On a new type of solitary surface waves in finite water depth

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Abstract

In this paper, a new type of solitary surface waves in a finite water depth is found by analytically solving the fully nonlinear wave equations. Using a new type of base functions which decays exponentially in the horizontal direction, this new type of solitary surface waves is gained first by means of linear wave equations, and then confirmed by the fully nonlinear wave equations. The new type of solitary surface waves have many unusual characteristics. First, it has a peaked crest. Secondly, it may be in the form of depression, which has been often reported for internal solitary waves but never for free-surface solitary ones, to the best of author's knowledge. Third, its phase speed has nothing to do with wave height, say, the peaked solitary waves are non-dispersive. Finally, its horizontal velocity at bottom is always larger than that on surface. All of these are so different from the traditional periodic and solitary waves that they clearly indicate the novelty of the peaked solitary waves. Based on the new peaked solitary surface waves, a new explanation to the so-called rogue waves and some theoretical predictions are given. All of these are helpful to deepen our understandings and enrich our knowledge about solitary waves.

Key words Solitary wave, peaked crest, progressive wave, fully nonlinear

1 Introduction

Since the solitary surface wave was discovered by John Scott Russell in 1834, various types of solitary waves have been found. The mainstream models of shallow water waves, such as the Boussinesq equation [1], the KdV equation [2], the BBM equation [3] and so on, admits dispersive *smooth* periodic and solitary waves of permanent form: the wave elevation is *infinitely* differentiable in the whole domain. Especially, the phase speed of these smooth water waves is closely related to the wave height: in general, a wave with higher amplitude travels faster than a lower one. Such kind of smooth periodic and solitary waves have been the mainstream of the teaching and investigating of water waves for quite a long time.

However, in theory, the discontinuity of water wave elevation appears accidentally. It is well-known that the limiting gravity wave has a corner crest with 120 degree, as pointed out by Stokes [4] in 1894. It is a pity that the importance of such kind of discontinuity is neglected since Stokes limiting gravity wave [4] is regarded to hardly appear in practice. About one hundred years later, Camassa & Holm [5] proposed a model for the shallow water waves, called today the Camassa-Holm (CH) equation

$$u_t + 2\omega u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1}$$

where $u(x,t)$ denotes the wave elevation, x and t are the spatial and temporal variables, $\omega = c_0/4$ is a constant related to the critical shallow water wave speed $c_0 = \sqrt{gD}$, g denotes the acceleration due to gravity and D the water depth, respectively. As pointed out by Camassa & Holm [5], the CH equation (1) has the solitary wave when $0 \le \omega < 1/2$. Especially, when $\omega = 0$, i.e. $c_0 = \sqrt{gD} \to 0$, the CH equation (1) admits the peaked solitary wave in the closed-form

$$u(x,t) = c\ \exp(-|x - ct|),$$

whose first derivative is *discontinuous* at the crest, where c denotes the phase speed. Unlike the KdV equation and Boussinesq equation, the CH equation (1) can model both phenomena of soliton interaction and wave breaking, as mentioned by Constantin [6]. Mathematically, the CH equation is integrable and bi-Hamiltonian, thus possesses an infinite number of conservation laws in involution, as pointed out by Camassa & Holm [5]. In addition, it is associated with the geodesic flow on the infinite dimensional Hilbert manifold of diffeomorphisms of line, as mentioned by Constantin [6]. Thus, the CH equation (1) has many intriguing physical and mathematical properties. As pointed out by Fushsstainer [7], the CH equation (1) "has the potential to become the new master equation for shallow water wave theory".

The peaked solitary waves of the CH equation (1) when $\omega = 0$ have been investigated in details, and hundreds of related articles have been published. However, Camassa & Holm [5] pointed out that the discontinuity of wave elevation appears only when $\omega \to 0$, i.e. $D \to 0$, and the solitary wave of the CH equation (1) "becomes C^∞ and there is no derivative discontinuity at its peak" in case of $\omega \ne 0$. It implies that the discontinuity of wave elevation is *not* a common property of water waves.

In contrast to the above-mentioned theoretical results, the discontinuity widely appears in practical flows, such as the dam break in hydrodynamics and shock wave in aerodynamics. In fact, such kind of discontinuous problems belong to the Riemann problem, a classic research field. Therefore, the discontinuity of wave elevation are reasonable not only in mathematics but also in physics.

Currently, the closed-form solutions of peaked solitary waves of the Boussinesq equation, the KdV equation, the BBM equation, and the modified KdV equation are found by Liao [8]. Besides, it is also found by Liao [9] that the CH equation (1) admits peaked solitary waves even when $\omega \ne 0$. Like the peaked solitary waves of the peaked solitary waves of the CH equation (when $\omega = 0$) found by Camassa & Holm, all of these solitary waves are reasonable not only in mathematics but also in physics. Therefore, nearly *all* mainstream models of the shallow water waves admit the peaked solitary waves. It indicates that the discontinuity and the peaked crest might be a common property of shallow water waves.

Where does this discontinuity come from? Are there any peaked waves in finite water depth?

Note that all of the above-mentioned mainstream models of shallow water waves are approximations of the fully nonlinear wave equations. So, the correct answers to these two questions should exist in the exact nonlinear water wave equations.

In this article, using the fully nonlinear wave equations, we indeed obtain a new type of solitary surface waves in *finite* water depth, which have a peaked crest and many unusual characteristics quite different from the traditional ones. So, the traditional nonlinear water wave equations admit two different kinds of waves: one is *infinitely* differentiable with phase speed closely related to the wave height, the other has a peaked crest whose phase speed has *nothing* to do with the wave height. Therefore, the discontinuity is a common property of water waves.

In § 2, the governing equation and boundary conditions for progressive waves with permanent form in finite water depth are described, which admit all traditional smooth periodic and solitary progressive waves. In § 3, a new type of peaked solitary surface waves are obtained, for the first time, by means of the linearized wave equations. In § 4, the existence of such kind of new solitary surface waves is confirmed by the fully nonlinear wave equations. This kind of new peaked solitary surface waves have unusual characteristics different from traditional ones, as described in § 5. The concluding remarks, discussions and some theoretical predictions are given in § 6.

2 Mathematical formulations

First of all, we provide the mathematical formulations for progressive waves with permanent form in finite water depth. Especially, we must be extremely careful not to lose the discontinuous solutions.

Consider a progressive surface gravity wave propagating on a horizontal bottom with a constant phase speed c and a permanent form. Assumed that the fluid is inviscid and incompressible, the flow is irrotational, the surface tension is neglected and the wave elevation has a symmetry. Let D denote the water depth and g the acceleration due to gravity. For simplicity, let us consider the problem in the frame moving with the same phase speed c, with ϕ denoting the velocity potential, ζ the free surface, x, z the horizontal and vertical co-ordinates with $x = 0$ corresponding to the wave crest, respectively. All of these variables are dimensionless by means of D and $\sqrt{gD}$ as the characteristic scales of length and velocity. The z axis is upward so that $z = -1$ corresponding to the bottom. Due to the symmetry $\zeta(-x) = \zeta(x)$, we only need consider the flow in the domain $x \in [0, +\infty)$, governed by

$$\nabla^2\phi(x, z) = 0, \qquad z \le \zeta(x), 0 \le x < +\infty, \tag{2}$$

subject to the boundary conditions on the unknown free surface $z = \zeta(x)$:

$$\alpha^2\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi}{\partial z} - \alpha\frac{\partial}{\partial x}\left(\nabla\phi\cdot\nabla\phi\right) + \nabla\phi\cdot\nabla\left(\frac{1}{2}\nabla\phi\cdot\nabla\phi\right) = 0, \qquad 0 \le x < +\infty, \tag{3}$$

$$\zeta - \alpha\frac{\partial\phi}{\partial x} + \frac{1}{2}\nabla\phi\cdot\nabla\phi = 0, \qquad 0 \le x < +\infty, \tag{4}$$

and the bottom condition

$$\frac{\partial\phi}{\partial z} = 0, \qquad z = -1, 0 \le x < +\infty, \tag{5}$$

where ∇^2 is a Laplace operator,

$$\alpha = \frac{c}{\sqrt{gD}} \tag{6}$$

is the dimensionless wave-speed, respectively. On the vertical boundary $x = 0$, we have the additional condition

$$\frac{\partial\phi}{\partial x} = U(z), \qquad x = 0, \ \ z \le \zeta(x), \tag{7}$$

where $U(z)$ is such an unknown horizontal velocity at $x = 0$ that the velocity potential $\phi(x, z)$ and the corresponding progressive wave elevation $\zeta(x)$ with permanent form exist. Besides, let H_w denote the dimensionless wave-elevation at $x = 0$, corresponding to the wave crest. For given H_w, one has an addition condition

$$\zeta(0) = H_w. \tag{8}$$

In addition, the wave elevation must be bounded, i.e.

$$|\zeta(x)| < C, \qquad 0 \le x < +\infty, \tag{9}$$

for a large enough constant C. The corresponding velocities $u(x, z)$ and $v(x, z)$ is given by

$$u(x, z) = \frac{\partial\phi}{\partial x}, \quad v(x, z) = \frac{\partial\phi}{\partial z}, \qquad 0 \le x < +\infty.$$

Due to the symmetry, in the domain $-\infty < x \le 0$, we have

$$\zeta(x) = \zeta(-x), \quad u(x, z) = u(-x, z), \quad v(x, z) = -v(-x, z). \tag{10}$$

In this way, the problem is well defined mathematically.

Note that, according to the above symmetry, the wave elevation $\zeta(x)$ and the horizontal velocity u are continuous at the vertical boundary $x = 0$. Besides, using the Bernoulli's principle and the symmetry (10) of the flow, it is easy to prove that the pressure is also continuous at the vertical boundary $x = 0$, too. In fact, due to the symmetry (10), the boundary condition (7) is equivalent to the continuous condition of the horizontal velocity u at the vertical boundary $x = 0$. It is well-known that the Laplace equation (2) needs only *one* boundary condition at each boundary. Therefore, at the vertical boundary $x = 0$, (7) is *sufficient* for the Laplace equation (2), and any other conditions for the vertical velocity v and the smoothness of the horizontal velocity u at $x = 0$ are unnecessary: at $x = 0$, there are no restrictions for the vertical velocity v since the fluid is inviscid, and besides the higher-order derivatives of the horizontal velocity

$$\frac{\partial^2\phi}{\partial x^2}, \ \frac{\partial^3\phi}{\partial x^3}, \ \frac{\partial^4\phi}{\partial x^4}, \cdots$$

are *unnecessary* to be continuous at $x = 0$, since *one and only one* boundary condition at $x = 0$ is enough for the Laplace equation (2). Thus, any other boundary conditions such as that ϕ and ζ should be *infinitely* differentiable at $x = 0$ may lead to the *loss* of the solutions and thus *must* be avoided. In other words, both of ϕ and ζ are *unnecessary* to be *infinitely* differentiable at $x = 0$.

Note that the two nonlinear boundary conditions (3) and (4) must be satisfied on the unknown free surface $z = \zeta(x)$. This leads to the mathematical difficulty to solve the nonlinear partial differential equations (PDEs). In case of small wave-amplitude, the linear boundary condition

$$\alpha^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial z} = 0, \qquad \text{on } z = 0, \tag{11}$$

is a good approximation of (3), and

$$\zeta = \alpha \frac{\partial \phi}{\partial x}\bigg|_{z=0} \tag{12}$$

is a good approximation of (4), respectively. The above two linearized free-surface boundary conditions, combined with the Laplace equation (2) and the bottom condition (5), provide us the so-called linear wave equations.

Based on the above linearized or fully nonlinear wave equations, hundreds of articles have been published for the periodic and solitary progressive waves. Nearly all of these traditional waves are based on the base functions

$$\cosh[nk(z+1)]\ \sin(nkx), \quad n \geq 1, \tag{13}$$

for the velocity potential ϕ, which automatically satisfy the Laplace equation (2) and the bottom condition (5), where k denotes the wave number and $n \geq 1$ is an integer. For periodic progressive waves with small wave-amplitude, substituting the velocity potential

$$\phi(x,z) = \alpha A_0 \cosh[k(z+1)]\ \sin(kx) \tag{14}$$

into the linear boundary condition (11), one has the dimensionless phase speed

$$\alpha = \sqrt{\frac{\tanh(k)}{k}} \leq 1, \tag{15}$$

say, the phase speed of a spatially periodic progressive wave in a finite water depth D is always less then $\sqrt{gD}$. Besides, substituting (14) into (12) gives the wave elevation with small amplitude

$$\zeta = \frac{H_w}{2}\ \cos(kx), \tag{16}$$

where $H_w = 2A_0 \sinh(k)$. The corresponding horizontal velocity reads

$$u(x,z) = \frac{\alpha H_w k \cosh[k(z+1)] \cos(kx)}{2\sinh(k)} = \frac{H_w}{2\alpha \cosh(k)} \cosh[k(z+1)] \cos(kx), \tag{17}$$

which gives

$$\frac{u}{U_0} = \frac{\cosh[k(z+1)]\ \cos(kx)}{\cosh(k)}, \tag{18}$$

where $U_0 = H_w/(2\alpha)$. At the left boundary $x = 0$, we have the corresponding horizontal velocity

$$U(z) = u(0,z) = \frac{H_w}{2\alpha\cosh(k)}\cosh[k(z+1)].$$

Note that the velocity potential and the wave elevation given by the above traditional linear wave theory automatically satisfies the symmetry (10). In other words, we can gain exactly the same results by first solving the PDEs (2) to (9) in the domain $0 \le x < +\infty$ and then expanding the result to the domain $-\infty < x \le 0$ by means of the symmetry (10). Note that, for given x, the horizontal velocity u of periodic progressive Airy's waves decreases *exponentially* as z varies from the surface ($z = 0$) to the bottom ($z = -1$). Especially, based on the base functions (13), the elevation of the Airy's wave and the corresponding velocities are *infinitely* differentiable, although such kind of smoothness conditions do *not* exist at all.

