDM systems which maintain the broken $\mathcal{N}$-fold SUSY.

Example 3.6. $A(z) = (z+\zeta)^2/2$, $[a_2 = 1/2, a_1 = \zeta = \sqrt{(\alpha-1)(\alpha+\mathcal{N}-1)}]$.
Potentials:

$$\begin{aligned} V^{(0)-}(q) = {} & \frac{1}{8}\mathrm{e}^{2q} - \frac{\mathcal{N}+1}{4}\mathrm{e}^{q} - \frac{(\mathcal{N}-1)(\mathcal{N}+2\alpha-2\zeta-1)\zeta}{4}\mathrm{e}^{-q} \\ & + \frac{\zeta^2[\mathcal{N}^2 + 2\mathcal{N}(4\alpha-2\zeta-3) + 4\alpha(2\alpha-2\zeta-3) + 4\zeta+5]}{8}\mathrm{e}^{-2q} \\ & - 2\left[\frac{(\alpha-\zeta-1)\mathrm{e}^{q}}{f(\mathrm{e}^q-\zeta;\alpha)} + \frac{2(\alpha-1)\mathrm{e}^{2q}}{f(\mathrm{e}^q-\zeta;\alpha)^2}\right] + V_0, \end{aligned} \tag{3.51}$$

$$\begin{aligned} V^{(0)+}(q) = {} & \frac{1}{8}\mathrm{e}^{2q} + \frac{\mathcal{N}-1}{4}\mathrm{e}^{q} + \frac{(\mathcal{N}+1)(\mathcal{N}+2\alpha-2\zeta-1)\zeta}{4}\mathrm{e}^{-q} \\ & + \frac{\zeta^2[\mathcal{N}^2 + 2\mathcal{N}(4\alpha-2\zeta-3) + 4\alpha(2\alpha-2\zeta-3) + 4\zeta+5]}{8}\mathrm{e}^{-2q} \\ & - 2\left[\frac{(\alpha+\mathcal{N}-\zeta-1)\mathrm{e}^{q}}{f(\mathrm{e}^q-\zeta;\alpha+\mathcal{N})} + \frac{2(\alpha+\mathcal{N}-1)\mathrm{e}^{2q}}{f(\mathrm{e}^q-\zeta;\alpha+\mathcal{N})^2}\right] + V_0, \end{aligned} \tag{3.52}$$

where V_0 is an irrelevant constant given by

$$V_0 = \frac{(\mathcal{N}+2\alpha)^2 + 2\zeta(\mathcal{N}-2\alpha) + 2(7\zeta-8\alpha+8)}{8} - c_0.$$

Solvable sectors:

$$\begin{aligned} \mathcal{V}_{\mathcal{N}}^{(0)-} = {} & \frac{\langle \tilde{\varphi}_1(\mathrm{e}^q-\zeta;\alpha), \dots, \tilde{\varphi}_{\mathcal{N}}(\mathrm{e}^q-\zeta;\alpha)\rangle}{f(\mathrm{e}^q-\zeta;\alpha)} \\ & \times \exp\left[-\frac{\mathrm{e}^q}{2} + \frac{2\zeta-2\alpha-\mathcal{N}+1}{2}\zeta\mathrm{e}^{-q} - \frac{\mathcal{N}-2}{2}q\right], \end{aligned} \tag{3.53}$$

$$\begin{aligned} \mathcal{V}_{\mathcal{N}}^{(0)+} = {} & \frac{\langle \bar{\chi}_1(\mathrm{e}^q-\zeta;\alpha+\mathcal{N}), \dots, \bar{\chi}_{\mathcal{N}}(\mathrm{e}^q-\zeta;\alpha+\mathcal{N})\rangle}{f(\mathrm{e}^q-\zeta;\alpha+\mathcal{N})} \\ & \times \exp\left[\frac{\mathrm{e}^q}{2} - \frac{2\zeta-2\alpha-\mathcal{N}+1}{2}\zeta\mathrm{e}^{-q} - \frac{\mathcal{N}}{2}q\right]. \end{aligned} \tag{3.54}$$

