

Complete Integrability for Hamiltonian Systems with a Cone Potential

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Abstract. It is known that, if a point in \mathbb{R}^n is driven by a bounded below potential \mathcal{V} , whose gradient is always in a closed convex cone which contains no lines, then the velocity has a finite limit as time goes to $+\infty$.

The components of the asymptotic velocity, as functions of the initial data, are trivially constants of motion. We find sufficient conditions for these functions to be C^k ($2 \leq k \leq +\infty$) first integrals, independent and pairwise in involution.

In this way we construct a large class of completely integrable systems. We can deal with very different asymptotic behaviours of the potential and we have persistence of the integrability under any small perturbation of the potential in an arbitrary compact set.

1. Introduction

Given a smooth real function $(p, q) \mapsto H(p, q)$ defined in an open domain Ω of $\mathbb{R}^n \times \mathbb{R}^n$ we can consider the associated *Hamiltonian system*, that is, the autonomous system of ordinary differential equations

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}. \quad (1.1)$$

The function H is called a (time-independent) Hamiltonian. We remind that the *Poisson brackets* of two smooth real functions $F, G: \Omega \rightarrow \mathbb{R}$ are

$$\{F, G\} := \sum_{i=1}^n \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right). \quad (1.2)$$

An $F \in C^1(\Omega, \mathbb{R})$ is a constant of motion (or first integral) for the system (1.1) if and only if $\{F, H\} = 0$.

Let us suppose that we find n functions $F_1, \dots, F_n: \Omega \rightarrow \mathbb{R}$ of class C^k , $2 \leq k \leq +\infty$, such that:

- i) $\{F_i, H\} = 0$ for all i (i.e., the F_i are first integrals of (1.1));
- ii) $\{F_i, F_j\} = 0$ for all i, j (i.e., the F_i are pairwise in involution);
- iii) $\nabla F_1, \dots, \nabla F_n$ are linearly independent in all of Ω (the F_i themselves are then said to be independent).

In this case a well known classical theorem says that the system (1.1) can be integrated by quadratures, in the usual sense of ordinary differential equations (see [2], Chapter 4, Section 1.1).

If H itself is one of the functions F_i and the solutions of the Hamiltonian systems associated with each F_i are all global (i.e., defined on \mathbb{R}), then the system (1.1) is called *C^k -completely integrable* (see [2], Chapter 4, Section 1.2). Analogously we define analytic integrability.

What is interesting about completely integrable systems is that the structure of the set of their solutions is very simple (see [2], Chapter 4, Section 1.2, Theorem 3). These properties, or rather the corresponding ones in a more general setting, are the foundations of a rich theory in the case when the level surfaces of the vector function (F_1, \dots, F_n) are compact. This paper does not deal with this last situation, but is concerned with proving the complete integrability of some systems with non-oscillatory behaviour, loosely related to scattering problems.

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Let $\mathcal{V}: \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth function (called potential) and consider the Hamiltonian $\mathcal{H}(p, q) := \frac{1}{2}|p|^2 + \mathcal{V}(q)$ with its associated system

$$\dot{q} = p, \quad \dot{p} = -\nabla \mathcal{V}(q). \quad (1.3)$$

Denote by $t \mapsto (p(t, \bar{p}, \bar{q}), q(t, \bar{p}, \bar{q}))$ the solution to (1.3) with (\bar{p}, \bar{q}) as initial data:

$$p(0, \bar{p}, \bar{q}) = \bar{p}, \quad q(0, \bar{p}, \bar{q}) = \bar{q}.$$

Our starting point is the following assumption on the potential \mathcal{V} .

Hypothesis 1.1 \mathcal{V} is a function in $C^2(\mathbb{R}^n; \mathbb{R})$ such that:

- i) \mathcal{V} is bounded below;
- ii) there is a basis $\{b_1, \dots, b_n\}$ for \mathbb{R}^n such that $-\nabla\mathcal{V}(q) \cdot b_i \geq 0$ for all $q \in \mathbb{R}^n$ and all b_i .

Of course, the system admits the first integral of energy

$$\frac{1}{2}|p(t, \bar{p}, \bar{q})|^2 + \mathcal{V}(q(t, \bar{p}, \bar{q})) = \frac{1}{2}|\bar{p}|^2 + \mathcal{V}(\bar{q}). \quad (1.4)$$

From i) we see that $|p(\cdot, \bar{p}, \bar{q})|$ must be bounded for each solution, so that by standard arguments in Ordinary Differential Equations we can prove that all solutions to (1.3) are defined for all times $t \in \mathbb{R}$.

On the other hand, property ii) implies that $t \mapsto p(t, \bar{p}, \bar{q}) \cdot b_i$ is a monotone function for all (\bar{p}, \bar{q}) and for all b_i .

The whole of Hypothesis 1.1 thus ensures the existence, along each solution, of the following limit, the *asymptotic velocity*:

$$p_\infty(\bar{p}, \bar{q}) := \lim_{t \rightarrow +\infty} p(t, \bar{p}, \bar{q}) \in \mathbb{R}^n. \quad (1.5)$$

The limit as $t \rightarrow -\infty$ exists as well.

These remarkably simple facts were pointed out by Gutkin in [5]. He called the potentials \mathcal{V} satisfying Hypothesis 1.1 ii) *cone potentials*. The reason for this name is as follows. Let \mathcal{C} be the convex cone in \mathbb{R}^n spanned by the forces $-\nabla\mathcal{V}$:

$$\mathcal{C} := \left\{ -\sum_{\alpha \in I} \lambda_\alpha \nabla\mathcal{V}(q_\alpha) : \emptyset \neq I \text{ finite set, } \lambda_\alpha \geq 0, q_\alpha \in \mathbb{R}^n \forall \alpha \in I \right\}. \quad (1.6)$$

and let \mathcal{D} be the dual cone of \mathcal{C} , defined by

$$\mathcal{D} := \{w \in \mathbb{R}^n : w \cdot v \geq 0 \quad \forall v \in \mathcal{C}\}. \quad (1.7)$$

Then Hypothesis 1.1 ii) means that \mathcal{D} has nonempty interior, or, equivalently, that the closure of \mathcal{C} contains no straight lines (such cones \mathcal{C} are called *proper*). We refer to Section 2 for more details about cones.

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Let us survey the content of the present paper. We are going to provide only hints to our assumptions and results. We will direct in each case to the precise statements scattered through the following Sections.

In Section 2 we give a few generalities about cones in \mathbb{R}^n and prove a formula (Proposition 2.4) that will be used extensively.

Section 3 presents three simple instances of cone potentials for which the asymptotic velocity does not depend continuously on the initial data. The analysis of these counterexamples leads in Section 4 to write down our basic assumptions (the only *global* ones)

on the potential \mathcal{V} , that, roughly speaking, amount to these:

- 1) every level set of the potential \mathcal{V} must be contained in a set of the form $q + \mathcal{D}$, so that the asymptotic velocity turn out to belong to \mathcal{D} (Hypothesis 4.1);
- 2) the force $-\nabla\mathcal{V}$ must push consistently toward the interior of \mathcal{D} ; somewhat less roughly, the component of $-\nabla\mathcal{V}(q)$ along any given direction of $\bar{\mathcal{C}}$ shall be bounded below by a positive constant, when q varies on a (possibly noncompact) set of a certain sort (Hypothesis 4.2).

Requirement 2) is actually the only severe limitation for our approach. In particular, it implies that \mathcal{C} is contained in \mathcal{D} , i.e., the scalar product of any two vectors from \mathcal{C} is nonnegative (i.e., \mathcal{C} has width not larger than $\pi/2$). Until Section 10 we will think of \mathcal{V} as being defined on all of \mathbb{R}^n , but everything runs just as well if \mathcal{V} is defined on a set of the form $q + \mathcal{D}^\circ$.