For the periodic progressive surface waves with large amplitude, the fully nonlinear wave equations must be considered. As pointed out by Cokelet [10], the phase speed c of the progressive periodic waves depends not only on the water depth D and the wave number k but also on the wave height H_w: in most cases, the larger the wave amplitude, the faster the periodic wave propagates. In other words, the traditional progressive periodic waves are dispersive. Besides, the periodic progressive surface waves have a smooth crest with the exponentially decaying velocity $u(x,z)$ from the surface to the bottom. Like the Airy's linear waves, the traditional nonlinear periodic progressive waves are also infinitely differentiable, although such kind of smoothness conditions do *not* exist at all. Note that the exactly same results for the traditional progressive periodic waves can be obtained by first solving the PDEs (2) to (9) in the domain $0 \le x < +\infty$ and then expanding the results to the domain $-\infty < x \le 0$ by means of the symmetry (10).

It should be emphasized that, in the frame of the linear wave theory, solitary waves have *never* be reported, to the best knowledge of the author. For details, please refer to Mei et al [11]. Solitary wave solutions for nonlinear and dispersive long waves had been found by Boussinesq [1] and Rayleigh [12]. For dispersive long waves of permanent form, the so-called KdV equation [2] gives the periodic cnoidal wave for a finite wavelength λ, which tends to the solitary wave

$$\zeta(\tilde{x}) = H_w\ \mathrm{sech}^2\left[\frac{1}{2}\sqrt{\frac{3H_w}{D}}\frac{(\tilde{x} - c\,t)}{2D}\right] \tag{19}$$

with the phase speed

$$c = \sqrt{g\,D\left(1 + \frac{H_w}{D}\right)}, \tag{20}$$

as $\lambda \to +\infty$. Note that these cnoidal and solitary waves have a smooth elevation, say, $\zeta(x)$ is *infinitely* differentiable for all $x \in (-\infty, +\infty)$. Besides, its phase speed c depends upon the wave height H_w: the larger the wave height, the faster the solitary wave propagates, as shown by (20). All of these results can be gained by first solving the KdV equation in the domain $0 \le x < +\infty$ and then expanding the results to the domain $-\infty < x \le 0$ by means of the symmetry (10).

Both of the solitary wave and the Airy linear waves are special cases of the so-called cnoidal waves. By means of perturbation methods and using the fully nonlinear wave equations, Fenton [13,14] gave respectively a high-order cnoidal wave theory and a ninth-order solution for the solitary wave in the form

$$\zeta(x) = \sum_{i=1}^{+\infty} \sum_{j=1}^{i} a_{i,j}\ \epsilon^i \left[\operatorname{sech}^2(\beta x)\right]^j, \tag{21}$$

where $a_{i,j}, \epsilon, \beta$ are constants determined by the physical parameters. It should be emphasized that all of these traditional cnoidal and solitary waves have a smooth crest: $\zeta(x)$ is *infinitely* differentiable for all $x \in (-\infty, +\infty)$. Besides, the velocity $u(x, z)$ at bottom is always larger than that at crest. Furthermore, the phase speed is dependent upon wave height. Finally, to the best of author's knowledge, all traditional solitary surface waves have a crest higher than the still water: the solitary waves in the form of depression have been reported for interfacial waves, but never for the surface waves. It should be emphasized that all of these traditional results can be gained by first similarly solving the PDEs (2) to (9) in the domain $0 \le x < +\infty$ and then expanding the results to the domain $-\infty < x \le 0$ by means of the symmetry (10). This indicates that the PDEs (2) to (9) with the symmetry condition (10) are indeed consistent with the traditional linear and nonlinear progressive waves with smooth wave crest.

Indeed, the traditional periodic and solitary progressive waves are infinitely differentiable. This kind of smoothness is however *unnecessary*, since no such kind of smoothness conditions are enforced to the PDEs (2) to (9). In essence, such kind of perfect smoothness of the wave elevation and velocities come from the base functions (13), which are infinitely differentiable at $x = 0$.

There exist a little thing in the traditional wave theory that should be reconsidered carefully. Note that the traditional cnoidal waves are periodic and thus have an infinite number of wave crests. As a special case of the cnoidal waves as the wavelength $\lambda \to +\infty$, the solitary waves (19) of the KdV equation should have an infinite number of wave crest, although the distance between the two crests is infinite. So, seriously speaking, the solitary waves (19) given by the KdV equation is not truly "solitary", since it might have an infinite number of crests.

Note that, like the base functions (13) that are widely used for the traditional periodic and solitary progressive waves, the following base functions

$$\cos[nk(z+1)]\ \exp(-nkx), \quad n \ge 1,\ k > 0,\ 0 \le x < +\infty, \tag{22}$$

also automatically satisfy the Laplace equation (2), the bottom condition (5) and the bounded condition (9). However, different from the traditional base functions (13), the above base functions decays exponentially in the x direction and thus seem to be more convenient to express a solitary wave that has truly only one crest. In addition, unlike the traditional base functions (13), the high-order derivatives of the above base functions with respect to x are not differentiable at $x = 0$. Mainly due to such kind of discontinuity, the base functions (22) have been completely neglected.

As mentioned above, mathematically speaking, no smoothness conditions are enforced to the PDEs (2) to (9). Physically, such kind of discontinuity widely appears in practice, such as dam break and shock waves, which have clear physical meanings. Thus, like the peaked solitary waves of the CH equation (1), the peaked solitary waves of the PDEs (2) to (9) should be reasonable and acceptable not only in mathematics but also in physics.

Can we find any kinds of solutions of peaked solitary surface waves of the fully nonlinear wave equations (2) to (9) by means of the new type of base functions (22)? The answer to the above question is positive: we demonstrate in the following part of this article that the same nonlinear wave equations (2) to (9) indeed admit such a new type of solitary surface waves with peaked crest and some unusual characteristics that are completely different from the traditional ones.

3 Peaked solitary waves by linear equations

As mentioned in § 2, *both* of the base functions (13) and (22) *automatically* satisfy the Laplace equation (2), the bottom condition (5) and the bounded condition (9). In the frame of linear wave theory, the former gives the well-known Airy wave, which is infinitely differentiable and horizontally periodic, as mentioned above. It is a pity that the base function (22) was neglected, which corresponds to a velocity potential decaying exponentially as $x \to +\infty$.

In the domain $0 \leq x < +\infty$, we have the velocity potential in the form

$$\phi = \alpha A_0 \cos[k(z+1)]\, e^{-kx}, \quad 0 \leq x < +\infty, \tag{23}$$

where $k > 0$ is a given parameter and A_0 is a constant to be determined. Note that the above expression automatically satisfies the Laplace equation (2), the bottom condition (5) and the bounded condition (9). Substituting (23) into the linear boundary condition (11) gives

$$\alpha k A_0 \left(\alpha^2 k \cos k - \sin k\right) \exp(-kx) = 0, \quad 0 \leq x < +\infty,$$

which leads to

$$\alpha^2 = \frac{\tan k}{k}. \tag{24}$$

Since $k > 0$ and $\alpha^2 > 0$, the above expression implies

$$n\pi < k < n\pi + \frac{\pi}{2}, \tag{25}$$

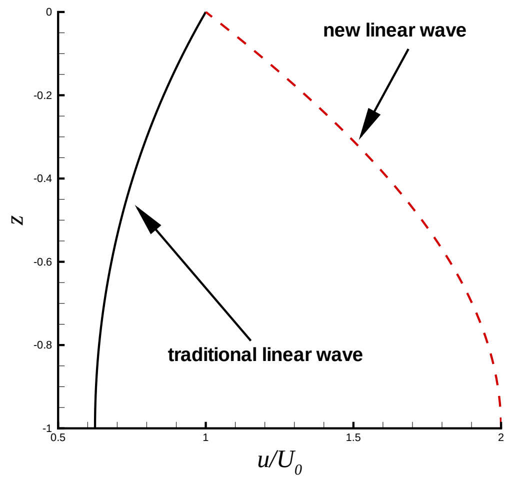

Figure 1: Velocity profile u/U_0 at $x = 0$ in case of $k = \pi/3$. Solid line: traditional linear wave theory; Dashed line: new linear wave theory.

where $n \geq 0$ is an integer. Therefore, the dimensionless phase speed reads

$$\alpha = \sqrt{\frac{\tan k}{k}}, \qquad n\pi < k < n\pi + \frac{\pi}{2}.$$

Obviously, $\alpha \geq 1$, i.e. $c \geq \sqrt{gD}$. This is quite different from the traditional Airy wave whose phase speed has the property $c \leq \sqrt{gD}$. Besides, using the linear boundary condition (12), we have the corresponding elevation of the solitary wave

$$\zeta(x) = \alpha \frac{\partial \phi}{\partial z}\bigg|_{z=0} = -k\alpha^2 A_0 \cos(k) e^{-kx} = H_w\ e^{-kx}, \quad 0 \leq x < +\infty, \tag{26}$$

where $H_w = -A_0 \sin(k)$ denotes the wave height. Then, using the symmetry condition (10), the wave elevation has a uniform expression

$$\zeta = H_w\ e^{-k|x|}, \qquad -\infty < x < +\infty. \tag{27}$$

This is a solitary wave that seriously has only one crest, with a discontinuous derivative $\zeta'(x)$ at crest! This is quite different from the traditional periodic and solitary progressive waves, which are infinitely differentiable. This clearly indicates the novelty of the new type of peaked solitary waves.

In the domain $0 \leq x < +\infty$, the corresponding horizontal velocity reads

$$u(x, z) \ = \ \frac{\partial \phi}{\partial x} = \frac{\alpha H_w k \cos[k(z+1)] e^{-kx}}{\sin(k)}. \tag{28}$$

Using the symmetry (10), we have

$$u(x, z) \ = \ u(-x, z) = \frac{\alpha H_w k \cos[k(z+1)] e^{kx}}{\sin(k)}, \quad -\infty < x \leq 0. \tag{29}$$

At the vertical boundary $x = 0$, u is continuous and we gain the corresponding horizontal velocity

$$U(z) = \frac{\alpha H_w k \cos[k(z+1)]}{\sin(k)}. \tag{30}$$

Thus, in the whole domain $-\infty < x < +\infty$, we have a uniform expression

$$\frac{u}{U_0} = \frac{\cos[k(z+1)]e^{-k|x|}}{\cos(k)}, \qquad x \in (-\infty, +\infty), \tag{31}$$

where $U_0 = H_w/\alpha$. So, for given x, the horizontal velocity u of the peaked solitary wave *increases* as z varies from the surface ($z = 0$) to the bottom ($z = -1$): in other words, u at bottom is always greater than that on surface. For example, when $k = \pi/3$, the horizontal velocity at bottom beneath crest of the peaked solitary wave is twice of that on surface, as shown in Fig. 1. This is quite different from the traditional ones whose horizontal velocity u at bottom is always less than that on surface. This also indicates the novelty of the new solitary waves.

Like Airy's wave, since the elevation (27) of the peaked solitary wave is gained by the linear wave equations, the value of H_w can be negative, corresponding to a peaked solitary wave in the form of depression. For example, $\zeta(x) = -\exp(-|x|)/10$ is a peaked solitary wave of depression. Such kind of peaked solitary waves have been never reported for surface waves. This once again indicates the novelty of the new solitary waves.

Note that, unlike the traditional linear wave theory, the parameter k of the peaked solitary waves does not denote the wave number, but the decaying rate of the wave elevation as $x \to +\infty$: the larger the value of k, more quickly the wave elevation decays to zero. According to (24), the new peaked solitary wave can propagate very quickly even if the water depth D and the wave height H_w are small, since $\tan(k)/k \to +\infty$ as $k \to \pi/2$.

Traditionally, it is widely believed that solitary waves are always governed by nonlinear differential equations. However, we illustrate here, for the first time, that the solitary waves exist even in the frame of the linear wave equations in finite water depth! Note also that the peaked solitary wave (27) is the same as the peaked solitary wave found by Casamma & Holm [5]. This reveals the origin of the peaked solitary waves of the CH equation (1) that is an approximation of the fully nonlinear wave equations in shallow water. However, the new peaked solitary wave (27) is valid not only in shallow water but also in finite water depth, with the detailed horizontal velocity (31), and thus is more general.

4 Peaked solitary waves by nonlinear equations

As shown in § 3, the new type of peaked solitary surface waves given by the linear wave equations has some unusual characteristics quite different from the traditional periodic and solitary ones. Does the fully nonlinear wave equations (2) to (9) indeed admit such kind of new peaked solitary waves? Does this kind of new peaked solitary waves have the same unusual characteristics as those given by the linear wave equations, if the answer of the above question is positive?

To answer these questions, we consider here the solitary surface waves with a finite wave-amplitude so that the nonlinear terms of the boundary conditions (3) and (4) are not negligible and besides $z = 0$ is not a good approximation of the free surface $\zeta(x)$. In other words, we had to solve the fully nonlinear wave equations (2) to (9) accurately.

Analytically, the fully nonlinear wave equations (2) to (9) are often solved by means of perturbation methods which are based on some assumptions such as small wave height, large wave length and so on. For example, in case of small wave height H_w, Fenton [13] gave a ninth-order approximation of dispersive solitary waves with a smooth crest by means of perturbation methods, whose phase speed increases as the wave height H_w enlarges.

In this paper, an analytic technique, namely the homotopy analysis method (HAM) proposed by Liao [15–20], is applied to solve the fully nonlinear wave equations (2) to (9). Unlike perturbation techniques, the HAM does not need any assumptions of small physical parameters, since it is based on the homotopy, a basic concept in topology. Besides, the HAM provides us great freedom to choose base functions for solutions of considered nonlinear equations. Especially, by means of the so-called "convergence-control parameter" that has no physical meanings, the HAM provides us a convenient way to guarantee the convergence of approximation series: in essence, it is the so-called "convergence-control parameter" that differs the HAM from all other analytic approximation techniques, as pointed out currently by Liao [20]. Therefore, the HAM is valid for highly nonlinear problems, as shown by many successfully applications in fluid mechanics, applied mathematics, physics and finance. For example, by means of the HAM, Liao [16] gained, for the first time, convergent series solution for Blasius and Falker-Skan boundary-layer flows, which are uniformly valid in the whole field of flow. Note that the traditional power series given by Blasius [21] is valid only in the near field, and thus had to be marched with another asymptotic approximation of flow in far field. Besides, using the HAM as a tool, the exact Navier-Stokes equations were solved by Turkyilmazoglu [22] for a compressible boundary layer flow due to a porous rotating disk, and by Xu et al [23] for viscous flows in a porous channel with orthogonally moving walls. Furthermore, the limit cycle of Duffing - van der Pol equation was solved by Turkyilmazoglu [24], and the two coupled Van der Pol equations were solved by Li et al [25]. Especially, by means of the HAM, some new boundary layer flows have been found by Liao [26] and by Liao & Magyari [27], which have been neglected by other analytic and even numerical techniques. In addition, the HAM has been also successfully applied to solve some nonlinear PDEs with moving boundary conditions, such as those about American put option. For example, Zhu [28] successfully applied the HAM to give a series approximation of the American put option, which gives optimal exercise boundary valid for a couple of years, while perturbative and/or asymptotic formulas are accurate only in a few days or weeks. All of these illustrate the potential and validity of the HAM for highly nonlinear problems.