This system is new and presented in this paper for the first time. The exponential-type $V_{\mathcal{N}}^{\pm}(q)$ are naturally defined on the whole real line $\mathbb{R}$ since they have no singularity on it, so the Hilbert space is $L^2(\mathbb{R})$. Noting that $2\zeta - 2\alpha - \mathcal{N} + 1 < 0$ for $\alpha > 1$, since

$$4\zeta^2 - (2\alpha+\mathcal{N}-1)^2 = -4\alpha - (\mathcal{N}-1)(\mathcal{N}+3) < -(\mathcal{N}+1)^2 < 0,$$

we see that $\mathcal{V}_{\mathcal{N}}^{(0)-}(\mathbb{R}) \subset L^2(\mathbb{R})$ and $\mathcal{V}_{\mathcal{N}}^{(0)+}(\mathbb{R}) \not\subset L^2(\mathbb{R})$ for $\zeta > 0$. Hence, it manifests unbroken $\mathcal{N}$-fold SUSY. For $\zeta < 0$, on the other hand, neither $\mathcal{V}_{\mathcal{N}}^{(0)-}(\mathbb{R})$ nor $\mathcal{V}_{\mathcal{N}}^{(0)+}(\mathbb{R})$ belongs to $L^2(\mathbb{R})$, so the $\mathcal{N}$-fold SUSY is broken in the constant-mass background.

In a PDM background, the solvable sectors $\mathcal{V}^{\pm}_{\mathcal{N}}$ of the type X_2 PDM Hamiltonians $H^{\pm}$ are deformed as [cf., Eq. (2.18c)]

$$\mathcal{V}^{-}_{\mathcal{N}} = \frac{\left\langle \tilde{\varphi}_1(\mathrm{e}^{u(q)} - \zeta; \alpha), \ldots, \tilde{\varphi}_{\mathcal{N}}(\mathrm{e}^{u(q)} - \zeta; \alpha) \right\rangle}{f(\mathrm{e}^{u(q)} - \zeta; \alpha)} \times m(q)^{1/4} \exp\left[-\frac{e^{u(q)}}{2} + \frac{2\zeta - 2\alpha - \mathcal{N} + 1}{2} \zeta \mathrm{e}^{-u(q)} - \frac{\mathcal{N} - 2}{2} u(q) \right], \tag{3.55}$$

$$\mathcal{V}^{+}_{\mathcal{N}} = \frac{\left\langle \bar{\chi}_1(\mathrm{e}^{u(q)} - \zeta; \alpha + \mathcal{N}), \ldots, \bar{\chi}_{\mathcal{N}}(\mathrm{e}^{u(q)} - \zeta; \alpha + \mathcal{N}) \right\rangle}{f(\mathrm{e}^{u(q)} - \zeta; \alpha + \mathcal{N})} \times m(q)^{1/4} \exp\left[\frac{\mathrm{e}^{u(q)}}{2} - \frac{2\zeta - 2\alpha - \mathcal{N} + 1}{2} \zeta \mathrm{e}^{-u(q)} - \frac{\mathcal{N}}{2} u(q) \right], \tag{3.56}$$

and the potentials $U^{\pm}(q)$ can be obtained using Eqs. (2.18a), (3.51), and (3.52). In this case, the choice of mass functions are as follows:

(i) $m(q) = (1-q^2)^{-1}$, $q \in (-1, 1)$, for which the solvable sectors of the PDM Hamiltonians $H^{\pm}$ are given by

$$\mathcal{V}^{-}_{\mathcal{N}} = \frac{\left\langle \tilde{\varphi}_1(\mathrm{e}^{u(q)} - \zeta; \alpha), \ldots, \tilde{\varphi}_{\mathcal{N}}(\mathrm{e}^{u(q)} - \zeta; \alpha) \right\rangle}{(1-q^2)^{1/4} f(\mathrm{e}^{u(q)} - \zeta; \alpha)} \times \exp\left[-\frac{\mathrm{e}^{\sin^{-1} q}}{2} + \frac{2\zeta - 2\alpha - \mathcal{N} + 1}{2} \zeta \mathrm{e}^{-\sin^{-1} q} - \frac{\mathcal{N} - 2}{2} \sin^{-1} q \right], \tag{3.57}$$