With the right hypotheses in hand, it becomes easy to prove that the asymptotic velocity always lies in the *interior* of the dual cone \mathcal{D} (Proposition 4.3), with certain *locally uniform* estimates on the trajectories (Proposition 4.4). Such information is first used in Section 5 to find general sufficient conditions (Hypothesis 5.1) on the decay rate of \mathcal{V} “at infinity” (in the direction of the cone \mathcal{D}) for p_∞ to be a continuous function of the initial data. The tools are the fact that p_∞ can be expressed as an integral:

$$p_\infty(\bar{p}, \bar{q}) = \bar{p} + \int_0^{+\infty} -\nabla\mathcal{V}(q(t, \bar{p}, \bar{q})) dt \quad (1.8)$$

and the theorems on uniform integrability.

The first order differentiability of p_∞ is less immediate. We get it in two different sets of assumptions. In Section 6 we impose an *exponential* decay on the second derivatives of \mathcal{V} (Hypothesis 6.1). This will permit to exploit a simple Gronwall estimate on the solutions of the first variational equations of our system, and to use the theorems on differentiation under the integral sign in (1.8). In Section 7 we allow far more general asymptotics for \mathcal{V} , but we add the side hypotheses of *convexity* on \mathcal{V} and a kind of *monotonicity* in the Hessian matrix (Hypothesis 7.1). A Liapunov function built on the Hessian matrix of \mathcal{V} will give a sharp control over the growth of the solutions of the first variational equation. As for the rest, Sections 6 and 7 run very much parallel to each other. Beside the mere regularity (Propositions 6.3 and 7.3), we also prove that p_∞ , as a function of the initial data, is asymptotic, in the C^1 norm, to the *projection* $(p, q) \mapsto p$ (Propositions 5.3, 6.4 and 7.4). This will be crucial in proving independence and involution in Section 9.

In Section 8 we show how to get higher order differentiability of p_∞ . This is not difficult, since the bulk of the job has already been done in Sections 6 and 7.

In Section 9 we reap the rewards of the regularity theory to prove that the components of the asymptotic velocity are first integrals, independent and in involution, and to state the *complete integrability* of our systems (Theorem 9.1). Furthermore, we show that the potentials \mathcal{V} satisfying our sufficient conditions for integrability can undergo arbitrary (small enough) perturbations on any compact set of \mathbb{R}^n without losing the property of yielding completely integrable systems (*Persistence Theorem* 9.2). The fact that the integrability is decided almost only on *asymptotic* behaviour and survives generic modifications in a bounded set seems to be unusual in the theory of integrable Hamiltonian systems.

In Section 10 we give some examples. Namely, we provide manageable conditions (Hypotheses 10.1) on the functions f_1, \dots, f_N and on the vectors v_1, \dots, v_N ($N \geq 1$, no relation to n) so that our theory applies to the system with the potential \mathcal{V} given by

$$\mathcal{V}(q) := \sum_{\alpha=1}^N f_{\alpha}(q \cdot v_{\alpha}), \quad q \in \mathbb{R}^n \quad (1.9)$$

(Proposition 10.5). A concrete instance is given in Corollary 10.6: if $v_{\alpha} \cdot v_{\beta} \geq 0$ for all α, β and if $r > 0$, then the Hamiltonian system with potential

$$\mathcal{V}(q) := \sum_{\alpha=1}^N \frac{1}{(q \cdot v_{\alpha})^r}, \quad q \in \{\bar{q} \in \mathbb{R}^n : \bar{q} \cdot v_{\alpha} > 0 \forall \alpha\} \quad (1.10)$$

is C^{∞} -completely integrable.

These cone potentials have polyhedral (that is, finitely generated) cone \mathcal{C} of the forces (Lemma 10.2). In a future paper (in preparation) we will provide an example where \mathcal{C} is *not polyhedral*. In fact, the present approach does not exploit such additional structures of \mathcal{V} as being finite sum of one-dimensional functions.

* * *

An important analytically integrable system with cone potential (and cone wider than $\pi/2$) is the classical nonperiodic Toda Lattice system. It describes the dynamics of n particles on the line with coordinates q_1, \dots, q_n interacting through an exponential potential

$$\mathcal{V}(q) := \sum_{i=1}^{n-1} e^{q_i - q_{i+1}}, \quad q = (q_1, \dots, q_n) \in \mathbb{R}^n. \quad (1.11)$$

This system was integrated by Hénon and Flaschka (with different methods, unrelated to cone properties)—see the deep Lecture Notes [11].

Subsequently, Gutkin in [5] introduced cone potentials, recognizing that a large subclass of the Toda-like potentials

$$\mathcal{V}(q) := \sum_{\alpha=1}^N c_{\alpha} e^{q \cdot v_{\alpha}}, \quad c_{\alpha} > 0, \quad q \in \mathbb{R}^n \quad (1.12)$$

have the cone property (precisely, if the convex cone $\bar{\mathcal{C}}$ spanned by the vectors v_1, \dots, v_N is proper). He noted that the components of the asymptotic velocity, which exists for all systems with bounded below cone potentials, are likely candidates for being n independent integrals in involution, as required by the definition of integrability. However this is not true in general, since they may even be discontinuous—see Section 3.

In the case when $\bar{\mathcal{C}}$ has amplitude not larger than $\pi/2$, the conjecture (namely, C^{∞} -integrability) was rigorously proved by Oliva and Castilla in [12] for the potentials (1.12) and also for (1.9), but only with functions f_{α} having exponential asymptotic behaviour at $+\infty$. Their method uses the finite-sum form of the potential to define a “compactifying” change of variables. Then they prove and apply a Lemma in Dynamical Systems (of independent interest too), concerning the differentiability of a foliation of invariant manifolds.

Oliva and Castilla drop the $\pi/2$ restriction for the following potentials:

$$\begin{aligned} \mathcal{V}(q) &:= e^{q \cdot v_0} + e^{q \cdot v_\theta}, & \theta \in \mathbb{R}, \quad q \in \mathbb{R}^3, \quad v_0 = (1, -1, 0), \\ & & v_\theta = \theta(0, 1, -1) + (1 - \theta)v_0 \end{aligned} \quad (1.13)$$

with three degrees of freedom, and

$$\mathcal{V}(q) := e^{(-q_1 - \alpha q_2)} + e^{(-q_1 + \alpha q_2)}, \quad \alpha \geq 0, \quad q = (q_1, q_2) \in \mathbb{R}^2 \quad (1.14)$$

with two freedoms, and some generalizations thereof, all having either two or three freedoms. We think that admitting wide cones needs such strong restriction as low dimensions and/or specialized proofs exploiting the particular structure of a potential, and cannot be covered by a general theory of integrable systems with cone potential. The special role of the angle $\pi/2$ is not so surprising if one thinks about the behaviour of a billiard ball in a wedge. The dynamics is fundamentally simpler if the wedge is wider than $\pi/2$, and this corresponds to a cone of the forces smaller than $\pi/2$.

As for the potential (1.14), Yoshida, Ramani, Grammaticos and Hietarinta [13], using Ziglin's methods [14], proved that the associated Hamiltonian system cannot be analytically integrable if $\alpha^2 \neq m(m-1)/2$ for m integer. Oliva and Castilla therefore remarked that these systems are C^∞ but not analytically integrable. For $0 \leq \alpha \leq 1$ the system defined by (1.14) does fit into our framework too (Corollary 10.7). In particular, *we cannot expect analytic integrability* in the present approach either.