It should be emphasized that the HAM has been successfully applied to solve the fully nonlinear wave equations. Using the traditional base functions (13), Liao & Cheung [29] applied the HAM to solve the periodic progressive surface waves in deep water and obtained convergent solutions for waves with high amplitude even close to the limiting case. Their analytic results agree quite well with those given by Schwartz [30] and Longuet-Higgins [31]. Besides, using the same traditional base functions (13), Tao et al [32] successfully applied the HAM to solve the fully nonlinear wave equations (2) - (9) for periodic progressive waves in finite water depth, and their analytic results agree well not only with the analytic ones given by Cokelet [10] and Fenton [33] but also with the experimental ones reported by Mehaute et al [34]. All of these demonstrate the validity of the HAM for the fully nonlinear wave equations (2) - (9).

4.1 Analytic approach based on the homotopy analysis

As shown below, the fully nonlinear wave equations (2) to (9) can be solved by means of the HAM and the new base functions (22) in a similar way as those by Liao & Cheung [29] and Tao et al [32], although they used the traditional base functions (13).

Due to the symmetry (10), we need consider the case $x \geq 0$ only. Since the solitary wave elevation $\zeta(x)$ decays to zero as $x \to +\infty$, it is natural and straightforward that $\phi(x,z)$ should be expressed in the form

$$\phi(x,z) \quad = \quad \sum_{n=1}^{+\infty} a_n \ \cos[nk(z+1)] \ \exp(-nkx), \quad x \geq 0, \ k > 0, \tag{32}$$

which automatically satisfies the governing equation (2), the bottom boundary condition (5) and the bounded condition (9), where $k > 0$ is a scale parameter and a_n is a coefficient to be determined. We search for the solitary surface waves in the form

$$\zeta(x) = \sum_{n=1}^{+\infty} b_n \exp\left(-nkx\right), \qquad x \geq 0, \ k > 0, \tag{33}$$

where b_n is a constant coefficient to be determined. The above expressions (32) and (33) provide us the so-called solution-expression of $\phi(x,z)$ and $\zeta(x)$, respectively, which play important role in the frame of the HAM, as shown below.

Let $\phi_0(x,z), \zeta_0(x)$ denote the initial guess of the velocity potential $\phi(x,z)$ and the wave elevation $\zeta(x)$ in $x \geq 0$, respectively. To apply the HAM, we should first of all construct two continuous variations from the initial guess $\phi_0(x,z), \zeta_0(x)$ to the exact solution $\phi(x,z), \zeta(x)$, respectively. This can be easily done by means of the homotopy, a basic concept in topology, as shown below.

First, according to the solution expression (32), we choose

$$\phi_0(x,z) = A_0 \cos[k(z+1)] \, e^{-kx}, \quad x \geq 0, \ k > 0, \tag{34}$$

as the initial guess of the velocity potential $\phi(x,z)$, where A_0 is a constant to be determined later. Note that, different from Liao & Cheung [29] and Tao et al [32], the new base function (22) is used here. Note also that $\phi_0(x,z)$ automatically satisfies the Laplace equation (2), the bottom condition (5) and the bounded condition (9). Besides, following Liao & Cheung [29] and Tao et al [32], we choose

$$\zeta_0(x) = 0 \tag{35}$$

as the initial guess of wave elevation $\zeta(x)$.

Secondly, according to (3), we define a nonlinear operator

$$\mathcal{N}\phi = \alpha^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial z} - \alpha \frac{\partial}{\partial x}\left(\nabla\phi \cdot \nabla\phi\right) + \nabla\phi \cdot \nabla\left(\frac{1}{2}\nabla\phi \cdot \nabla\phi\right). \tag{36}$$

Let $q \in [0,1]$ denote an embedding parameter, c_ϕ and c_η be two non-zero auxiliary parameters without physical meanings, called the convergence-control parameters, and $\mathcal{L}$ denote an auxiliary linear operator, respectively. Following Liao & Cheung [29] and Tao et al [32], we construct the so-called zeroth-order deformation equation

$$\nabla^2 \Phi(x,z;q) = 0, \qquad z \le \eta(x;q), \tag{37}$$

subject to the boundary conditions on the unknown free surface $z = \eta(x;q)$:

$$(1-q)\mathcal{L}\left[\Phi(x,z;q) - \phi_0(x,z)\right] = c_\phi \, q \, \mathcal{N}\left[\Phi(x,z;q)\right], \tag{38}$$

$$(1-q)\eta(x;q) = c_\eta \, q \, \left[\eta(x;q) - \alpha\frac{\partial \Phi}{\partial x} + \frac{1}{2}\nabla\Phi \cdot \nabla\Phi\right], \tag{39}$$

and the boundary condition at the bottom

$$\frac{\partial \Phi}{\partial z} = 0, \qquad z = -1. \tag{40}$$

If wave height H_w is given, there exists the additional condition:

$$\eta(0;q) = H_w. \tag{41}$$

Note that $\Phi(x,z;q)$ and $\eta(x;q)$ depend not only on the original physical variables x, z but also on the embedding parameter $q \in [0,1]$ and the two convergence-control parameter c_ϕ, c_η that have no physical meanings at all. It should be emphasized that we have great freedom to choose the values of the convergence-control parameters c_ϕ and c_η. Following Liao & Cheung [29] and Tao et al [32], we choose the auxiliary linear operator

$$\mathcal{L}\phi = \alpha^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial z}, \tag{42}$$

which has the property $\mathcal{L}[0] = 0$. Note that $\mathcal{L}$ is exactly the linear part of the nonlinear operator $\mathcal{N}$ defined by (36). In this way, the zeroth-order deformation equations (37) to (41) are well defined.

When $q = 0$, we have from (39) that

$$\eta(x;0) = 0 = \zeta_0(x), \tag{43}$$

and then the corresponding zeroth-order deformation equations become

$$\nabla^2 \Phi(x,z;0) = 0, \qquad z \le 0, \ \ 0 \le x < +\infty, \tag{44}$$

subject to the boundary conditions on the known free surface

$$\mathcal{L}\left[\Phi(x,z;0) - \phi_0(x,z)\right] = 0, \qquad \text{when } z = 0, \ \ 0 \le x < +\infty, \tag{45}$$

and the boundary condition at the bottom

$$\frac{\partial \Phi(x,z;0)}{\partial z} = 0, \qquad z = -1, \ \ 0 \le x < +\infty. \tag{46}$$

Since the auxiliary linear operator $\mathcal{L}$ has the property $\mathcal{L}[0] = 0$ and besides the initial guess $\phi_0(x,z)$ defined by (34) satisfies the Laplace equation (2) and the bottom condition (5), it is straightforward that

$$\Phi(x,z;0) = \phi_0(x,z). \tag{47}$$

When $q = 1$, since $c_\phi \neq 0$ and $c_\eta \neq 0$, the zeroth-order deformation equations (37) to (41) are equivalent to the original fully nonlinear wave equations (2) to (8), respectively, so that we have the relationship

$$\Phi(x,z;1) = \phi(x,z), \quad \eta(x;1) = \zeta(x). \tag{48}$$

Thus, as the embedding parameter q increases from 0 to 1, $\Phi(x,z;q)$ and $\eta(x;q)$ indeed vary continuously from the initial guess $\phi_0(x,z), \zeta_0(x)$ to the exact solution $\phi(x,z), \zeta(x)$ of the fully nonlinear wave equations (2) to (8), respectively. Therefore, the zeroth-order deformation equations (37) to (41) truly construct such a kind of continuous variation that provides a base of our analytic approach, as shown below.

Since both of $\Phi(x,z;q)$ and $\eta(x;q)$ are dependent upon the embedding parameter $q \in [0,1]$, we can expand them in Maclaurin series with respect to q to gain the so-called homotopy-Maclaurin series

$$\Phi(x,z;q) = \phi_0(x,z) + \sum_{m=1}^{+\infty} \phi_m(x,z)\, q^m, \tag{49}$$

$$\eta(x;q) = \sum_{m=1}^{+\infty} \zeta_m(x)\, q^m, \tag{50}$$

where

$$\phi_m(x,z) = \frac{1}{m!} \left. \frac{\partial^m \Phi(x,z;q)}{\partial q^m} \right|_{q=0}, \quad \zeta_m(x) = \frac{1}{m!} \left. \frac{\partial^m \eta(x;q)}{\partial q^m} \right|_{q=0}$$

and the relationship (43) and (47) are used. However, it is well known that a Maclaurin series often has a finite radius of convergence. Fortunately, both of $\Phi(x,z;q)$ and $\eta(x;q)$ contain the two convergence-control parameters c_ϕ and c_η, which have great influence on the convergence of the Maclaurin series of $\Phi(x,z;q)$ and $\eta(x;q)$, as shown by Liao & Cheung [29] and Tao et al [32]. Here, it should be emphasized once again that we have great freedom to choose the values of c_ϕ and c_η. Thus, if the convergence-control parameters c_ϕ, c_η are properly chosen so that the above homotopy-Maclaurin series are convergent at $q = 1$, we have the homotopy-series solution

$$\phi(x,z) = \phi_0(x,z) + \sum_{m=1}^{+\infty} \phi_m(x,z), \tag{51}$$

$$\zeta(x) = \sum_{m=1}^{+\infty} \zeta_m(x). \tag{52}$$

The equations for the unknown $\phi_m(x,z)$ and $\zeta_m(x)$ can be derived directly from the zeroth-order deformation equations. Like Liao & Cheung [29] and Tao et al [32], substituting the series (49) and (50) into the zeroth-order deformation equations (37) to (41), then equating the like-power of q, we gain

$$\zeta_m(x) = \left\{ c_\eta \, \Delta^\eta_{m-1} + \chi_m \, \zeta_{m-1} \right\}\Big|_{z=0}, \quad m \geq 1, \;\; 0 \leq x < +\infty, \tag{53}$$

where

$$\Delta^\eta_m = \zeta_m - \alpha \, \bar{\phi}_{m,1} + \Gamma_{m,0}, \tag{54}$$

and the mth-order deformation equation

$$\nabla^2 \phi_m(x,z) = 0, \quad m \geq 1, \;\; z \leq 0, \;\; 0 \leq x < +\infty, \tag{55}$$

subject to the boundary condition on the known free surface $z = 0$:

$$\bar{\mathcal{L}}\left(\phi_m\right) = \left(\alpha^2 \, \frac{\partial^2 \phi_m}{\partial x^2} + \frac{\partial \phi_m}{\partial z} \right)\Bigg|_{z=0} = R_m(x), \;\; 0 \leq x < +\infty, \tag{56}$$

and the bottom condition

$$\frac{\partial \phi_m}{\partial z} = 0, \qquad z = -1, \;\; 0 \leq x < +\infty, \tag{57}$$

where

$$R_m(x) = \left\{ c_\phi \, \Delta^\phi_{m-1} + \chi_m \, S_{m-1} - \bar{S}_m \right\}\Big|_{z=0}, \;\; 0 \leq x < +\infty, \tag{58}$$

$$\chi_n = \begin{cases} 0, & \text{when } n \leq 1, \\ 1, & \text{when } n > 1. \end{cases} \tag{59}$$

The detailed derivations of $\Delta^\eta_{m-1}, \Delta^\phi_{m-1}, S_{m-1}, \bar{S}_m$ with all related formulas are given explicitly in the Appendix. Note that, unlike Liao & Cheung [29] and Tao et al [32], we explicitly give all formulas in details so that high-order approximations can be gained more efficiently.

Note that the dimensionless phase speed α of the new peaked solitary waves is unknown up to now. According to the linear wave theory mentioned in § 3, the peaked solitary waves exist only when

$$\alpha^2 = \frac{\tan k}{k}, \qquad n\pi < k < n\pi + \frac{\pi}{2}, \tag{60}$$

where $n \geq 0$ is an integer. If the above expression also holds for the fully nonlinear wave equations, the auxiliary linear operator defined by (42) have the property

$$\mathcal{L}\left\{\cos[k(z+1)]e^{-kx}\right\} = 0, \quad x \geq 0,\ k > 0, \tag{61}$$

and the corresponding inverse operator of $\bar{\mathcal{L}}$ defined by (56) has the property

$$\bar{\mathcal{L}}^{-1}\left\{\exp\left(-nkx\right)\right\} = \frac{\cos[nk(z+1)]\exp(-nkx)}{(nk)\left[\alpha^2(nk)\cos(nk) - \sin(nk)\right]}, \quad k > 0,\ n \neq 1,\ x \geq 0, \tag{62}$$

where $n \geq 2$ is an integer. Note that the above expression does not hold when $n = 1$. Fortunately, it is found that $R_m(x)$ indeed does not contain the term $\exp(-kx)$ as long as the phase speed is given by $\alpha^2 = \tan(k)/k$. Mathematically, this is because the nonlinear terms of (36) do not contain the term $\exp(-kx)$ at all, since

$$\exp(-mkx) \times \exp(-nkx) = e^{-(m+n)kx}$$

with $m + n \geq 2$ for any integers $m \geq 1$ and $n \geq 1$. So do the linear terms of (36), since

$$\begin{aligned} & \left.\left\{\left(\alpha^2\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial z}\right)\sum_{n=1}^{+\infty} b_n\ \cos[nk(z+1)]\ \exp(-nkx)\right\}\right|_{z=0} \\ = & \sum_{n=1}^{+\infty}(nk)\left[\alpha^2(nk)\cos(nk) - \sin(nk)\right]b_n\exp(-nkx) \\ = & \sum_{n=2}^{+\infty}(nk)\left[\alpha^2(nk)\cos(nk) - \sin(nk)\right]b_n\exp(-nkx) \end{aligned}$$

does not contain the term $\exp(-kx)$, too. This is the essential reason why the phase speed

$$\alpha = \sqrt{\frac{\tan(k)}{k}} \tag{63}$$

given by the linear wave equations still holds for the fully nonlinear wave equations (2) to (8). Physically speaking, the dimensionless phase speed $c/\sqrt{gD}$ of the new solitary waves has nothing to do with the wave height: this is quite different from the traditional periodic and solitary progressive waves. It indicates that the new peaked solitary waves are non-dispersive. We will illustrate this point later.