$$\mathcal{V}^{+}_{\mathcal{N}} = \frac{\left\langle \bar{\chi}_1(\mathrm{e}^{u(q)} - \zeta; \alpha + \mathcal{N}), \ldots, \bar{\chi}_{\mathcal{N}}(\mathrm{e}^{u(q)} - \zeta; \alpha + \mathcal{N}) \right\rangle}{(1-q^2)^{1/4} f(\mathrm{e}^{u(q)} - \zeta; \alpha + \mathcal{N})} \times \exp\left[\frac{\mathrm{e}^{\sin^{-1} q}}{2} - \frac{2\zeta - 2\alpha - \mathcal{N} + 1}{2} \zeta \mathrm{e}^{-\sin^{-1} q} - \frac{\mathcal{N}}{2} \sin^{-1} q \right]. \tag{3.58}$$

From the above solvable sectors, it is clear that both $\mathcal{V}^{\pm}_{\mathcal{N}}(-1, 1)$ do not belong to $L^2(-1, 1)$, so it manifests broken $\mathcal{N}$-fold SUSY irrespective of the sign of ζ. Hence, comparing the normalizability conditions in both the constant and position-dependent mass cases, we conclude that the mass function $m(q) = (1 - q^2)^{-1}$ affects dynamical breaking of $\mathcal{N}$-fold SUSY for $\zeta > 0$.

(ii) $m(q) = 2\mathrm{e}^{-2q^2}/\pi$, $q \in (-\infty, \infty)$, for which the $\mathcal{N}$-fold SUSY remains unbroken, which is evident from the corresponding solvable sectors given by

$$\mathcal{V}^{-}_{\mathcal{N}} = \frac{\left\langle \tilde{\varphi}_1(\mathrm{e}^{u(q)} - \zeta; \alpha), \ldots, \tilde{\varphi}_{\mathcal{N}}(\mathrm{e}^{u(q)} - \zeta; \alpha) \right\rangle}{f(\mathrm{e}^{u(q)} - \zeta; \alpha)} \exp\left[-\frac{q^2}{4} - \frac{\mathrm{e}^{\mathrm{Erf}\, q}}{2} + \frac{2\zeta - 2\alpha - \mathcal{N} + 1}{2} \zeta \mathrm{e}^{-\mathrm{Erf}\, q} - \frac{\mathcal{N} - 2}{2} \mathrm{Erf}\, q \right], \tag{3.59}$$

$$\mathcal{V}^{+}_{\mathcal{N}} = \frac{\left\langle \bar{\chi}_1(\mathrm{e}^{u(q)} - \zeta; \alpha + \mathcal{N}), \ldots, \bar{\chi}_{\mathcal{N}}(\mathrm{e}^{u(q)} - \zeta; \alpha + \mathcal{N}) \right\rangle}{f(\mathrm{e}^{u(q)} - \zeta; \alpha + \mathcal{N})} \times \exp\left[-\frac{q^2}{4} + \frac{\mathrm{e}^{\mathrm{Erf}\, q}}{2} - \frac{2\zeta - 2\alpha - \mathcal{N} + 1}{2} \zeta \mathrm{e}^{-\mathrm{Erf}\, q} - \frac{\mathcal{N}}{2} \mathrm{Erf}\, q \right]. \tag{3.60}$$

From the normalizability conditions in the constant and position-dependent mass cases, we see that the mass function $m(q) = 2\mathrm{e}^{-2q^2}/\pi$ affects the dynamical breaking of $\mathcal{N}$-fold SUSY for $\zeta < 0$.

IV. SUMMARY AND PERSPECTIVES

In this paper, we have investigated effect of position-dependent mass background on dynamical breaking of type B and type X_2 $\mathcal{N}$-fold SUSY. We have selected three different models in the constant mass background for each type, and then examined whether some of the physically relevant effective mass profiles can affect the pattern of $\mathcal{N}$-fold SUSY breaking in each model. We summarize the results in Table I. We can easily see from Table I that, except for the rational potentials, some of the PDM profiles can actually affect and change the patterns of dynamical $\mathcal{N}$-fold SUSY breaking in all the trigonometric, hyperbolic, and exponential potentials. Although we have selected the specific types of $\mathcal{N}$-fold SUSY to develop physical applicability of the new mathematical concept of exceptional polynomial subspaces, we can of course make a similar analysis on other types of $\mathcal{N}$-fold SUSY such as type A and type C to find out positive effect of PDM on SUSY breaking in some models.