Another well-known cone potential (also with cone wider than $\pi/2$) yielding an analytically completely integrable system is

$$\mathcal{V}(q) := \sum_{1 \leq i < j \leq n} \frac{1}{(q_i - q_j)^2}, \quad q \in \{(q_1, \dots, q_n) \in \mathbb{R}^n : q_1 < q_2 < \dots < q_n\}. \quad (1.15)$$

It was introduced by Calogero and Marchioro (see [3], [8], and [4]) as the classical counterpart to a certain quantum mechanical system. Marchioro proved (among many other results) the integrability by explicit calculation in the case $n = 3$. The integrability in the general case was conjectured by Calogero and proved by Moser [10] through isospectral deformations.

Moauero, Negrini and Oliva [private communication] introduced the potentials (1.10) with exponent $r = 2$ and put them into the framework of cone potentials. They proved the C^∞ -integrability for $n = 2$ and 3, even when the cone $\bar{\mathcal{C}}$ has amplitude larger than $\pi/2$. The compactification procedure, already successful in [12] for the exponential case (1.12), had to be supplemented with new ideas to overcome the degeneracies arising in this different situation. In particular, they use some interesting techniques that had been developed in [9] in connection with a Liapunov Stability problem.

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2. Cones

Definition 2.1 A cone in \mathbb{R}^n is a nonempty subset C of \mathbb{R}^n such that $v \in C, \lambda \geq 0 \Rightarrow \lambda v \in C$.

All the cones we will consider are convex, and many of them closed, too. Note that the closure of a (convex) cone is also a (convex) cone. We denote the closure of a subset A of \mathbb{R}^n either as \bar{A} or as $\text{cl}(A)$.

Definition 2.2 Given a convex cone C in \mathbb{R}^n , the dual of C is the set

$$C^* := \{w \in \mathbb{R}^n : w \cdot v \geq 0 \quad \forall v \in C\}.$$

It turns out that C^* is a closed convex cone and that $(C^*)^* = \bar{C}$.

Definition 2.3 A convex cone C in \mathbb{R}^n is called proper if its dual C^* has nonempty interior.

It is easy to see that a convex cone is proper if and only if its closure contains no straight lines. In fact, \bar{C} contains a straight line iff it contains both v and $-v$ for some $v \neq 0$. And this is equivalent to $C^* \subset \{w : w \cdot v = 0\}$.

There is a neat way to express the distance of a point of a convex cone from the boundary in terms of the dual cone. We will repeatedly use this formula.

Proposition 2.4 Let C be a convex cone in \mathbb{R}^n , not reduced to $\{0\}$, and let D be its dual. If $w \in D$ then

$$\text{dist}(w, \partial D) = \min_{\substack{v \in \bar{C} \\ |v|=1}} w \cdot v.$$

Proof. The distance of w from ∂D is the same as the distance from $\text{cl}(\mathbb{R}^n \setminus D)$, since $w \in D$. On the other hand, D is a closed convex cone, so we have

$$\text{cl}(\mathbb{R}^n \setminus D) = \bigcup_{\substack{v \in \bar{C} \\ |v|=1}} \{u \in \mathbb{R}^n : u \cdot v \leq 0\}.$$

In fact, from Hahn-Banach Theorem, for each $u \in \text{cl}(\mathbb{R}^n \setminus D)$, there is an affine function $z \mapsto z \cdot v + a$, with $|v| = 1$, that is nonnegative on the convex D and nonpositive on the half-line $\{\theta u : \theta \geq 0\}$. Since 0 belongs to both sets, the constant a is zero. Since $z \cdot v \geq 0$ for all $z \in D$, we have $v \in D^* = (C^*)^* = \bar{C}$. This proves the inclusion \subset . On the other hand, if $u \in \mathbb{R}^n$ and there exists $v \in \bar{C}, |v| = 1$, with $u \cdot v \leq 0$, then $u = \lim u_n$, where $u_n \cdot v < 0$. Hence $u_n \in \mathbb{R}^n \setminus D$ and $u \in \text{cl}(\mathbb{R}^n \setminus D)$.

We can write

$$\text{dist}(w, \partial D) = \inf_{\substack{v \in \bar{C} \\ |v|=1}} \text{dist}(w, \{u : u \cdot v \leq 0\}).$$

This last distance is the distance of w from a half-space. It thus coincides with $w \cdot v$. The infimum is finally a minimum because the set $\{v \in \bar{C} : |v| = 1\}$ is compact and $v \mapsto w \cdot v$ is continuous.

◇

In the case of a polyhedral cone, the formula becomes a minimum over a finite set. We will use it in Section 10.

Proposition 2.5 *Let \bar{C} be the closed cone generated by the vectors $v_1, \dots, v_N \in \mathbb{R}^n \setminus \{0\}$. Then the dual cone $D := C^*$ is given by $D = \{w : w \cdot v_\alpha \geq 0 \ \forall \alpha\}$ and for all $w \in D$ we have*

$$\text{dist}(w, \partial D) = \min_{\alpha} w \cdot \frac{v_\alpha}{|v_\alpha|}.$$

Proof. The formula for D is easy. Therefore $\mathbb{R}^n \setminus D = \bigcup_{\alpha} \{u : u \cdot v_\alpha < 0\}$. The union being finite, we can simply take the closure and write

$$\text{cl}(\mathbb{R}^n \setminus D) = \bigcup_{\alpha} \{u : u \cdot v_\alpha \leq 0\}.$$

The rest of the proof is the same as for the Proposition 2.4.

◇

Remark 2.6 Let D be a convex cone in \mathbb{R}^n , with nonempty interior and not coincident with all of \mathbb{R}^n . From Hahn-Banach Theorem, we can separate the origin $0 \in \mathbb{R}^n$ from the interior of D , i.e., there exists $v \in \mathbb{R}^n \setminus \{0\}$ such that $v \cdot w > 0 \ \forall w \in D^\circ$.

3. Counterexamples to Continuity

Consider the system (1.3) for the following potentials.

Counterexample 3.1 Let $\mathcal{V}: \mathbb{R} \rightarrow \mathbb{R}$, $q \mapsto e^{-q^3}$.

Obviously, this is a cone potential, that is, the conditions of Hypothesis 1.1 hold. In fact $\mathcal{V} > 0$ and $\bar{\mathcal{C}} = \mathbb{R}_+$, where \mathcal{C} is the cone generated by the forces—see (1.6). Moreover there is the equilibrium $(p, q) = (0, 0)$.

For $\lambda > 0$, let us consider $(p(\cdot, 0, \lambda), q(\cdot, 0, \lambda))$, i.e., the solution of (1.1) with $\bar{p} = 0$ and $\bar{q} = \lambda$ as initial conditions. Then $q(t, 0, \lambda) \rightarrow +\infty$ as $t \rightarrow +\infty$ for any $\lambda > 0$.

The first integral of energy (1.4) gives (for any t)

$$|p(t, 0, \lambda)|^2 + \mathcal{V}(q(t, 0, \lambda)) = e^{-\lambda^3}.$$

Therefore there is a discontinuity of the asymptotic velocity $(\bar{p}, \bar{q}) \mapsto p_\infty(\bar{p}, \bar{q})$ at $(\bar{p}, \bar{q}) = (0, 0)$ since $|p_\infty(0, \lambda)|^2 \rightarrow 1$ as $\lambda \rightarrow 0$ instead of $0 = |p_\infty(0, 0)|^2$ as for the equilibrium. \diamond

In the previous counterexample the lack of continuity is related with the presence of an equilibrium. However, the absence of equilibria is not sufficient to guarantee the continuity, as the following example shows.

Counterexample 3.2 Let $\mathcal{V}: \mathbb{R} \rightarrow \mathbb{R}$, $q \mapsto -\arctan q$.