Keeping (63) in mind and using the property (62) of the inverse operator $\bar{\mathcal{L}}^{-1}$, it is straightforward to gain the common solution of the high-order deformation equation (55) to (57):

$$\phi_m(x,z) = \phi_m^*(x,z) + A_m \cos\left[k\,(1+z)\right] e^{-kx}, \quad x \geq 0, \tag{64}$$

where $\phi_m^*(x,z) = \bar{\mathcal{L}}^{-1}[R_m(x)]$ is a special solution, and the coefficient A_m is determined by the given wave height

$$\sum_{n=1}^{m+1} \zeta_n(0) = H_w. \tag{65}$$

This is mainly because, according to (53), $\zeta_{m+1}(x)$ is dependent upon $\phi_m(x,z)$ that contains the unknown parameter A_m, where $m \geq 1$. Note that, according to (62), $\phi_m(x,z)$ is in the form of (32) and thus automatically satisfies the Laplace equation (2), the bottom condition (5) and the bounded condition (9). Thus, using the explicit formulas given in the Appendix, it is computationally efficient to gain high-order analytic approximations successively, especially by means of computer algebra system such as Mathematica and Maple, since our approach needs only algebra computations.

For example, using the initial guess (34) and (53), we directly have

$$\begin{aligned} \zeta_1(x) &= -c_\eta \left(\alpha \frac{\partial \phi_0}{\partial x} - \frac{1}{2} \nabla\phi_0 \cdot \nabla\phi_0 \right)\Big|_{z=0} \\ &= c_\eta A_0 k \left[\alpha \cos(k) e^{-kx} + \frac{A_0 k}{2} e^{-2kx} \right], \quad x \geq 0. \end{aligned} \tag{66}$$

Thus, at the first-order of approximation, we have an algebraic equation for the given wave height

$$H_w = c_\eta k A_0 \left(\alpha \cos k + \frac{1}{2} k A_0 \right),$$

which gives two different solutions

$$A_0 = k^{-1} \left[-\alpha \cos k \pm \sqrt{\alpha^2 \cos^2(k) + 2H_w/c_\eta} \right]. \tag{67}$$

We simply choice

$$A_0 = -k^{-1} \left[\alpha \cos k - \sqrt{\alpha^2 \cos^2(k) + 2H_w/c_\eta} \right] \tag{68}$$

to calculate A_0 for a given H_w, since it has a smaller absolute value.

Furthermore, using the initial guess (34), we have

$$\Delta_0^\phi = k A_0 \left(\alpha^2 k \cos k - \sin k \right) e^{-kx} + 2\alpha k^3 A_0^2 e^{-2kx} + k^4 A_0^3 \cos(k) e^{-3kx}.$$

Using the phase speed (63), the term $\exp(-kx)$ of the above expression disappears, say,

$$\Delta_0^\phi = 2\alpha k^3 A_0^2 e^{-2kx} + k^4 A_0^3 \cos(k) e^{-3kx}, \quad x \geq 0.$$

Thus, the first-order deformation equation reads

$$\nabla^2\phi_1(x,z)=0, \qquad z\leq 0,\ \ 0\leq x<+\infty, \tag{69}$$

subject to the boundary condition on the known free surface $z=0$:

$$\bar{\mathcal{L}}\,(\phi_m)=\left(\alpha^2\,\frac{\partial^2\phi_m}{\partial x^2}+\frac{\partial\phi_m}{\partial z}\right)\bigg|_{z=0}=c_\phi\left[2\alpha k^3A_0^2e^{-2kx}+k^4A_0^3\cos(k)e^{-3kx}\right], \tag{70}$$

and the bottom condition

$$\frac{\partial\phi_1}{\partial z}=0, \qquad z=-1,\ \ 0\leq x<+\infty. \tag{71}$$

Using the property of the inverse operator (62), it is easy to gain the common solution

$$\begin{aligned}\phi_1(x,z) &= c_\phi\left\{\frac{\alpha k^2A_0^2\cos[2k(z+1)]e^{-2kx}}{2\alpha^2k\cos(2k)-\sin(2k)}+\frac{k^3A_0^3\cos k\cos[3k(z+1)]e^{-3kx}}{3[3\alpha^2k\cos(3k)-\sin(3k)]}\right\}\\ &+ A_1\cos[k(z+1)]e^{-kx}, \quad 0\leq x<+\infty,\end{aligned} \tag{72}$$

where A_1 is an unknown constant to be determined. Similarly, using (53), we gain $\zeta_2(x)$, which contains the unknown constant A_1. Then, for the given wave height H_w, we have a linear algebraic equation

$$H_w=\zeta_1(0)+\zeta_2(0),$$

which determines A_1. In this way, $\phi_1(x,z)$ is completely determined. Similarly, we further gain $\phi_2(x,z)$, $\zeta_3(x)$, and so on. Finally, using the symmetry (10), it is easy to gain the corresponding wave elevation $\zeta(x)$ and the velocities $u(x,z), v(x,z)$ in the whole domain $-\infty<x<+\infty$.

Our computations confirm that, for all $m\geq 0$, $R_m(x)$ in (56) indeed does not contain the term $\exp(-kx)$ at all. Thus, the fully nonlinear wave equations (2) - (9) indeed give the *same* dimensionless phase speed $\alpha=\sqrt{\tan k/k}$ as that by the linear ones. Therefore, the phase speed of the new peaked solitary wave has nothing to do with the wave height H_w, say, the peaked solitary waves are non-dispersive! This is indeed completely different from the traditional periodic and solitary waves with smooth crest. This unusual characteristic clearly demonstrates the novelty of the new peaked solitary surface waves. We will confirm and discuss this interesting characteristic of the new peaked solitary waves later.

Note that our HAM-based analytic approach mentioned above is rather similar to those by Liao & Cheung [29] and Tao et al [32] for the traditional progressive waves in deep and finite water, except that we use here the new base function (22) and the explicit formulas given in Appendix A, and besides regard the dimensionless phase speed α as a constant.

Finally, we should emphasize that, unlike perturbation methods, our HAM-based analytic approach does not need any assumptions about small/large physical parameters. More importantly, both of $\phi(x,z)$ and $\zeta(x)$ contain the two convergence-control parameters c_ϕ and c_η, which provide us a convenient way to guarantee the convergence of approximation series, as illustrated below.

Order of approx.	$U(-1)$	$U(-0.5)$	$U(-0.25)$	$U(H_w)$	$\zeta'(0_+)$
1	0.07222	0.06570	0.05762	0.04289	-0.04690
3	0.06833	0.06236	0.05466	0.04205	-0.04859
5	0.06796	0.06219	0.05489	**0.04213**	**-0.04823**
10	**0.06799**	**0.06221**	**0.05490**	**0.04213**	**-0.04823**
15	**0.06799**	**0.06221**	**0.05490**	**0.04213**	**-0.04823**
20	**0.06799**	**0.06221**	**0.05490**	**0.04213**	**-0.04823**
25	**0.06799**	**0.06221**	**0.05490**	**0.04213**	**-0.04823**

Table 1: Analytic approximations of $U(z) = u(0, z)$ and $\zeta'(0_+)$ in case of $H_w = 1/20$ and $k = 1$ by means of $c_\phi = -1$ and $c_\eta = -1$.

Order of approx.	$U(-1)$	$U(-0.5)$	$U(-0.25)$	$U(H_w)$	$\zeta'(0_+)$
1	-0.07047	-0.06377	-0.05101	-0.03840	0.05248
3	-0.08133	-0.06737	-0.05218	-0.03779	0.05220
5	-0.08140	-0.06749	-0.05218	**-0.03772**	0.05192
10	**-0.08145**	**-0.06750**	**-0.05218**	**-0.03772**	**0.05183**
20	**-0.08145**	**-0.06750**	**-0.05218**	**-0.03772**	**0.05183**
25	**-0.08145**	**-0.06750**	**-0.05218**	**-0.03772**	**0.05183**

Table 2: Analytic approximations of $U(z) = u(0, z)$ and $\zeta'(0_+)$ in case of $H_w = -1/20$ and $k = 1$ by means of $c_\phi = -1$ and $c_\eta = -1$.

4.2 Convergence of series solution

Note that, unlike perturbation results, $\phi_m(x, z)$ and $\zeta_m(x)$ gained in above-mentioned analytic approach contain two convergence-control parameters c_ϕ and c_η, which provide us a convenient way to guarantee the convergence of the series (51) and (52), as shown below. Obviously, the convergence rate of the series (51) and (52) is greatly influenced by c_ϕ and c_η. As pointed out by Liao & Cheung [29] and Tao et al [32], one can choose $c_\phi = -1$ and $c_\eta = -1$ for weakly nonlinear waves.

First, let us consider the case of $k = 1$ and $H_w = 1/20$, with the corresponding dimensionless phase velocity $\alpha = c/\sqrt{gD} = \sqrt{\tan(1)} = 1.24796$. Since the wave height is only 5% of the water depth D, the nonlinearity is weak. Thus, as suggested by Liao & Cheung [29] and Tao et al [32], we choose $c_\phi = -1$ and $c_\eta = -1$ for such a kind of weakly nonlinear wave problem. It is found that, the corresponding series of analytic approximation indeed converges quickly, as shown for examples in Table 1 for $\zeta'(0_+)$ and the horizontal velocity $U(z) = u(0, z)$ beneath the wave crest at $z = -1, z = -1/2, z = -1/4$ and $z = H_w$, respectively, where 0_+ denotes $x \to 0$ from the right along the x axis. It is found that the velocity potential $\phi(x, z)$ converges

Order of approx.	$U(-1)$	$U(-0.5)$	$U(-0.25)$	$U(H_w)$	$\zeta'(0_+)$
1	0.1696	0.1543	0.1347	0.09362	-0.08561
3	0.1246	0.1180	0.1090	0.08837	-0.09332
5	0.1270	0.1196	0.1098	0.08813	-0.09142
10	0.1254	0.1183	0.1090	0.08788	-0.09285
15	**0.1254**	**0.1183**	**0.1090**	**0.08789**	**-0.09299**
20	**0.1254**	**0.1183**	**0.1090**	**0.08789**	**-0.09299**
25	**0.1254**	**0.1183**	**0.1090**	**0.08789**	**-0.09299**

Table 3: Analytic approximations of $U(z) = u(0, z)$ and $\zeta'(0_+)$ in case of $H_w = 1/10$ and $k = 1$ by means of $c_\phi = -1/2$ and $c_\eta = -1$.

Order of approx.	$U(-1)$	$U(-0.5)$	$U(-0.25)$	$U(H_w)$	$\zeta'(0_+)$
1	-0.1418	-0.1191	-0.09328	-0.07474	0.1091
3	-0.1704	-0.1343	-0.09684	-0.07211	0.1130
5	-0.1768	-0.1368	-0.09664	-0.07070	0.1107
10	-0.1801	-0.1379	-0.09650	-0.07016	0.1079
15	-0.1805	-0.1380	-0.09649	-0.07013	0.1075
20	**-0.1806**	**-0.1380**	**-0.09648**	**-0.07012**	**0.1075**
25	**-0.1806**	**-0.1380**	**-0.09648**	**-0.07012**	**0.1075**

Table 4: Analytic approximations of $U(z) = u(0, z)$ and $\zeta'(0_+)$ in case of $H_w = -1/10$ and $k = 1$ by means of $c_\phi = -3/4$ and $c_\eta = -1$.

quickly in the whole domain $x \in [0, +\infty)$ and $z \leq \zeta(x)$, as shown for examples in Fig. 2 for the corresponding horizontal velocity profile $U(z) = u(0, z)$ beneath the crest. This confirms that the new solitary waves is indeed a solution of the fully nonlinear wave equations (2) - (9)!

Secondly, let us consider the case with $k = 1$ and $H_w = -1/20$, with the same dimensionless phase velocity $\alpha = c/\sqrt{gD} = 1.24796$. It is found that the corresponding series of analytic approximations given by $c_\phi = -1$ and $c_\eta = -1$ converge quickly in the whole domain $x \geq 0$, as shown for examples in Table 2 for $\zeta'(0_+)$ and the horizontal velocity $U(z) = u(0, z)$ beneath the crest at $z = -1, -0.5, -0.25$ and $z = H_w$, respectively. Besides, the corresponding velocity potential $\phi(x, z)$ converges quickly in the whole domain $x \in [0, +\infty)$ and $z \leq \zeta(x)$, as shown for examples in Fig. 3 for the horizontal velocity profile $U(z) = u(0, z)$ beneath the crest. This confirms that the new peaked solitary wave in the form of depression is truly a solution of the fully nonlinear wave equations (2) - (9)!

Similarly, in case of $k = 1/2$ and $H_w = \pm 1/20$, with the corresponding dimen-

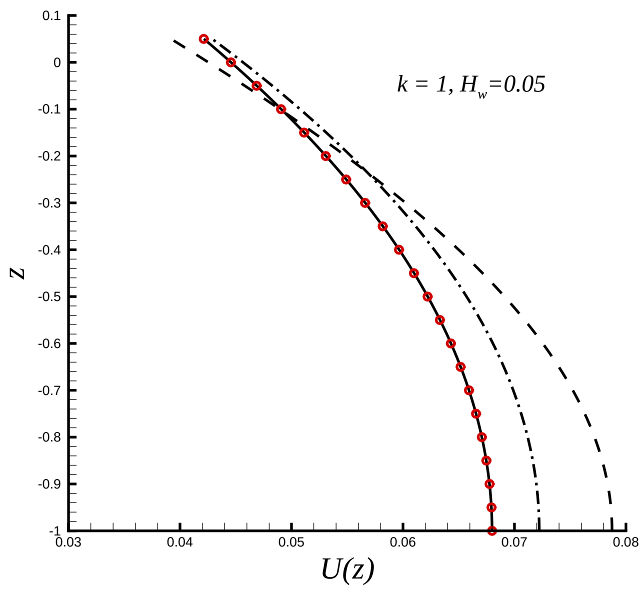

Figure 2: Analytic approximations of the dimensionless horizontal velocity profile $U(z) = u(0, z)$ beneath the crest in case of $k = 1$ and $H_w = 0.05$ given by $c_\phi = -1$ and $c_\eta = -1$. Dashed-line: zeroth-order of approx.; Dash-dotted line: 1st-order of approx.; Solid line: 4th-order of approx.; Symbols: 25th-order of approximation.