Hence, it would be possible to observe experimentally transition between a broken and an unbroken phases if an effective mass can be controlled experimentally such that the constant mass limit can be also realized in an experimental setting. The physical meanings of a position-dependent mass depend on each physical system under consideration, for instance, the curvature of the local band structure of an alloy in the momentum space for electrons in a crystal with graded composition [1], the particle densities of 3He and 4He in pure and mixed helium clusters with doping atoms or molecules [4], the effective electron mass for electrons confined in a quantum dot [3] and for dipole excitations of sodium clusters [5], and so on. Thus, if we can prepare such an atomic, molecular, or condensed matter system which is described by a certain PDM quantum model subjected to an $\mathcal{N}$-fold SUSY potential with mass profiles, e.g., $m(q) = \mathrm{e}^{-\nu^2 q^2}$ or $(1 - \nu^2 q^2)^{-1}$ where ν is an experimentally adjustable parameter such that $\nu \to 0$ is realizable, then the spectral change of the system could be observed at $\nu = 0$ due to the phase transition. The essence and novelty of our idea rely on the observation that the physically controllable PDM can cause the phase transition by changing the normalizability of the solvable sector although the latter is superficially a simple mathematical aspect. Hence, it is quite important to note that the normalizability of wave functions can play much more roles than the quantization of energy spectrum which is referred to by any standard textbook on quantum mechanics.

We note that this experimental observability might have impact not only on some atomic, molecular, and condensed matter problems from which PDM quantum theory originated, but also on high-energy physics. Until now many high-energy physicists have believed that SUSY is realized at the GUT or Planck scale as a resolution of the naturalness and the hierarchy problem but is broken at least at the electroweak scale. Unfortunately, however, theoretical analysis on dynamical SUSY breaking in field theoretical models are extremely difficult on the one hand, and it is virtually impossible to make a GUT scale experiment on the other hand. The aforementioned experimental observability suggests that we might extract some clues to understand dynamical SUSY breaking in high-energy physics from realistic eV scale experiments in atomic, molecular, and condensed matter physics. It is because the Witten's work [31] has indicated that the mechanism of dynamical SUSY breaking in quantum field theory and quantum mechanics is essentially the same. We also note that the careful non-

TABLE I: The effects of PDM profiles on dynamical breaking of $\mathcal{N}$-fold SUSY in various type B and type X_2 models.

Types of potentials		Dynamical breaking of $\mathcal{N}$-fold SUSY	
		Constant mass	PDM
Type B	rational	unbroken	no effect
	trigonometric	broken	unbroken for $m(q) \propto e^{-2q^2}$
	exponential	partially broken for $-\mathcal{N}/2 < b_1 < (\mathcal{N}+1-2k)/2$ and fully broken otherwise	broken for $m(q) = (1-q^2)^{-1}$ unbroken for $m(q) \propto e^{-2q^2}$
Type X_2	rational	unbroken	no effect
	hyperbolic	broken	unbroken for $m(q) \propto \operatorname{sech}^2 q$, e^{-2q^2}
	exponential	unbroken for $\zeta > 0$ broken for $\zeta < 0$	broken for $m(q) = (1-q^2)^{-1}$, $\forall\zeta$ unbroken for $m(q) \propto e^{-2q^2}$, $\forall\zeta$

perturbative analyses in Refs. [59, 60] have shown that the mechanism of dynamical breaking of ordinary and $\mathcal{N}$-fold SUSY is also the same. Hence, dynamical aspects of SUSY quantum field theoretical models would be mimicked in $\mathcal{N}$-fold SUSY quantum mechanical toy models, regardless of whether or not $\mathcal{N}$-fold SUSY can be realized in higher dimensions. Therefore, we believe that further studies in this direction are worth pursuing both theoretically and experimentally. From a theoretical point of view it is a challenging issue to investigate both a perturbation theory and the non-renormalization theorem in PDM quantum systems.

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