It is a cone potential because $\mathcal{V} > -1$ and $\bar{\mathcal{C}} = \mathbb{R}_+$. Furthermore, there are no equilibria. If the initial position is $\bar{q} = 0$, and we conveniently choose the initial velocity $\bar{p} < 0$, then the corresponding solution $(p(t, \bar{p}, 0), q(t, \bar{p}, 0)) \rightarrow (0, -\infty)$ as $t \rightarrow +\infty$. From the conservation of energy

$$\frac{1}{2}|p(t, \bar{p}, 0)|^2 - \arctan q(t, \bar{p}, 0) = \frac{1}{2}|\bar{p}|^2,$$

we see that the good choice for the initial velocity is $\bar{p} = -\sqrt{\pi}$.

Now, let us reduce the initial speed, so that the motion reverses its direction at a certain time. We easily see that the solution $(p(\cdot, -\sqrt{\pi} + \lambda, 0), q(\cdot, -\sqrt{\pi} + \lambda, 0))$, for any $\lambda > 0$, has the following asymptotic behaviour:

$$q(t, -\sqrt{\pi} + \lambda, 0) \rightarrow +\infty, \quad |p(t, -\sqrt{\pi} + \lambda, 0)|^2 \rightarrow 2\pi - 2\lambda\sqrt{\pi} + \lambda^2 \quad \text{as } t \rightarrow +\infty.$$

The further limit as $\lambda \rightarrow 0+$ proves the discontinuity of $(\bar{p}, \bar{q}) \mapsto p_\infty(\bar{p}, \bar{q})$ at the point $(\bar{p}, \bar{q}) = (-\sqrt{\pi}, 0)$ (which are initial data of a solution with a vanishing asymptotic velocity, as we saw above). \diamond

Of course the preceding counterexample is possible because \mathcal{V} does not go to $+\infty$ as $q \rightarrow -\infty$, i.e., because we do not have a “barrier” in the direction opposite to the force.

So far we have seen that we must avoid the equilibria and we need a “barrier” in order that any motion eventually “points in the sense of the forces”. We shall give a precise formulation of these concepts in the next Section. Now let us give a last example to show that the barrier is not yet sufficient. For this we need two degrees of freedom.

Counterexample 3.3 $\mathcal{V}: \mathbb{R}^2 \rightarrow \mathbb{R}$, $(q_1, q_2) \mapsto e^{-q_1^3} + e^{-q_2}$.

This is a cone potential, $\{(v_1, v_2) : v_1 \geq 0, v_2 > 0\} \cup \{(0, 0)\}$ being the cone \mathcal{C} of the forces, and the dual cone coinciding in this case too with $\bar{\mathcal{C}}$. We do not have equilibria. The behaviour of the solutions can be easily investigated because the two degrees of freedom are separate. By Counterexample 3.1, we see at once that a discontinuity in p_∞ arises at the origin.

◇

What seems to go wrong in the third counterexample is that the force $-\nabla\mathcal{V}(q)$ does not drive toward the *interior* of the dual cone for the q along the axis $q_1 = 0$.

4. Geometrical Bounds for the Asymptotic Velocity

The basic ingredient of this work is the potential function \mathcal{V} , about which we started off with Hypothesis 1.1. From \mathcal{V} we constructed the Hamiltonian function \mathcal{H} , the Hamiltonian system (1.3), its solutions $(p(t, \bar{p}, \bar{q}), q(t, \bar{p}, \bar{q}))$, the asymptotic velocity $p_\infty(\bar{p}, \bar{q})$, the convex cone \mathcal{C} spanned by the forces (1.6) and its dual $\mathcal{D} := \mathcal{C}^*$ (1.7).

The next assumptions on \mathcal{V} are the first steps toward integrability.

Hypothesis 4.1 *For each $M > 0$ there exists a $q_M \in \mathbb{R}^n$ such that*

$$q \in \mathbb{R}^n \setminus (q_M + \mathcal{D}) \quad \Rightarrow \quad \mathcal{V}(q) \geq M.$$

Hypothesis 4.2 *For each $q', q'' \in \mathbb{R}^n$ such that $q'' \in q' + \mathcal{D}$, and for each $v \in \bar{\mathcal{C}} \setminus \{0\}$ there exists $\varepsilon > 0$ such that*

$$\left(q \in q' + \mathcal{D} \quad \text{and} \quad q \cdot v \leq q'' \cdot v \right) \quad \Rightarrow \quad -\nabla \mathcal{V}(q) \cdot v \geq \varepsilon.$$

Hypothesis 4.2 implies in particular that $-\nabla \mathcal{V}(q) \cdot v > 0$ for all $q \in \mathbb{R}^n$, and all $v \in \bar{\mathcal{C}} \setminus \{0\}$. Therefore $\mathcal{C} \subset \mathcal{D}$ and \mathcal{C} has amplitude not larger than $\pi/2$.

Proposition 4.3 *If Hypotheses 1.1, 4.1 and 4.2 hold, then*

$$p_\infty(\bar{p}, \bar{q}) \cdot v > 0 \tag{4.1}$$

for all $v \in \bar{\mathcal{C}} \setminus \{0\}$ and all initial data (\bar{p}, \bar{q}) . This is the same as saying that $p_\infty(\bar{p}, \bar{q})$ lies in the interior of the dual cone \mathcal{D} .

Proof. Let us fix the initial data (\bar{p}, \bar{q}) . The potential is bounded above along the trajectory:

$$\mathcal{V}(q(t, \bar{p}, \bar{q})) \leq \frac{1}{2} |\bar{p}|^2 + \mathcal{V}(\bar{q}) := M \quad \forall t \in \mathbb{R} \tag{4.2}$$

by the conservation of energy. Hypothesis 4.1 alone guarantees that $q(t, \bar{p}, \bar{q})$ remains in $q_M + \mathcal{D}$ for all times. This already implies that $p_\infty(\bar{p}, \bar{q})$ belongs to the closed set \mathcal{D} . Let $v \in \bar{\mathcal{C}} \setminus \{0\}$. As we remarked immediately after Hypothesis 4.2,

$$\dot{p}(t, \bar{p}, \bar{q}) \cdot v = -\nabla \mathcal{V}(q(t, \bar{p}, \bar{q})) \cdot v > 0$$

and so $t \mapsto p(t, \bar{p}, \bar{q}) \cdot v$ is an increasing function.

We argue by contradiction. If $p_\infty(\bar{p}, \bar{q}) \cdot v$ happened to be nonpositive, then

$$\dot{q}(t, \bar{p}, \bar{q}) \cdot v = p(t, \bar{p}, \bar{q}) \cdot v < 0 \quad \forall t \in \mathbb{R}$$

and $t \mapsto q(t, \bar{p}, \bar{q}) \cdot v$ would be decreasing. Hence

$$t \geq 0 \quad \Rightarrow \quad q(t, \bar{p}, \bar{q}) \cdot v \leq \bar{q} \cdot v.$$

Hypothesis 4.2 now yields that for $t \geq 0$ the scalar product $-\nabla\mathcal{V}(q(t, \bar{p}, \bar{q})) \cdot v$ is not less than some $\varepsilon > 0$. Thus we can write:

$$\begin{aligned} p(t, \bar{p}, \bar{q}) \cdot v &= \bar{p} \cdot v + \int_0^t -\nabla\mathcal{V}(q(s, \bar{p}, \bar{q})) \cdot v \, ds \geq \\ &\geq \bar{p} \cdot v + \varepsilon t \quad \rightarrow +\infty \quad \text{as } t \rightarrow +\infty, \end{aligned}$$

and this contradicts the assumption $p_\infty(\bar{p}, \bar{q}) \cdot v \leq 0$. Finally, formula (4.1) is equivalent to $p_\infty(\bar{p}, \bar{q}) \in \mathcal{D}^\circ$ because of Proposition 2.4. ◇

In the sequel we will need the following information about the trajectories: locally uniformly in the initial data,

- 1) the velocity $p(t, \bar{p}, \bar{q})$ is eventually in the interior of the dual cone, its distance from the boundary remains larger than a positive number γ , and
- 2) the position $q(t, \bar{p}, \bar{q})$ enters and no longer quits any set of the form $q_0 + \mathcal{D}$.