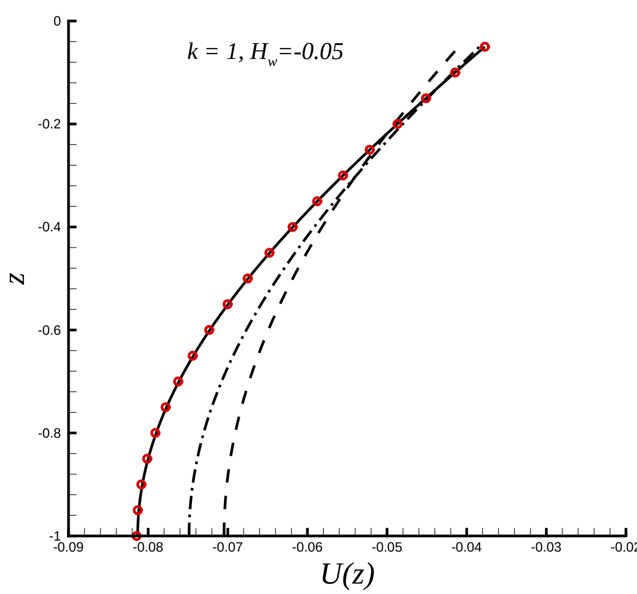

Figure 3: Analytic approximations of the dimensionless horizontal velocity profile $U(z) = u(0, z)$ beneath the crest in case of $k = 1$ and $H_w = -0.05$ given by $c_\phi = -1$ and $c_\eta = -1$. Dashed-line: zeroth-order of approx.; Dash-dotted line: 1st-order of approx.; Solid line: 4th-order of approx.; Symbols: 25th-order of approximation.

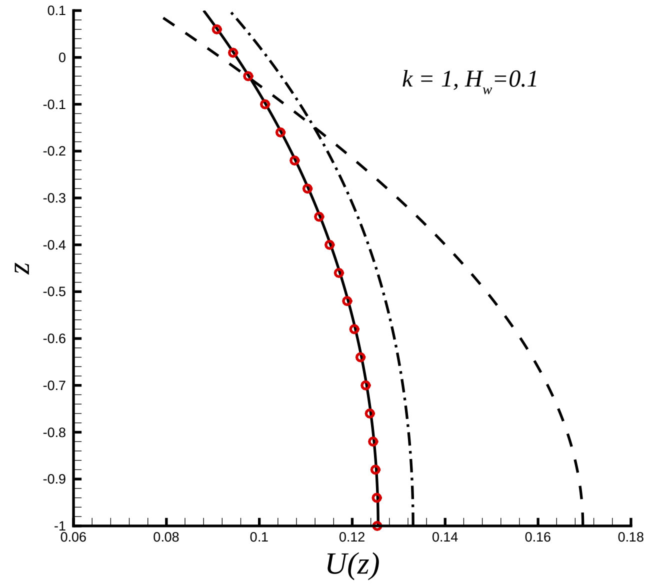

Figure 4: Analytic approximations of the dimensionless horizontal velocity profile $U(z) = u(0, z)$ beneath the crest in case of $k = 1$ and $H_w = 0.1$ given by $c_\phi = -1/2$ and $c_\eta = -1$. Dashed-line: zeroth-order of approx.; Dash-dotted line: 2nd-order of approx.; Solid line: 6th-order of approx.; Symbols: 25th-order of approximation.

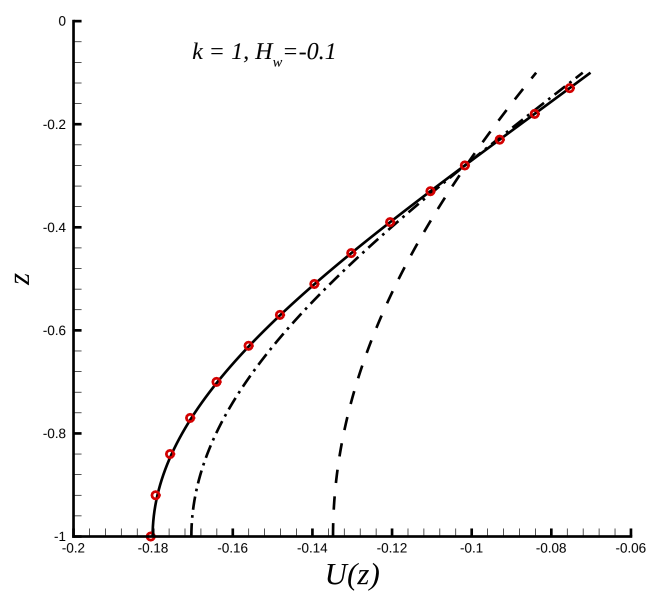

Figure 5: Analytic approximations of the dimensionless horizontal velocity profile $U(z) = u(0, z)$ beneath the crest in case of $k = 1$ and $H_w = -0.1$ given by $c_\phi = -3/4$ and $c_\eta = -1$. Dashed-line: zeroth-order of approx.; Dash-dotted line: 1st-order of approx.; Solid line: 10th-order of approx.; Symbols: 25th-order of approximation.

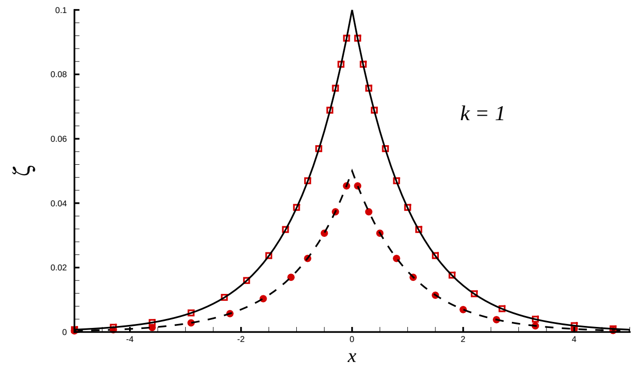

Figure 6: Analytic approximations of elevation of the new solitary waves when $k = 1$ (corresponding to $c/\sqrt{gD} = 1.24796$). Solid line: 5th-order approximation when $H_w = 0.1$ given by $c_\phi = -0.5$ and $c_\eta = -1$; Filled circles: 25th-order approximation when $H_w = 0.1$ given by $c_\phi = -0.5$ and $c_\eta = -1$; Dashed line: 5th-order approximation when $H_w = 0.05$ given by $c_\phi = -1$ and $c_\eta = -1$; Open circles: 25th-order approximation when $H_w = 0.05$ given by $c_\phi = -1$ and $c_\eta = -1$.

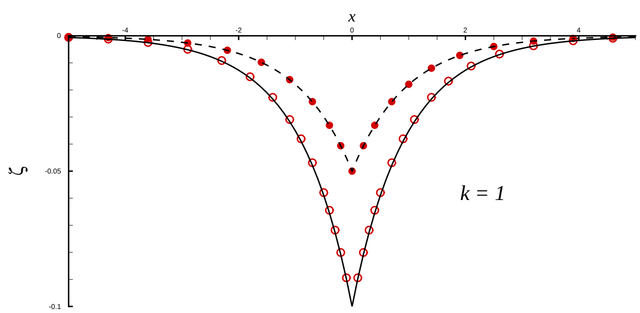

Figure 7: Analytic approximations of elevation of the new solitary waves when $k = 1$ (corresponding to $c/\sqrt{gD} = 1.24796$). Solid line: 5th-order approximation when $H_w = -0.1$ given by $c_\phi = -0.75$ and $c_\eta = -1$; Filled circles: 25th-order approximation when $H_w = -0.1$ given by $c_\phi = -0.75$ and $c_\eta = -1$; Dashed line: 5th-order approximation when $H_w = -0.05$ given by $c_\phi = -1$ and $c_\eta = -1$; Open circles: 25th-order approximation when $H_w = -0.05$ given by $c_\phi = -1$ and $c_\eta = -1$.

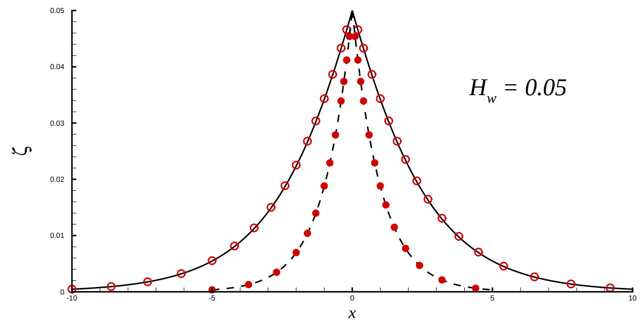

Figure 8: Analytic approximations of $\zeta(x)$ of the new soliton waves when $H_w = 0.05$ by means of $c_\phi = -1$ and $c_\eta = -1$. Solid line: 5th-order approximation when $k = 1/2$ (corresponding to $c/\sqrt{gD} = 1.04528$); Filled circles: 25th-order approximation when $k = 1/2$; Dashed line: 5th-order approximation when $k = 1$ (corresponding to $c/\sqrt{gD} = 1.24796$); Open circles: 25th-order approximation when $k = 1$.

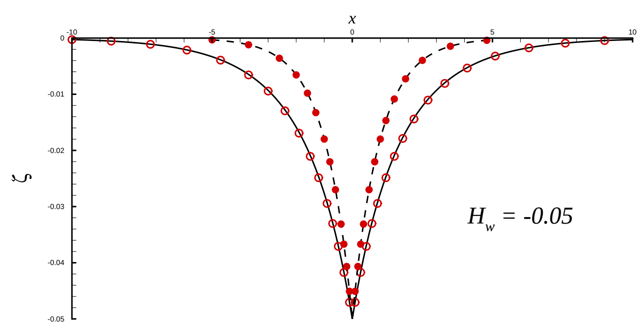

Figure 9: Analytic approximations of $\zeta(x)$ of the new soliton waves when $H_w = -0.05$ by means of $c_\phi = -1$ and $c_\eta = -1$. Solid line: 5th-order approximation when $k = 1/2$ (corresponding to $c/\sqrt{gD} = 1.04528$); Filled circles: 25th-order approximation when $k = 1/2$; Dashed line: 5th-order approximation when $k = 1$ (corresponding to $c/\sqrt{gD} = 1.24796$; Open circles: 25th-order approximation when $k = 1$.

sionless phase speed $\alpha = c/\sqrt{gD} = 1.04528$, we gain convergent series of analytic approximations of $\phi(x,z)$ and $\zeta(x)$ in the whole domain $x \in [0,+\infty)$ and $z \leq \zeta(x)$ by means of $c_\phi = -1$ and $c_\eta = -1$, respectively.

Furthermore, let us consider the case of $k = 1$ and $H_w = 0.1$, with the corresponding dimensionless phase velocity $\alpha = c/\sqrt{gD} = 1.24796$. Since the wave weight increases to 10% of the water depth, the nonlinearity becomes stronger. As suggested by Liao & Cheung [29] and Tao et al [32], we should choose convergence-control parameters c_ϕ and c_η with smaller absolute values for the higher nonlinearity. It is found that the series of analytic approximations given by $c_\phi = -1/2$ and $c_\eta = -1$ converges quickly, as shown in Table 3 and Fig. 4 for the horizontal velocity profile $U(z) = u(0,z)$ beneath the wave crest. Similarly, in case of $k = 1$ and $H_w = -0.1$, we gain convergent series of analytic approximation by means of $c_\phi = -3/4$ and $c_\eta = -1$, as shown in Table 4 and Fig. 5. This illustrates that the two convergence-control parameters c_ϕ and c_η indeed provide us a convenient way to guarantee the convergence of approximation series. Note that the absolute value of the horizontal velocity at bottom in case of $H_w = -0.1$ is 44% larger than that in case of $H_w = 0.1$. So, there does not exist symmetry between these two wave elevations for $H_w = 0.1$ and $H_w = -0.1$, respectively.

It should be emphasized that, in case of $k = 1$, we gain convergent series solutions of the new peaked solitary waves with the *same* phase speed but *different* positive and negative values of H_w such as $H_w = \pm 0.05$ and $H_w = \pm 0.1$. This confirms that the phase speed of the new peaked solitary waves indeed does not depend upon the wave height, say, the peaked solitary waves are non-dispersive. This is an unusual characteristic of the new peaked solitary waves.

Finally, using the symmetry (10), it is straightforward to gain the wave elevation in the whole domain $-\infty < x < +\infty$. As shown in Figs. 6 to 9, the wave elevation $\zeta(x)$ also converges quickly in all of above-mentioned cases. In case of $k = 1$, the wave elevations when $H_w = \pm 0.1$ are compared with those with $H_w = \pm 0.05$, as shown in Figs. 6 and 7. It is found that, for the same value of k, the larger the value of $|H_w|$, the faster $\zeta(x)$ decays to zero. Note also that the wave elevation $\zeta(x)$ with larger k decays to 0 more quickly, as shown in Figs. 8 and 9. In other words, the larger the value of k, the faster $\zeta(x) \to 0$. This provides us a physical meaning of the parameter k. For this reason, we call k the decaying-rate parameter.

Note that our HAM-based analytic approach contains two convergence-control parameters c_ϕ and c_η. As illustrated above, we can gain convergent series of analytic approximations by choosing proper values of c_ϕ and c_η, which indeed provide us a convenient way to guarantee the convergence of approximation series.

Note that, as proved in general by Liao [17, 20], each series solution given by the HAM satisfies its original equations as long as it is convergent. So, all of these convergent series of $\phi(x,z)$ and $\zeta(x)$ are solutions of the fully nonlinear wave equations (2) - (9), as further confirmed below.