Proposition 4.4 *In the hypotheses of Proposition 4.3, for each $(\bar{p}_0, \bar{q}_0) \in \mathbb{R}^n \times \mathbb{R}^n$ and each $q_0 \in \mathbb{R}^n$ there exist $\gamma > 0$, $t_0 \in \mathbb{R}$ and a bounded neighbourhood U of (\bar{p}_0, \bar{q}_0) in $\mathbb{R}^n \times \mathbb{R}^n$ such that, for all $t \geq t_0$ and $(\bar{p}, \bar{q}) \in U$ we have*

$$\begin{aligned} p(t, \bar{p}, \bar{q}) &\in \mathcal{D}^\circ, & \text{dist}(p(t, \bar{p}, \bar{q}), \partial\mathcal{D}) &\geq \gamma, \\ q(t, \bar{p}, \bar{q}) &\in q_0 + \mathcal{D}, & \text{dist}(q(t, \bar{p}, \bar{q}), q_0 + \partial\mathcal{D}) &\geq \gamma(t - t_0). \end{aligned}$$

Proof. Since $p_\infty(\bar{p}_0, \bar{q}_0) \in \mathcal{D}^\circ$, let $\gamma := (1/2)\text{dist}(p_\infty(\bar{p}_0, \bar{q}_0), \partial\mathcal{D}) > 0$. Because of Proposition 2.4 and the continuity of the distance function, there exist $t_1 \in \mathbb{R}$ and a bounded neighbourhood U of (\bar{p}_0, \bar{q}_0) such that

$$p(t_1, \bar{p}, \bar{q}) \in \mathcal{D}^\circ, \quad \text{dist}(p(t_1, \bar{p}, \bar{q}), \partial\mathcal{D}) \geq \gamma$$

for all $(\bar{p}, \bar{q}) \in U$. But for all (\bar{p}, \bar{q}) and all $v \in \bar{\mathcal{C}} \setminus \{0\}$ the function $t \mapsto p(t, \bar{p}, \bar{q}) \cdot v$ is increasing, so that the velocity $p(t, \bar{p}, \bar{q})$ lies in \mathcal{D}° for all $t \geq t_1$ and all $(\bar{p}, \bar{q}) \in U$, and its distance from the boundary is not less than γ . For all $(\bar{p}, \bar{q}) \in U$, $t \geq t_1$, $v \in \bar{\mathcal{C}}$, $|v| = 1$, we have

$$\begin{aligned} (q(t, \bar{p}, \bar{q}) - q_0) \cdot v &= (q(t_0, \bar{p}, \bar{q}) - q_0) \cdot v + \int_{t_1}^t p(s, \bar{p}, \bar{q}) \cdot v \, ds \geq \\ &\geq \inf_{(\bar{p}, \bar{q}) \in U} \inf_{\substack{w \in \bar{\mathcal{C}} \\ |w|=1}} (q(t_1, \bar{p}, \bar{q}) - q_0) \cdot w + \gamma(t - t_1) := a + \gamma(t - t_1), \end{aligned}$$

and finally, for $t \geq t_0 := \max\{t_1, t_1 - (a/\gamma)\}$ the point $q(t, \bar{p}, \bar{q})$ belongs to $q_0 + \mathcal{D}$ and

$$\text{dist}(q(t, \bar{p}, \bar{q}), q_0 + \partial\mathcal{D}) = \min_{\substack{v \in \bar{\mathcal{C}} \\ |v|=1}} (q(t, \bar{p}, \bar{q}) - q_0) \cdot v \geq \gamma(t - t_0). \quad \diamond$$

We may ask what happens of \mathcal{V} and $\nabla\mathcal{V}$ along the trajectories.

Proposition 4.5 *In the hypotheses of Proposition 4.3, for all initial data $(\bar{p}, \bar{q}) \in \mathbb{R}^n \times \mathbb{R}^n$ we have*

$$\lim_{t \rightarrow +\infty} \mathcal{V}(q(t, \bar{p}, \bar{q})) = \inf \mathcal{V}.$$

Proof. Fix $\varepsilon > 0$ and pick $q_\varepsilon \in \mathbb{R}^n$ such that $\mathcal{V}(q_\varepsilon) \leq \inf \mathcal{V} + \varepsilon$. Let $q \in q_\varepsilon + \mathcal{D}$. Then

$$\begin{aligned} \mathcal{V}(q) - \mathcal{V}(q_\varepsilon) &= \int_0^1 \frac{d}{d\theta} \mathcal{V}(q_\varepsilon + \theta(q - q_\varepsilon)) d\theta = \\ &= \int_0^1 \nabla \mathcal{V}(q_\varepsilon + \theta(q - q_\varepsilon)) \cdot (q - q_\varepsilon) d\theta \leq 0, \end{aligned}$$

because $-\nabla \mathcal{V}(q_\varepsilon + \theta(q - q_\varepsilon)) \in \mathcal{C}$ and $q - q_\varepsilon \in \mathcal{D} = \mathcal{C}^*$. Hence we can write

$$q \in q_\varepsilon + \mathcal{D} \quad \Rightarrow \quad \mathcal{V}(q) \leq \inf \mathcal{V} + \varepsilon.$$

On the other hand, Proposition 4.4 guarantees, in particular, that for all (\bar{p}, \bar{q}) there exists $t_\varepsilon \in \mathbb{R}$ such that

$$t \geq t_\varepsilon \quad \Rightarrow \quad q(t, \bar{p}, \bar{q}) \in q_\varepsilon + \mathcal{D}.$$

This concludes the proof. ◇

Corollary 4.6 *In the hypotheses of Proposition 4.3, the following identity holds:*

$$\mathcal{H}(\bar{p}, \bar{q}) = \frac{1}{2} |\bar{p}|^2 + \mathcal{V}(\bar{q}) = \frac{1}{2} |p_\infty(\bar{p}, \bar{q})|^2 + \inf \mathcal{V}.$$

Within the assumptions of this Section, the gradient $\nabla\mathcal{V}$ does not need to be infinitesimal along the trajectories. Already in one dimension, it is not difficult to figure out a $\mathcal{V} \in C^2(\mathbb{R}, \mathbb{R})$ that decreases from $+\infty$ to 0, and whose graph has infinitely many smooth, but steep steps (whose height will obviously tend to zero):

$$\inf \mathcal{V} = 0, \quad \sup \mathcal{V} = +\infty, \quad \mathcal{V}' < 0, \quad \liminf_{q \rightarrow +\infty} \mathcal{V}'(q) < 0.$$

All the $q(t)$ will go to $+\infty$ as $t \rightarrow +\infty$ because our hypotheses are verified, so that they will never stop undergoing jerks ($\nabla\mathcal{V}$ does not converge). The assumptions of the next Section will rule out this possibility.

5. Continuity

This Section deals with the continuity of the asymptotic velocity with respect to the initial data. Gutkin in [7] already studied the problem, but in his setting he had no guarantee that p_∞ belonged to the interior of the dual cone for all the trajectories. Much less did he obtain such crucial estimates as the ones in Proposition 4.4. So he obtained the continuity in a nonempty, open subset of the space of the initial data, defined in terms of p_∞ itself.

In our assumptions, we get *global* continuity. We will also prove an asymptotic property of p_∞ , that will enable us later to determine the exact range of the mapping p_∞ (Proposition 7.5).