4.3 Validation check of analytic approximations

Note that the velocity potential $\phi(x,z)$ is expressed in the form (32), which *automatically* satisfies the Laplace equation (2), the bottom condition (5), and the bounded condition (9). Thus, it is only necessary for us to check the two nonlinear boundary conditions (3) and (4) which are satisfied on the unknown wave elevation $\zeta(x)$.

To check the validation of our analytic approximations, we define the averaged residual squares of the two free surface boundary conditions

$$\mathcal{E}_m^{\phi}(c_\phi, c_\eta) = \frac{1}{(1+M)} \sum_{n=0}^{M} \left(\mathcal{N}\left[\check{\phi}(x,z)\right]\right)^2 \Big|_{x=x_n, z=\zeta(x_n)}, \quad (73)$$

$$\mathcal{E}_m^{\zeta}(c_\phi, c_\eta) = \frac{1}{(1+M)} \sum_{n=0}^{M} \left[\check{\zeta}(x) - \alpha \frac{\partial \check{\phi}}{\partial x} + \frac{1}{2}\nabla\check{\phi}\cdot\nabla\check{\phi}\right]^2 \Bigg|_{x=x_n, z=\zeta(x_n)}, \quad (74)$$

where

$$\check{\phi}(x,z) = \sum_{n=0}^{m} \phi_n(x,z), \quad \check{\zeta} = \sum_{n=1}^{m} \zeta_n(x)$$

are the mth-order approximation of $\phi(x,z)$ and $\zeta(x)$, respectively, and

$$x_n = n\left(\frac{x_R}{M}\right), \quad 0 \le i \le M,$$

with large enough x_R and M. For all results given below, we choose $x_R = 10$ and $M = 100$, if not mentioned. Since the potential velocity $\phi(x,z)$ and the wave elevation $\zeta(x)$ decay exponentially, $x_R = 10$ is large enough.

In case of $k = 1$ and $H_w = \pm 0.05$, the averaged residual squares $\mathcal{E}_m^{\phi}$ and $\mathcal{E}_m^{\zeta}$ of the corresponding analytic approximations obtained by $c_\phi = -1$ and $c_\eta = -1$ decay quickly to the level 10^{-25} as the order of approximation increases to 25, as shown in Table 5. In other words, our 25th-order approximation of $\phi(x,z)$ and $\zeta(x)$ satisfies the Laplace equation (2), the bottom condition (5) and the bounded condition (9) *exactly*, and besides the two nonlinear free surface boundary conditions (3) and (4) *very accurately* (to the level 10^{-25}). In addition, $u(0,z)$ converges quickly so that $U(z) = u(0,z)$ corresponding to the peaked solitary waves is uniquely determined. Therefore, without doubt, our convergent analytic approximation is a very accurate solution of the fully nonlinear wave equations (2) to (9). Similarly, $\mathcal{E}_m^{\phi}$ and $\mathcal{E}_m^{\zeta}$ decays to the level 10^{-13} in case of $k = 1$ and $H_w = \pm 0.1$, and to the level 10^{-18} in case of $k = 1/2$ and $H_w = \pm 0.05$, respectively, as shown in Tables 6 and 7. Theses guarantee that the corresponding analytic approximations of $\phi(x,z)$ and $\zeta(x)$ are indeed quite accurate solutions of the fully nonlinear wave equations (2) to (9), respectively. All of these confirm once again the mathematical proof of Liao [17,20] that each convergent series solution given by the HAM satisfies its original equations in general.

It should be emphasized that, in case of $k = 1$, correspond to the *same* dimensionless phase speed $c/\sqrt{gD} = 1.24796$, we gain very accurate solutions of solitary waves with *different* positive and negative wave heights such as $H_w = \pm 0.05$ and $H_w = \pm 0.1$,

Order of approx.	$H_w = 0.05$		$H_w = -0.05$	
m	$\mathcal{E}_m^\phi$	$\mathcal{E}_m^\zeta$	$\mathcal{E}_m^\phi$	$\mathcal{E}_m^\zeta$
1	6.59×10^{-7}	9.21×10^{-9}	2.25×10^{-6}	9.39×10^{-9}
3	1.40×10^{-8}	1.57×10^{-9}	8.09×10^{-9}	5.90×10^{-10}
5	3.32×10^{-11}	2.90×10^{-12}	7.80×10^{-11}	1.53×10^{-11}
10	9.71×10^{-15}	7.00×10^{-16}	8.96×10^{-16}	2.42×10^{-16}
15	7.33×10^{-19}	1.68×10^{-19}	6.97×10^{-20}	1.05×10^{-20}
20	4.43×10^{-22}	4.83×10^{-23}	2.40×10^{-24}	6.45×10^{-25}
25	2.23×10^{-25}	2.09×10^{-27}	4.56×10^{-28}	3.96×10^{-29}

Table 5: Averaged residual squares of the two nonlinear free boundary conditions (3) and (4) in case of $k = 1$ and $H_w = \pm 0.05$ by means of $c_\phi = -1$ and $c_\eta = -1$, with the corresponding dimensionless phase speed $c/\sqrt{gD} = 1.24796$.

Order of approx.	$H_w = 0.1$ ($c_\phi = -0.5, c_\eta = -1$)		$H_w = -0.1$ ($c_\phi = -0.75, c_\eta = -1$)	
m	$\mathcal{E}_m^\phi$	$\mathcal{E}_m^\zeta$	$\mathcal{E}_m^\phi$	$\mathcal{E}_m^\zeta$
1	1.48×10^{-4}	7.21×10^{-7}	5.89×10^{-5}	3.67×10^{-7}
3	1.63×10^{-7}	8.63×10^{-8}	5.84×10^{-7}	1.21×10^{-7}
5	1.39×10^{-7}	3.96×10^{-10}	6.83×10^{-7}	1.11×10^{-8}
10	5.96×10^{-10}	3.09×10^{-11}	8.63×10^{-9}	2.31×10^{-10}
15	1.30×10^{-12}	3.70×10^{-14}	3.29×10^{-11}	1.95×10^{-12}
20	4.34×10^{-13}	1.88×10^{-15}	1.17×10^{-12}	3.69×10^{-14}
25	2.25×10^{-13}	4.52×10^{-16}	2.51×10^{-14}	8.11×10^{-16}

Table 6: Averaged residual squares of the two nonlinear free boundary conditions (3) and (4) in case of $k = 1$ and $H_w = \pm 0.1$, with the corresponding dimensionless phase speed $c/\sqrt{gD} = 1.24796$.

Order of approx.	$H_w = 0.05$		$H_w = -0.05$	
m	$\mathcal{E}_m^{\phi}$	$\mathcal{E}_m^{\zeta}$	$\mathcal{E}_m^{\phi}$	$\mathcal{E}_m^{\zeta}$
1	2.75×10^{-8}	1.04×10^{-6}	3.74×10^{-7}	9.09×10^{-7}
3	3.76×10^{-10}	4.85×10^{-8}	4.48×10^{-10}	9.61×10^{-9}
5	4.13×10^{-12}	2.00×10^{-9}	4.85×10^{-13}	4.18×10^{-12}
10	1.80×10^{-14}	2.21×10^{-12}	1.08×10^{-16}	3.16×10^{-17}
15	5.23×10^{-15}	1.55×10^{-15}	7.93×10^{-20}	1.06×10^{-19}
20	1.86×10^{-16}	1.50×10^{-16}	1.67×10^{-23}	1.52×10^{-24}
25	2.56×10^{-18}	9.46×10^{-18}	3.87×10^{-29}	2.57×10^{-28}

Table 7: Averaged residual squares of the two nonlinear free boundary conditions (3) and (4) in case of $k = 1/2$ and $H_w = \pm 0.05$ by means of $c_\phi = -1$ and $c_\eta = -1$, with the corresponding dimensionless phase speed $c/\sqrt{gD} = 1.04528$.

respectively. This confirms that the phase speed of the new solitary waves is indeed independent of the wave height H_w.

In fact, one can choose the optimal values of c_ϕ and c_η by the minimum of $\mathcal{E}_m^{\phi}(c_\phi, c_\eta)$ and $\mathcal{E}_m^{\zeta}(c_\phi, c_\eta)$, say,

$$\frac{\partial \mathcal{E}_m^{\phi}(c_\phi, c_\eta)}{\partial c_\phi} = 0, \quad \frac{\partial \mathcal{E}_m^{\zeta}(c_\phi, c_\eta)}{\partial c_\zeta} = 0. \tag{75}$$

It is found that, by means of the optimal values of c_ϕ and c_η, the corresponding series of analytic approximations often converge quickly.

In addition, we also apply perturbation method to check the validity of our analytic solutions. It is found that the first-order perturbation approximation of $\phi(x, z)$ is exact the same as (72) given by our HAM-based approach in case of $c_\phi = -1$ and $c_\eta = -1$. This confirms the validity of our analytic approximations in a different way.

All of these demonstrate that the convergent series obtained by our HAM-based approach are indeed the solutions of the fully nonlinear wave equations (2) to (9).

5 Characteristics of new solitary surface waves

Based on the base functions (22), the new solitary waves have some unusual characteristics that are quite different from those of the traditional waves with smooth crest.

First, the new solitary waves have a peaked wave crest, since $\zeta'(x)$ is discontinuous at $x = 0$, i.e. $\zeta'(0_+) \neq \zeta'(0_-)$, where 0_+ and 0_- denote $x \to 0$ from the right and left along the x axis, respectively. For example, in case of $H_w = 0.1$ and $k = 1$,

$\zeta'(0_+) = -0.09299$, but $\zeta'(0_-) = 0.09299$. This is obviously different from traditional periodic and solitary waves which are infinitely differentiable.

Secondly, the new peaked solitary waves may be in the form of depression, which has been reported for internal waves but never for surface ones. Mathematically, it is straightforward to gain such kind of solitary waves of depression even by means of the linear wave equations, as shown in § 3.

Third, unlike traditional periodic and solitary waves, the dimensionless phase speed of the new peaked solitary waves depends only upon k, the so-called decaying-rate parameter, but has nothing to do with the wave height H_w. So, in the same water depth D, the new solitary waves with the same k but different wave height H_w may propagate with the same phase speed, where H_w may be either positive or negative. For example, it is found that, in case of $k = 1$, all of the peaked solitary waves with $H_w = \pm 0.1$ or $H_w = \pm 0.5$ propagate with the *same* phase speed $c = 1.24796\sqrt{gD}$: in these cases, we gain *different* convergent series solution with the *same* phase speed, as shown in § 4.2. On the other side, the peaked solitary waves with the same wave height H_w but different decay-rate parameter k may propagate with different phase speed! These are quite different from the traditional periodic and solitary waves whose phase speed strongly depends upon wave height. In other words, the traditional periodic and solitary waves with smooth crest are dispersive, but the peaked solitary waves are non-dispersive.

Finally, as shown in Tables 1 to 4 and Figs. 2 to 5, the horizontal bottom velocity beneath the crest of the new peaked solitary waves is always larger than that at crest. For example, in case of $k = 1$ and $H_w = 0.1$, the horizontal velocity at bottom beneath the crest is 43% larger than that at crest, as shown in Table 3. Especially, as shown in Table 4, in case of $k = 1$ and $H_w = -0.1$, the horizontal velocity at bottom beneath the crest is even 158% larger than that at crest! In general, for the same x, the horizontal velocity $u(x, -1)$ (at bottom) has always a larger absolute value than $u(x, \zeta(x))$ on the free surface, as shown for example in Figs. 10 and 11. This is quite different from the traditional periodic and solitary waves whose horizontal velocity at bottom is always less than that on surface. Besides, these also illustrate that there does not exist symmetry between the two kinds of peaked solitary waves with $H_w > 0$ and $H_w < 0$, respectively.

Furthermore, as shown in Figs. 6 and 7, in case of the same k, the peaked solitary wave with larger value of $|H_w|$ is sharper at crest. It is found that, in general, the larger the value of $|H_w|$, the sharper the peaked solitary waves at crest, as shown in Fig. 12. Note that all of these peaked solitary waves with the *same* k but *different* H_w propagate with the *same* phase speed! In addition, as shown in Figs. 8 and 9, in case of the same value of H_w, the larger the value of k, the sharper the peaked solitary wave at crest. This also holds in general: the larger the value of k, the shaper the peaked solitary waves at crest, as shown in Fig.13. Therefore, generally speaking, the larger the values of k and $|H_w|$, the sharper the peaked solitary waves.

All of these unusual characteristics clearly indicate the novelty of the peaked solitary waves. It should be emphasized that these so-called peaked solitary waves

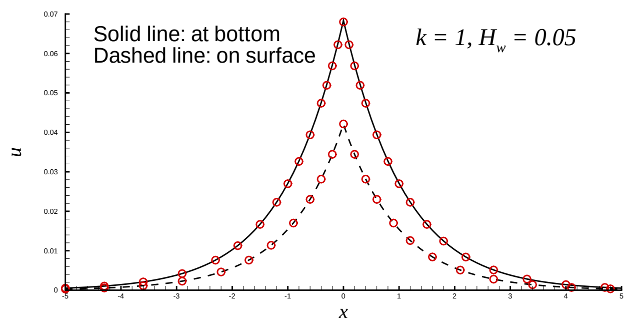

Figure 10: Horizontal velocity at bottom and on free surface when $k = 1$ and $H_w = 0.05$ by means of $c_\phi = -1$ and $c_\eta = -1$. Solid line: 3rd-order approx. of $u(x, -1)$ (at bottom); Dashed line: 3rd-order approx. of $u(x, \zeta(x))$ (on free surface); Symbols: the corresponding 25th-order approximations.

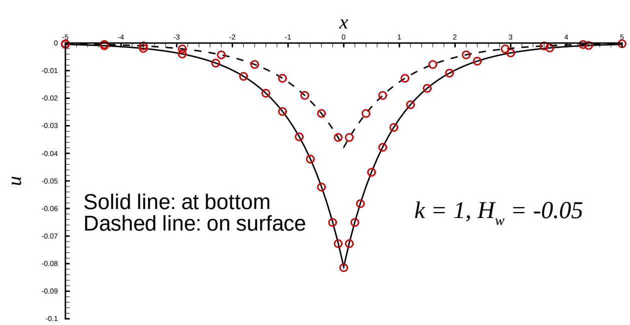

Figure 11: Horizontal velocity at bottom and on free surface when $k = 1$ and $H_w = -0.05$ by means of $c_\phi = -1$ and $c_\eta = -1$. Solid line: 3rd-order approx. of $u(x, -1)$ (at bottom); Dashed line: 3rd-order approx. of $u(x, \zeta(x))$ (on free surface); Symbols: the corresponding 25th-order approximations.