The asymptotic velocity $p_\infty(\bar{p}, \bar{q}) = \lim_{t \rightarrow +\infty} p(t, \bar{p}, \bar{q})$ can be expressed in terms of an integral:

$$p_\infty(\bar{p}, \bar{q}) = \bar{p} + \int_0^{+\infty} -\nabla\mathcal{V}(q(s, \bar{p}, \bar{q})) ds. \quad (5.1)$$

In the hypotheses of the last section, we know that $q(\cdot, \bar{p}, \bar{q})$ is always contained in $q_M + \mathcal{D}$ (see formula (4.2)). Moreover, $p_\infty(\bar{p}, \bar{q})$ is in the *interior* of the dual cone \mathcal{D} , so that the distance of $q(t, \bar{p}, \bar{q})$ from the boundary of $q_M + \mathcal{D}$ grows linearly as $t \rightarrow +\infty$, as we saw in Proposition 4.4. We may expect p_∞ to be a continuous function of (\bar{p}, \bar{q}) if the norm $|\nabla\mathcal{V}(q)|$ is dominated by an integrable function of the distance between q and $q_M + \partial\mathcal{D}$. We may thus use a uniform integrability theorem on the integral (5.1).

Hypothesis 5.1 *There exist $q_0 \in \mathbb{R}^n$ and an weakly decreasing, integrable function $h_0: \mathbb{R}_+ \rightarrow \mathbb{R}$ such that*

$$q \in q_0 + \mathcal{D} \quad \Rightarrow \quad |\nabla\mathcal{V}(q)| \leq h_0\left(\text{dist}(q, q_0 + \partial\mathcal{D})\right).$$

Proposition 5.2 *If Hypotheses 1.1, 4.1, 4.2 and 5.1 hold, then p_∞ is a continuous function of the initial data.*

Proof. Let (\bar{p}_0, \bar{q}_0) be fixed and pick $\gamma > 0$, $t_0 \in \mathbb{R}$ and U from Proposition 4.4:

$$q(t, \bar{p}, \bar{q}) \in q_0 + \mathcal{D}, \quad \text{dist}(q(t, \bar{p}, \bar{q}), q_0 + \partial\mathcal{D}) \geq \gamma(t - t_0)$$

for all $t \geq t_0$, $(\bar{p}, \bar{q}) \in U$. Using Hypothesis 5.1,

$$|\nabla\mathcal{V}(q(t, \bar{p}, \bar{q}))| \leq h_0\left(\text{dist}(q(t, \bar{p}, \bar{q}), q_0 + \partial\mathcal{D})\right) \leq h_0(\gamma(t - t_0)),$$

so that we can apply the theorems on uniform integrability to the formula

$$p_\infty(\bar{p}, \bar{q}) = p(t_0, \bar{p}, \bar{q}) + \int_{t_0}^{+\infty} -\nabla\mathcal{V}(q(s, \bar{p}, \bar{q})) ds$$

and obtain our continuity result. ◇

We are now provided with n continuous integrals of motion: the components of the asymptotic velocity $p_\infty(\bar{p}, \bar{q})$.

Roughly speaking, if we find a region $q + \mathcal{D}$ where the driving force $-\nabla\mathcal{V}$ is utterly negligible, we may expect that, if we start the motion there, with a velocity \bar{p} in the interior of \mathcal{D} , then the motion has approximately constant speed:

$$p(t, \bar{p}, \bar{q}) \approx \bar{p}$$

so that, for those initial data (\bar{p}, \bar{q}) we have

$$p_\infty(\bar{p}, \bar{q}) \approx \bar{p}. \tag{5.3}$$

Proposition 5.3 *In the hypotheses of Proposition 5.2, for each $\mu > 0$ and each $\gamma > 0$, there exists $q'_0 \in \mathbb{R}^n$ such that*

$$\left(\bar{p} \in \mathcal{D}^\circ, \quad \text{dist}(\bar{p}, \partial\mathcal{D}) \geq \gamma, \quad \bar{q} \in q'_0 + \mathcal{D} \right) \Rightarrow |p_\infty(\bar{p}, \bar{q}) - \bar{p}| \leq \mu.$$

Proof. Let $\mu > 0$, $\gamma > 0$ be fixed and pick q_0 from Hypothesis 5.1. Let $d_0 \geq 0$ be such that

$$\int_{d_0}^{+\infty} h_0(\gamma t) dt \leq \mu.$$

Choose $q'_0 \in q_0 + \mathcal{D}$ such that $\text{dist}(q'_0, q_0 + \partial\mathcal{D}) \geq d_0$. Then, for all $\bar{p} \in \mathcal{D}^\circ$ such that $\text{dist}(\bar{p}, \partial\mathcal{D}) \geq \gamma$ and $\bar{q} \in q'_0 + \mathcal{D}$ we have

$$\text{dist}(q(t, \bar{p}, \bar{q}), q_0 + \partial\mathcal{D}) \geq \gamma t + d_0$$

and so

$$\begin{aligned} |p_\infty(\bar{p}, \bar{q}) - \bar{p}| &= \left| \int_0^{+\infty} -\nabla\mathcal{V}(q(t, \bar{p}, \bar{q})) dt \right| \leq \\ &\leq \int_0^{+\infty} h_0(\gamma t + d_0) dt = \int_{d_0}^{+\infty} h_0(\gamma t) dt \leq \mu. \end{aligned}$$

◇

6. First Order Differentiability without Convexity on the Potential

In the Hypotheses 1.1 we know that the velocity $p(t, \bar{p}, \bar{q})$ is a differentiable function of the initial data at all finite times t . If we differentiate the equation

$$p(t, \bar{p}, \bar{q}) = p(t_0, \bar{p}, \bar{q}) + \int_{t_0}^t -\nabla \mathcal{V}(q(s, \bar{p}, \bar{q})) ds \quad (6.1)$$

with respect to an arbitrary component of \bar{p} or \bar{q} , we obtain

$$Dp(t, \bar{p}, \bar{q}) = Dp(t_0, \bar{p}, \bar{q}) + \int_{t_0}^t -\text{H}\mathcal{V}(q(s, \bar{p}, \bar{q})) Dq(s, \bar{p}, \bar{q}) ds, \quad (6.2)$$

where we denote by $\text{H}\mathcal{V}$ the Hessian matrix of \mathcal{V} and by D the partial derivative (we reserve the character D for the Jacobian matrix).

In the hypotheses of Section 4, we know that formula (6.1) holds with $t = +\infty$ and the integrability is uniform. Can we expect the same for (6.2)? What we seem to need is:

- 1) an a priori bound on the growth of $Dq(t, \bar{p}, \bar{q})$, locally uniform on (\bar{p}, \bar{q}) ;
- 2) a rapid enough decrease of the norm of the Hessian $\text{H}\mathcal{V}$ along the trajectories $t \mapsto q(t, \bar{p}, \bar{q})$.

If the two estimates match appropriately, we can use the theorems on the differentiation under the integral sign.

The function $z(t) = Dq(t, \bar{p}, \bar{q})$ satisfies the linear differential equation

$$\ddot{z}(t) = -\text{H}\mathcal{V}(q(t, \bar{p}, \bar{q})) z(t), \quad (6.3)$$

that can also be rewritten as a first-order system:

$$\frac{d}{dt} \begin{pmatrix} z(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -\text{H}\mathcal{V}(q(t, \bar{p}, \bar{q})) & 0 \end{pmatrix} \begin{pmatrix} z(t) \\ \dot{z}(t) \end{pmatrix}.$$

Let us denote by $R(t, s, \bar{p}, \bar{q})$ the evolution operator of the system, i.e., the $2n \times 2n$ matrix solution of

$$\frac{\partial}{\partial t} R(t, s, \bar{p}, \bar{q}) = \begin{pmatrix} 0 & I_n \\ -\text{H}\mathcal{V}(q(t, \bar{p}, \bar{q})) & 0 \end{pmatrix} R(t, s, \bar{p}, \bar{q}), \quad R(s, s, \bar{p}, \bar{q}) = I_{2n}.$$

Let Π and Π' be the two projections $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ defined as $\Pi(x, y) = x$, $\Pi'(x, y) = y$. Since

$$Dq(t, \bar{p}, \bar{q}) = \Pi R(t, 0, \bar{p}, \bar{q}) \begin{pmatrix} Dq(0, \bar{p}, \bar{q}) \\ Dp(0, \bar{p}, \bar{q}) \end{pmatrix} = \Pi R(t, 0, \bar{p}, \bar{q}) \begin{pmatrix} D\Pi' \\ D\Pi \end{pmatrix},$$

what we are interested in is the behaviour of $\|\Pi R(t, s, \bar{p}, \bar{q})\|$ as $t \rightarrow +\infty$.