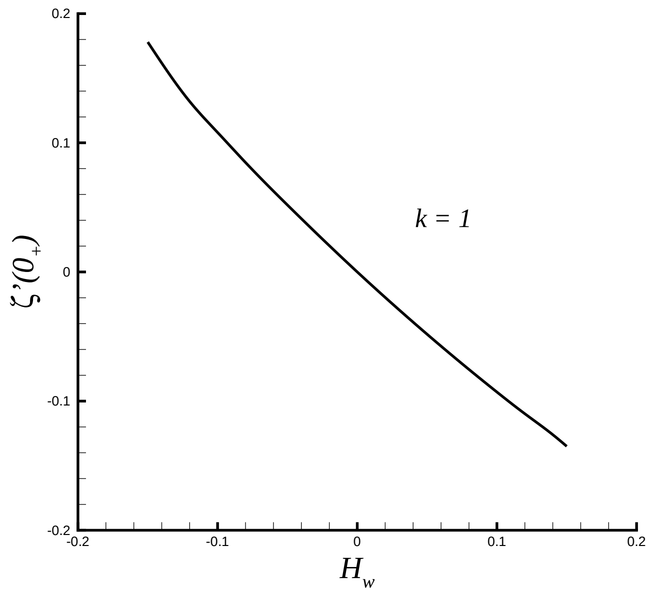

Figure 12: $\zeta'(0_+)$ (as $x \to 0$ from right) versus wave height H_w in case of $k = 1$ with the same wave speed $c/\sqrt{gD} = 1.24796$.

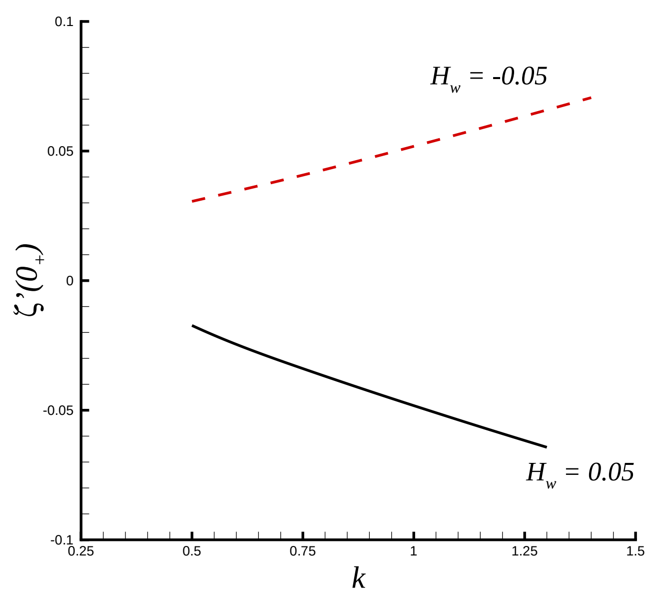

Figure 13: $\zeta'(0_+)$ (as $x \to 0$ from right) versus k. Solid line: $H_w = 0.05$; Dashed line: $H_w = -0.05$.

given by the linear wave equations in § 3 also have the *same* unusual characteristics.

6 Concluding remarks, discussions and some theoretical predictions

In this article, a new type of peaked solitary surface waves is found first by means of the linear wave equations in § 3 and then confirmed by using the fully nonlinear wave equations (2) to (9) in § 4. Following Liao & Cheung [29] and Tao et al [32], we successfully apply the homotopy analysis method (HAM) to gain convergent series of the velocity potential $\phi(x, z)$ and the wave elevation $\zeta(x)$ for different k and positive/negative wave height H_w by meas of the new base functions (22). The validity of these convergent analytic approximations are carefully checked: all linear governing equations and boundary conditions are automatically satisfied, and besides the two nonlinear free surface boundary conditions are satisfied very accurately, as shown in Tables 5 to 7. So, we are quite sure that our convergent series of analytic approximations of the peaked solitary waves are indeed the solutions of the fully nonlinear wave equations (2) to (9).

It is found that the peaked solitary waves have many unusual characteristics different from the traditional periodic and solitary ones. First, it has a peaked crest. Secondly, it may be in the form of depression, corresponding to a negative wave height H_w, which has been often reported for internal solitary waves but never for free-surface solitary ones, to the best of author's knowledge. Third, its phase speed has nothing to do with wave weight H_w, say, the peaked solitary waves are non-dispersive. Finally, its horizontal velocity at bottom is always larger than that on free surface. All of these are so different from the traditional periodic and solitary waves with smooth crest that they clearly indicate the novelty of the peaked solitary ones.

All of these unusual characteristics come from the base functions instantaneously of the peaked solitary waves, which is quite different from the base functions (13) of the traditional periodic and solitary waves, although both of them automatically satisfy the Laplace equation (2), the bottom condition (5) and the bounded condition (9). Note that the traditional base functions (13) are infinitely differentiable everywhere, but the high-order derivatives of the new base functions (22) with respect to x are not differentiable at $x = 0$. This difference of the two kinds of base functions (13) and (22) is the origin of the completely different characteristics between the smooth and peaked waves.

Note that all traditional periodic and solitary progressive waves can be derived by means of the fully nonlinear wave equations (2) to (9) with the symmetry (10). In other words, Eqs. (2) to (9) with the symmetry (10) is consistent with the traditional wave theory. Therefore, it is quite reasonable that the fully nonlinear wave equations (2) to (9) contain two different types of solutions: one (the traditional progressive periodic and solitary wave) has an infinitely differentiable wave elevation $\zeta(x)$ with $\zeta'(0) = 0$ and the infinitely differentiable velocity u and v everywhere, the other (the peaked solitary wave found in this paper) has a peaked crest with the discontinuous $\zeta'(0)$ and a discontinuous first derivative of the horizontal velocity $u(0,z)$ at $x=0$, corresponding to the two different base functions (13) and (22), respectively. It should be emphasized that, according to the theory of differential equations, the Laplace equation (2) needs *one and only one* boundary condition (7) at $x=0$, so that any other smoothness conditions at $x=0$ such as the infinitely differentiable velocity potential is *unnecessary* and must be avoided since they may lead to the loss of the peaked solitary waves.

In theory, such kind of gravity waves with peaked crest are not new. It is well-known that the limiting gravity wave has a corner crest with 120 degree, as pointed out by Stokes [4] in 1894. In 1993, Camassa & Holm [5] found the peaked solitary waves by means of the CH equation (1) in the special case of $\omega=0$, which is the same as (27) found in this article by means of the linear wave equations. Currently, it is found by Liao [9] that the CH equation also admits peaked solitary waves even in case of $\omega \neq 0$. Besides, the closed-form solutions of the peaked solitary waves of the KdV equation [2] , the Boussinesq equation [1], the BBM equation [3] and modified KdV equation are currently found by Liao [8]. Therefore, in theory, nearly all mainstream models of shallow water waves admit the peaked solitary waves. Note that all of these equations are approximations of the fully nonlinear wave equations in shallow water, so that our peaked solitary waves derived from the fully nonlinear wave equations (2) to (10) well explain why all of these shallow water equations admit peaked solitary waves. This also indicates that the peaked crest is a common property of water waves.

Besides, in practice, it is well-known that solutions related to dam break and shock waves are *discontinuous*. Such kind of discontinuous problems belong to the so-called Riemann problems [35–38], a classic field of fluid mechanics. Such kind of discontinuity (or singularity) of solutions of water wave equations have clear physical meanings, which are often solved in different sub-domains by many numerical and analytic methods. For such kind of Riemann problems, it is *unnecessary* for the related differential equations and boundary conditions to be satisfied at points with discontinuity. The numerical and analytic results of many Riemann problems agree well with experimental results, indicating that such kind of discontinuity (or singularity) is reasonable not only in mathematics but also in physics. So, physically, the peaked solitary waves are not strange at all even from the traditional view-points. Therefore, the new peaked solitary waves are reasonable and acceptable in hydrodynamics.

On the other hand, such kind of discontinuity (or singularity) of the peaked solitary waves can be removed in the following way. Let $U(z)=u(x,z)$ denote the horizontal velocity at $x=0$, corresponding to a progressive wave governed by the fully nonlinear wave equations (2) to (9). Assume that, in the frame moving with the solitary wave, one can instantaneously replace the boundary $x=0$ by a porous vertical plate, and at the same time enforce a horizontal velocity $U(z)=u(0,z)$ through the porous plate. Then, the corresponding velocity potential ϕ and wave elevation $\zeta(x)$ in the domain $x \in [0,+\infty)$ are governed by the *same* wave equations (2) to (9). Therefore, for a properly given $U(z)$, one gains either the traditional periodic and solitary waves with smooth crest (in the domain $0 \leq x < +\infty$), if the traditional base

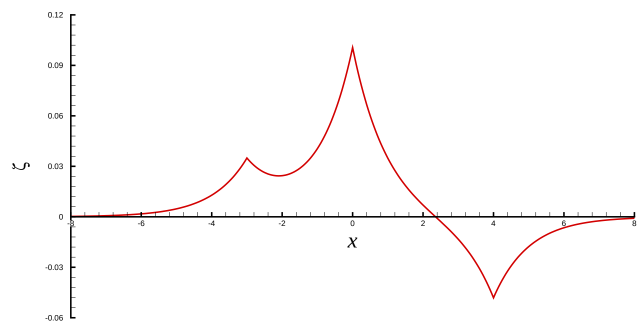

Figure 14: Possible complicated elevation of three peaked solitary waves with the same phase speed and the permanent form $\zeta(x) = e^{-|x|}/10 - e^{-|x-4|}/20 + 3e^{-|x+3|}/100$, predicted by the linear theory in § 3.

functions (13) are used, or the new peaked solitary waves reported in this article, if the new base functions (22) are used, respectively. In this case, the Laplace equation (2) is satisfied in the *whole* domain $x \in [0, +\infty)$ and $z \leq \zeta(x)$ so that *no* discontinuity (or singularity) exists at all. If necessary, using the symmetry (10), it is straightforward to gain the wave elevation $\zeta(x)$ in the whole domain $-\infty < x < +\infty$.

The peaked solitary surface waves may provide us new explanations for a few natural phenomenon and some theoretical predictions. For example, the peaked solitary wave has an unusual and interesting characteristic: its phase speed has nothing to do with the wave height H_w that may be either positive or negative. According to (24), the peaked solitary waves with small wave height H_w may propagate very quickly, since $\tan(k)/k \to +\infty$ as $x \to \pi/2$. Thus, all of these peaked solitary waves with small wave height and different phase speed may create a huge solitary surface wave somewhere: this gives a new theoretical explanation about the so-called rogue wave that can suddenly appear on ocean even when "the weather was good, with clear skies and glassy swells", as reported by Graham [39] and mentioned by Kharif [40]. On the other hand, several waves with different wave heights H_w but the same k can propagate together with a *permanent* form and the same phase speed, as shown for example in Fig. 14. However, such kind of complicated solitary waves have never been observed in experiment and practice: this provides us a new theoretical prediction.

In addition, the new peaked solitary waves may be in the form of depression, corresponding to a negative wave height H_w. To the best of the author's knowledge, such kind of solitary waves of depression have been reported only for internal waves but never for surface waves. Obviously, such kind of peaked solitary surface waves of depression should be more difficult to create than the traditional ones. However, if this theoretical prediction is physically correct, sooner or later, we should be able to observe it in laboratory experiments and/or in practice. This is an interesting but challenging work: it could enrich and deepen our understanding about solitary waves, no matter the final conclusions are positive or not.

Possibly, the new peaked solitary waves might change some traditional viewpoints. Note that solitary waves are often regarded as a nonlinear phenomenon. However, we illustrate in this article, for the first time, that solitary surface waves may exist even in a system of linear differential equations! Besides, it is also widely believed that solitary waves exist only in shallow water. However, we indicates in this article that solitary waves can exist even in a finite water depth, say, D is *unnecessary* to be small. For example, in case of $D = 100$ meter, $k = 1$ and the dimensionless wave height $H_w = 0.05$, the corresponding new peaked solitary surface wave propagates with the 5 meter wave-height in the phase speed $c = 1.24796\sqrt{gD} \approx 39.1$ meter per second, which is not very dangerous. However, in case of $D = 1000$ meter, $k = 1$ and $H_w = 0.05$, the corresponding peaked solitary wave propagates with the 50 meter wave-height and the phase speed $c = 123.5$ meter per second, which is deadly destructive if it indeed could occur on the earth.

According to the traditional wave theories, the velocity of fluid decreases exponentially in the vertical direction (from surface to bottom) so that a submarine far enough beneath ocean surface is safe even if there are huge waves on surface. However, different from traditional periodic and solitary waves with smooth crest, the horizontal bottom velocity of the peaked solitary waves is always larger than that on free surface. Certainly, due to the viscosity of fluid, the horizontal velocity at bottom of all water waves must be zero, so that such kind of the peaked solitary waves might not exist exactly in its theoretical form reported in this article, since there exists a thin viscous boundary layer near the bottom. Thus, such kind of solitary waves with a peaked crest and larger bottom velocity might be more difficult to create not only in laboratory experiments but also in nature. This might be the reason why such kind of peaked solitary surface waves have never been reported and observed. However, if such kind of peaked solitary waves could indeed exist in nature even not in the exactly same form, it would be quite dangerous to submarines, platforms and equipments in underwater engineering.

Note that the peaked solitary waves found in this article are obtained under the assumptions that the fluid is inviscid and incompressible, the flow is irrotational, the surface tension is neglected and the wave elevation has a symmetry. Although the same assumptions are widely used for the traditional periodic and solitary waves with smooth crest, their physical reasonableness for the peaked solitary waves should be reconsidered carefully in future.