We will carry out this program in two sets of hypotheses. In the rest of this Section our assumptions will be as follows.

Hypothesis 6.1 *There exist $q_1 \in \mathbb{R}^n$, $A_1 \geq 0$, $\lambda_1 > 0$ such that*

$$q \in q_1 + \mathcal{D} \quad \Rightarrow \quad \|\mathbf{H}\mathcal{V}(q)\| \leq A_1 \exp\left(-\lambda_1 \operatorname{dist}(q, q_1 + \partial\mathcal{D})\right).$$

On one hand, the mere fact that the Hessian is infinitesimal along the trajectories $t \mapsto q(t, \bar{p}, \bar{q})$ ensures, via a Gronwall lemma, that $Dq(t, \bar{p}, \bar{q})$ must grow *less than exponentially* as $t \rightarrow +\infty$ (i.e., it is $o(e^{\varepsilon t})$ for all $\varepsilon > 0$). On the other hand the actual *exponential* decrease of the Hessian compensates for the other growth and yields the uniform integrability of (6.2).

Lemma 6.2 *Suppose that Hypotheses 1.1, 4.1, 4.2 and 6.1 hold. Then, for all $\varepsilon > 0$, and for all $t \geq 0$, $x, y \in \mathbb{R}^n$ we have*

$$\begin{aligned} \left(\bar{p} \in \mathcal{D}, \quad \bar{q} \in q_1 + \mathcal{D}, \quad \operatorname{dist}(\bar{q}, q_1 + \partial\mathcal{D}) \geq -\frac{1}{\lambda_1} \ln \frac{\varepsilon^2}{A_1} \right) &\Rightarrow \\ \Rightarrow \quad \left| \Pi R(t, 0, \bar{p}, \bar{q}) \begin{pmatrix} x \\ y \end{pmatrix} \right| &\leq \frac{1}{2\varepsilon} \left((\varepsilon|x| + |y|)e^{\varepsilon t} + (\varepsilon|x| - |y|)e^{-\varepsilon t} \right). \end{aligned}$$

Proof. Choose $\varepsilon > 0$, $\bar{q} \in q_1 + \mathcal{D}$ such that $\operatorname{dist}(\bar{q}, q_1 + \partial\mathcal{D}) \geq (-1/\lambda_1) \ln(\varepsilon^2/A_1)$. For any $\bar{p} \in \mathcal{D}$, we have

$$\operatorname{dist}(q(t, \bar{p}, \bar{q}), q_1 + \partial\mathcal{D}) \geq \operatorname{dist}(\bar{q}, q_1 + \partial\mathcal{D})$$

and so

$$\begin{aligned} \|\mathbf{H}\mathcal{V}(q(t, \bar{p}, \bar{q}))\| &\leq A_1 \exp\left(-\lambda_1 \operatorname{dist}(q(t, \bar{p}, \bar{q}), q_1 + \partial\mathcal{D})\right) \leq \\ &\leq A_1 \exp\left(-\lambda_1 \operatorname{dist}(\bar{q}, q_1 + \partial\mathcal{D})\right) \leq \varepsilon^2. \end{aligned}$$

For any $x, y \in \mathbb{R}^n$, the function $z(t) = \Pi R(t, 0, \bar{p}, \bar{q}) \begin{pmatrix} x \\ y \end{pmatrix}$ is a solution of (6.3), that can be rewritten in integral form:

$$z(t) = x + ty + \int_0^t -(t-s) \mathbf{H}\mathcal{V}(q(s, \bar{p}, \bar{q})) z(s) ds.$$

Taking the norms,

$$\begin{aligned} |z(t)| &\leq |x| + t|y| + \int_0^t (t-s) \|\mathbf{H}\mathcal{V}(q(s, \bar{p}, \bar{q}))\| |z(s)| ds \leq \\ &\leq |x| + t|y| + \varepsilon \int_0^t (t-s) |z(s)| ds. \end{aligned}$$

A standard Gronwall argument yields that $|z(t)| \leq \varphi(t)$ for $t \geq 0$, where φ is the solution of

$$\varphi(t) = |x| + t|y| + \varepsilon \int_0^t (t-s) \varphi(s) ds,$$

and is precisely the expression appearing in the statement of the Lemma. ◇

Proposition 6.3 *Suppose that Hypotheses 1.1, 4.1, 4.2, and 6.1 hold. Then the asymptotic velocity p_∞ is a C^1 function of the initial data.*

Proof. What we need is local uniform integrability of the integral in (6.2) for some $t_0 \in \mathbb{R}$. Choose an initial condition (\bar{p}_0, \bar{q}_0) . From Proposition 4.4, there exist $\gamma > 0$, $t_1 \in \mathbb{R}$ and a bounded neighbourhood U of (\bar{p}_0, \bar{q}_0) such that

$$\begin{aligned} p(t, \bar{p}, \bar{q}) &\in \mathcal{D}^\circ, & \text{dist}(p(t, \bar{p}, \bar{q}), \partial\mathcal{D}) &\geq \gamma, \\ q(t, \bar{p}, \bar{q}) &\in q_1 + \mathcal{D}, & \text{dist}(q(t, \bar{p}, \bar{q}), q_1 + \partial\mathcal{D}) &\geq \gamma(t - t_1) \end{aligned}$$

for all $t \geq t_1$, $(\bar{p}, \bar{q}) \in U$. So we have

$$\| \text{HV}(q(t, \bar{p}, \bar{q})) \| \leq A_1 e^{-\lambda_1 \gamma (t - t_1)}$$

for all $t \geq t_1$, $(\bar{p}, \bar{q}) \in U$. Now choose $\varepsilon > 0$ and $t_0 \geq t_1$ such that

$$0 < \varepsilon < \lambda_1 \gamma \quad \text{and} \quad A_1 e^{-\lambda_1 \gamma (t_0 - t_1)} \leq \varepsilon^2.$$

Since

$$Dq(t, \bar{p}, \bar{q}) = \Pi R(s, 0, \bar{p}, \bar{q}) \begin{pmatrix} D\Pi' \\ D\Pi \end{pmatrix} = \Pi R(s - t_0, 0, p(t_0, \bar{p}, \bar{q}), q(t_0, \bar{p}, \bar{q})) \begin{pmatrix} Dq(t_0, \bar{p}, \bar{q}) \\ Dp(t_0, \bar{p}, \bar{q}) \end{pmatrix}, \quad (6.4)$$

from Lemma 6.2 we get that

$$|Dq(t, \bar{p}, \bar{q})| \leq a_1 e^{\varepsilon(t - t_0)} + a_2 \quad (6.5)$$

for all $t \geq t_0$ and all $(\bar{p}, \bar{q}) \in U$, where

$$a_1 := \sup_{(\bar{p}, \bar{q}) \in U} \frac{1}{2\varepsilon} \left(\varepsilon |Dq(t_0, \bar{p}, \bar{q})| + |Dp(t_0, \bar{p}, \bar{q})| \right), \quad a_2 := \sup_{(\bar{p}, \bar{q}) \in U} \frac{1}{2} |Dq(t_0, \bar{p}, \bar{q})|.$$

We can finally write, for all $t \geq t_0$, $(\bar{p}, \bar{q}) \in U$:

$$| -\text{HV}(q(t, \bar{p}, \bar{q})) Dq(t, \bar{p}, \bar{q}) | \leq A_1 e^{-\lambda_1 \gamma (t - t_1)} (a_1 e^{\varepsilon(t - t_0)} + a_2)$$

and we are done. ◇

The approximate equality (5.3) extends to the derivatives of the functions involved. The character D stands for the Jacobian matrix.