Therefore, without doubts, further theoretical, numerical and experimental studies and especially practical observations about this new type of solitary surface waves with peaked crest and many unusual characteristics are needed in future: all of these could deepen our understandings and enrich our knowledge about solitary waves.

Finally, it should be emphasized that, the discontinuity and/or singularity exist widely in natural phenomena, such as dam break in hydrodynamics, shock waves in aerodynamics, black holes in general relativity equation and so on. Indeed, the discontinuity and/or singularity are difficult to handle by traditional methods. But, they can greatly enrich and deepen our understandings about the real world, and therefore should not be evaded.

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A Detailed derivation of formulas (54) - (59)

In this appendix, we explicitly give the formulas of all terms in (54) - (59).

Write

$$\left(\sum_{i=1}^{+\infty} \zeta_i \, q^i\right)^m = \sum_{n=m}^{+\infty} \mu_{m,n} \, q^n, \tag{76}$$

with the definition

$$\mu_{1,n}(x) = \zeta_n(x), \quad n \geq 1. \tag{77}$$

Then,

$$\left(\sum_{i=1}^{+\infty} \zeta_i \, q^i\right)^{m+1} = \left(\sum_{n=m}^{+\infty} \mu_{m,n} \, q^n\right)\left(\sum_{i=1}^{+\infty} \zeta_i \, q^i\right) = \sum_{n=m+1}^{+\infty} \mu_{m+1,n} \, q^n, \tag{78}$$

which gives

$$\mu_{m,n}(x) = \sum_{i=m-1}^{n-1} \mu_{m-1,i}(x) \, \zeta_{n-i}(x), \quad m \geq 2, \; n \geq m. \tag{79}$$

Define

$$\psi_i^{n,m}(x) = \frac{\partial^i}{\partial x^i}\left(\frac{1}{m!} \left.\frac{\partial^m \phi_n}{\partial z^m}\right|_{z=0}\right).$$

By Taylor series, we have for *any* z that

$$\phi_n(x,z) = \sum_{m=0}^{+\infty}\left(\frac{1}{m!} \left.\frac{\partial^m \phi_n}{\partial z^m}\right|_{z=0}\right) z^m = \sum_{m=0}^{+\infty} \psi_0^{n,m} \, z^m \tag{80}$$

and

$$\frac{\partial^i \phi_n}{\partial x^i} = \sum_{m=0}^{+\infty} \frac{\partial^i}{\partial x^i}\left(\frac{1}{m!} \left.\frac{\partial^m \phi_n}{\partial z^m}\right|_{z=0}\right) z^m = \sum_{m=0}^{+\infty} \psi_i^{n,m} \, z^m. \tag{81}$$

Then, on the *unknown* free surface $z = \eta(x;q)$, we have using (76) that

$$\begin{aligned} \frac{\partial^i \phi_n}{\partial x^i} &= \sum_{m=0}^{+\infty} \psi_i^{n,m} \left(\sum_{s=1}^{+\infty} \zeta_s \, q^s\right)^m = \psi_i^{n,0} + \sum_{m=1}^{+\infty} \psi_i^{n,m} \left(\sum_{s=m}^{+\infty} \mu_{m,s} \, q^s\right) \\ &= \sum_{m=0}^{+\infty} \beta_i^{n,m}(x) \, q^m, \end{aligned} \tag{82}$$

where

$$\begin{aligned} \beta_i^{n,0} &= \psi_i^{n,0}, & (83)\\ \beta_i^{n,m} &= \sum_{s=1}^{m} \psi_i^{n,s}\ \mu_{s,m}, \quad m \geq 1. & (84) \end{aligned}$$

Similarly, on the unknown free surface $z = \eta(x; q)$, it holds

$$\begin{aligned} \frac{\partial^i}{\partial x^i}\left(\frac{\partial \phi_n}{\partial z}\right) &= \sum_{m=0}^{+\infty} \gamma_i^{n,m}(x)\ q^m, & (85)\\ \frac{\partial^i}{\partial x^i}\left(\frac{\partial^2 \phi_n}{\partial z^2}\right) &= \sum_{m=0}^{+\infty} \delta_i^{n,m}(x)\ q^m, & (86) \end{aligned}$$

where

$$\begin{aligned} \gamma_i^{n,0} &= \psi_i^{n,1}, & (87)\\ \gamma_i^{n,m} &= \sum_{s=1}^{m} (s+1)\psi_i^{n,s+1}\ \mu_{s,m}, \quad m \geq 1, & (88)\\ \delta_i^{n,0} &= 2\psi_i^{n,2}, & (89)\\ \delta_i^{n,m} &= \sum_{s=1}^{m} (s+1)(s+2)\psi_i^{n,s+2}\ \mu_{s,m}, \quad m \geq 1. & (90) \end{aligned}$$

Then, on the unknown free surface $z = \eta(x; q)$, it holds using (82) that

$$\begin{aligned} \Phi(x, \zeta; q) &= \sum_{n=0}^{+\infty} \phi_n(x, \zeta)\ q^n = \sum_{n=0}^{+\infty} q^n \left[\sum_{m=0}^{+\infty} \beta_0^{n,m}(x)\ q^m\right]\\ &= \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \beta_0^{n,m}(x)\ q^{m+n} = \sum_{s=0}^{+\infty} q^s \left[\sum_{m=0}^{s} \beta_0^{s-m,m}(x)\right]\\ &= \sum_{n=0}^{+\infty} \bar{\phi}_{n,0}(x)\ q^n, & (91) \end{aligned}$$

where

$$\bar{\phi}_{n,0}(x) = \sum_{m=0}^{n} \beta_0^{n-m,m}. \tag{92}$$

Similarly, we have

$$\begin{aligned} \frac{\partial^i \Phi}{\partial x^i} &= \sum_{n=0}^{+\infty} \bar{\phi}_{n,i}(x)\ q^n, & (93)\\ \frac{\partial^i}{\partial x^i}\left(\frac{\partial \Phi}{\partial z}\right) &= \sum_{n=0}^{+\infty} \bar{\phi}_{n,i}^{z}(x)\ q^n, & (94)\\ \frac{\partial^i}{\partial x^i}\left(\frac{\partial^2 \Phi}{\partial z^2}\right) &= \sum_{n=0}^{+\infty} \bar{\phi}_{n,i}^{zz}(x)\ q^n, & (95) \end{aligned}$$

where

$$\bar{\phi}_{n,i}(x) = \sum_{m=0}^{n} \beta_i^{n-m,m}, \tag{96}$$

$$\bar{\phi}^z_{n,i}(x) = \sum_{m=0}^{n} \gamma_i^{n-m,m}, \tag{97}$$

$$\bar{\phi}^{zz}_{n,i}(x) = \sum_{m=0}^{n} \delta_i^{n-m,m}. \tag{98}$$

Then, on the unknown free surface $z = \eta(x;q)$, it holds using (93) and (94) that

$$\begin{aligned} f &= \frac{1}{2}\nabla\Phi\cdot\nabla\Phi \\ &= \frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right] \\ &= \sum_{m=0}^{+\infty} \Gamma_{m,0}(x)\ q^m, \end{aligned} \tag{99}$$

where

$$\Gamma_{m,0}(x) = \frac{1}{2}\sum_{n=0}^{m}\left(\bar{\phi}_{n,1}\ \bar{\phi}_{m-n,1} + \bar{\phi}^z_{n,0}\ \bar{\phi}^z_{m-n,0}\right). \tag{100}$$

Similarly, it holds on $z = \eta(x;q)$ that

$$\begin{aligned} \frac{\partial f}{\partial x} &= \nabla\Phi\cdot\nabla\left(\frac{\partial\Phi}{\partial x}\right) \\ &= \frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial\Phi}{\partial z}\frac{\partial}{\partial x}\left(\frac{\partial\Phi}{\partial z}\right) \\ &= \sum_{m=0}^{+\infty} \Gamma_{m,1}(x)\ q^m, \end{aligned} \tag{101}$$

where

$$\Gamma_{m,1}(x) = \sum_{n=0}^{m}\left(\bar{\phi}_{n,1}\ \bar{\phi}_{m-n,2} + \bar{\phi}^z_{n,0}\ \bar{\phi}^z_{m-n,1}\right). \tag{102}$$

Besides, on $z = \eta(x;q)$, we have by means of (93), (94) and (95) that

$$\begin{aligned} \frac{\partial f}{\partial z} &= \nabla\Phi\cdot\nabla\left(\frac{\partial\Phi}{\partial z}\right) \\ &= \frac{\partial\Phi}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial\Phi}{\partial z}\right) + \frac{\partial\Phi}{\partial z}\frac{\partial^2\Phi}{\partial z^2} \\ &= \sum_{m=0}^{+\infty} \Gamma_{m,3}(x)\ q^m, \end{aligned} \tag{103}$$

where

$$\Gamma_{m,3}(x) \;=\; \sum_{n=0}^{m} \left(\bar{\phi}_{n,1}\ \bar{\phi}^{z}_{m-n,1} + \bar{\phi}^{z}_{n,0}\ \bar{\phi}^{zz}_{m-n,0}\right). \tag{104}$$

Furthermore, using (93), (101) and (103), we have on $z = \eta(x;q)$ that

$$\nabla\Phi \cdot \nabla f \;=\; \frac{\partial \Phi}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial z} = \sum_{m=0}^{+\infty} \Lambda_m(x)\ q^m, \tag{105}$$

where

$$\Lambda_m(x) \;=\; \sum_{n=0}^{m} \left(\bar{\phi}_{n,1}\ \Gamma_{m-n,1} + \bar{\phi}^{z}_{n,0}\ \Gamma_{m-n,3}\right) \tag{106}$$

Then, using (93), (94), (101) and (105), we have on $z = \eta(x;q)$ that

$$\begin{aligned} & \mathcal{N}\left[\Phi(x,z;q)\right] \\ =\; & \alpha^2 \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial \Phi}{\partial z} - 2\alpha \frac{\partial f}{\partial x} + \nabla\Phi \cdot \nabla f \\ =\; & \sum_{m=0}^{+\infty} \Delta^{\phi}_m(x)\ q^m, \end{aligned} \tag{107}$$

where

$$\Delta^{\phi}_m(x) \;=\; \alpha^2\ \bar{\phi}_{m,2} + \bar{\phi}^{z}_{m,0} - 2\alpha\ \Gamma_{m,1} + \Lambda_m \tag{108}$$

for $m \geq 0$.

Using (49) and (82), we have on $z = \eta(x;q)$ that

$$\begin{aligned} \frac{\partial^2}{\partial x^2}\left(\Phi - \phi_0\right) \;&=\; \sum_{n=1}^{+\infty} \frac{\partial^2 \phi_n(x,\eta)}{\partial x^2}\ q^n = \sum_{n=1}^{+\infty} q^n \left(\sum_{m=0}^{+\infty} \beta_2^{n,m}\ q^m\right) \\ &=\; \sum_{n=1}^{+\infty} q^n \left(\sum_{m=0}^{n-1} \beta_2^{n-m,m}\right), \end{aligned} \tag{109}$$

and similarly

$$\begin{aligned} \frac{\partial}{\partial z}\left(\Phi - \phi_0\right) \;&=\; \sum_{n=1}^{+\infty} \frac{\partial \phi_n(x,\eta)}{\partial z}\ q^n = \sum_{n=1}^{+\infty} q^n \left(\sum_{m=0}^{+\infty} \gamma_0^{n,m}\ q^m\right) \\ &=\; \sum_{n=1}^{+\infty} q^n \left(\sum_{m=0}^{n-1} \gamma_0^{n-m,m}\right), \end{aligned} \tag{110}$$

respectively. Then, on $z = \eta(x;q)$, it holds due to the linear property of the operator (42) that

$$\mathcal{L}\left(\Phi - \phi_0\right) = \sum_{n=1}^{+\infty} S_n(x)\ q^n, \tag{111}$$

where

$$S_n(x) = \sum_{m=0}^{n-1} \left(\alpha^2 \ \beta_2^{n-m,m} + \ \gamma_0^{n-m,m}\right). \tag{112}$$

Then, on $z = \eta(x;q)$, it holds

$$(1-q)\mathcal{L}\left(\Phi - \phi_0\right) = (1-q) \ \sum_{n=1}^{+\infty} S_n \ q^n = \sum_{n=1}^{+\infty} \left(S_n - \chi_n \ S_{n-1}\right) q^n, \tag{113}$$

where

$$\chi_n = \begin{cases} 0, & \text{when } n \le 1, \\ 1, & \text{when } n > 1. \end{cases} \tag{114}$$

Substituting (113), (107) into (38) and equating the like-power of q, we have the boundary condition:

$$S_m(x) - \chi_m \ S_{m-1}(x) = c_\phi \ \Delta_{m-1}^{\phi}(x), \quad m \ge 1. \tag{115}$$

Define

$$\bar{S}_n(x) = \sum_{m=1}^{n-1} \left(\alpha^2 \ \beta_2^{n-m,m} + \gamma_0^{n-m,m}\right). \tag{116}$$

Then,

$$S_n \ = \ \left(\alpha^2 \ \beta_2^{n,0} + \gamma_0^{n,0}\right) + \bar{S}_n = \left(\alpha^2 \ \frac{\partial^2 \phi_n}{\partial x^2} + \frac{\partial \phi_n}{\partial z}\right)\bigg|_{z=0} + \bar{S}_n. \tag{117}$$

Substituting the above expression into (115) gives the boundary condition on $z = 0$:

$$\left(\alpha^2 \ \frac{\partial^2 \phi_m}{\partial x^2} + \frac{\partial \phi_m}{\partial z}\right)\bigg|_{z=0} = \left\{c_\phi \ \Delta_{m-1}^{\phi} + \chi_m \ S_{m-1} - \bar{S}_m\right\}\Big|_{z=0}, \quad m \ge 1. \tag{118}$$

Substituting the series (50), (93) and (99) into (39), equating the like-power of q, we have on $z = 0$ that

$$\zeta_m(x) = \left\{c_\eta \ \Delta_{m-1}^{\eta} + \chi_m \ \zeta_{m-1}\right\}\big|_{z=0}, \quad m \ge 1, \tag{119}$$

where

$$\Delta_m^{\eta} = \zeta_m - \alpha \ \bar{\phi}_{m,1} + \Gamma_{m,0}.$$

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