Proposition 6.4 *In the hypotheses of Proposition 6.3, for each $\mu > 0$ and for each $\gamma > 0$ there exists $d_0 \geq 0$ such that*

$$\left(\begin{array}{l} \bar{p} \in \mathcal{D}^\circ, \quad \text{dist}(\bar{p}, \partial\mathcal{D}) \geq \gamma, \\ \bar{q} \in q_1 + \mathcal{D}, \quad \text{dist}(\bar{q}, q_1 + \partial\mathcal{D}) \geq d_0 \end{array} \right) \Rightarrow \left\| Dp_\infty(\bar{p}, \bar{q}) - D\Pi(\bar{p}, \bar{q}) \right\| \leq \mu.$$

Proof. Fix $\gamma > 0$. For any $d_0 \geq 0$ such that

$$(A_1 e^{-\lambda_1 d_0})^{1/2} \leq \frac{\lambda_1 \gamma}{2}$$

we set $\varepsilon = (A_1 e^{-\lambda_1 d_0})^{1/2}$. Applying Lemma 6.2 we get that for all $\bar{p} \in \mathcal{D}$, $\bar{q} \in q_1 + \mathcal{D}$ such that

$$\text{dist}(\bar{q}, q_1 + \partial\mathcal{D}) \geq -\frac{1}{\lambda_1} \ln \frac{\varepsilon^2}{A_1} = d_0$$

and for all $t \geq 0$, we have

$$\begin{aligned} |Dq(t, \bar{p}, \bar{q})| &= \left| \Pi R(t, 0, \bar{p}, \bar{q}) \begin{pmatrix} D\Pi' \\ D\Pi \end{pmatrix} \right| \leq \frac{1}{2\varepsilon} (\varepsilon |D\Pi'| + |D\Pi|) e^{\varepsilon t} + \frac{1}{2} |D\Pi'| \leq \\ &\leq \frac{1}{2\varepsilon} \left(\frac{\lambda_1 \gamma}{2} |D\Pi'| + |D\Pi| \right) e^{\lambda_1 \gamma t/2} + \frac{1}{2} |D\Pi'|. \end{aligned}$$

On the other hand, if moreover $\text{dist}(\bar{p}, \partial\mathcal{D}) \geq \gamma$, we have

$$\text{dist}(q(t, \bar{p}, \bar{q}), q_1 + \partial\mathcal{D}) \geq d_0 + \gamma t,$$

so that $\|\text{HV}(q(t, \bar{p}, \bar{q}))\| \leq A_1 e^{-\lambda_1 \gamma t - \lambda_1 d_0}$. Putting the pieces together, and reminding that $\varepsilon = (A_1 e^{-\lambda_1 d_0})^{1/2}$:

$$\begin{aligned} \left| Dp_\infty(\bar{p}, \bar{q}) - D\Pi(\bar{p}, \bar{q}) \right| &= \left| \int_0^{+\infty} -\text{HV}(q(t, \bar{p}, \bar{q})) Dq(t, \bar{p}, \bar{q}) dt \right| \leq \\ &\leq A_1^{1/2} \left(\frac{|D\Pi'|}{2} + \frac{|D\Pi|}{\lambda_1 \gamma} \right) e^{-\lambda_1 d_0/2} + \frac{A_1 |D\Pi'|}{2\lambda_1 \gamma} e^{-\lambda_1 d_0}. \end{aligned}$$

It is clear that we can choose d_0 so large that the last quantity is as small as we wish. \diamond

7. First Order Differentiability with convexity on the potential

If we assume that the potential \mathcal{V} is a *convex* function, then the Hessian matrix $\mathbf{H}\mathcal{V}$ is nonnegative definite, and, from equation (6.3),

$$\ddot{z}(t) = -\mathbf{H}\mathcal{V}(q(t, \bar{p}, \bar{q})) z(t), \quad (7.1)$$

it follows that $\ddot{z} \cdot z \leq 0$. This lets us hope that we can derive a much sharper estimate on $|Dq(t, \bar{p}, \bar{q})|$ than the mere less-than-exponential of Section 6. We will also assume that the quadratic form associated with $\mathbf{H}\mathcal{V}(q)$ behaves monotonically with respect to q . Supposing \mathcal{V} to be three times differentiable is not strictly necessary, but will simplify the proofs.

Hypothesis 7.1 \mathcal{V} is a C^3 function and there exists $q_1 \in \mathbb{R}^n$ such that

- i) \mathcal{V} is convex on $q_1 + \mathcal{D}$;
- ii) for all $q', q'' \in q_1 + \mathcal{D}$ and all $z \in \mathbb{R}^n$ we have

$$q'' \in q' + \mathcal{D} \quad \Rightarrow \quad \mathbf{H}\mathcal{V}(q'')z \cdot z \leq \mathbf{H}\mathcal{V}(q')z \cdot z;$$

- iii) there exists a weakly decreasing function $h_1: \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $\int_0^{+\infty} x h_1(x) dx < +\infty$ and

$$q \in q_1 + \mathcal{D} \quad \Rightarrow \quad \|\mathbf{H}\mathcal{V}(q)\| \leq h_1\left(\text{dist}(q, q_1 + \partial\mathcal{D})\right).$$

Lemma 7.2 Suppose that Hypotheses 1.1, 4.1, 4.2 and 7.1 i), ii) hold. Then, for all $x, y \in \mathbb{R}^n$ and $t \geq 0$ we have

$$\left(\bar{p} \in \mathcal{D}, \quad \bar{q} \in q_1 + \mathcal{D}\right) \quad \Rightarrow \quad \begin{cases} |\Pi R(t, 0, \bar{p}, \bar{q})\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)| \leq |x| + t(|y| + \|\mathbf{H}\mathcal{V}(q_1)\|^{1/2} |x|), \\ |\Pi' R(t, 0, \bar{p}, \bar{q})\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)| \leq |y| + \|\mathbf{H}\mathcal{V}(q_1)\|^{1/2} |x|. \end{cases}$$

Proof. Let $\bar{p} \in \mathcal{D}$, $\bar{q} \in q_1 + \mathcal{D}$ be fixed. Let $z(t) = \Pi R(t, 0, \bar{p}, \bar{q})\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$, and consider the following Liapunov function:

$$L(t, \bar{p}, \bar{q}) = |\dot{z}(t)|^2 + \mathbf{H}\mathcal{V}(q(t, \bar{p}, \bar{q}))z(t) \cdot z(t).$$

L is a nonnegative quantity because \mathcal{V} is convex in the points where the Hessian is evaluated. We are going to prove that L is decreasing in t for $t \geq 0$. Take the derivative with respect to t , and remind equation (7.1):

$$\begin{aligned} \frac{d}{dt}L(t, \bar{p}, \bar{q}) &= 2\dot{z}(t) \cdot \ddot{z}(t) + 2\mathbf{H}\mathcal{V}(q(t, \bar{p}, \bar{q}))z(t) \cdot \dot{z}(t) + \left(\frac{d}{dt}(\mathbf{H}\mathcal{V}(q(t, \bar{p}, \bar{q})))\right)z(t) \cdot z(t) = \\ &= \lim_{s \searrow t} \frac{1}{s-t} (\mathbf{H}\mathcal{V}(q(s, \bar{p}, \bar{q})) - \mathbf{H}\mathcal{V}(q(t, \bar{p}, \bar{q})))z(t) \cdot z(t). \end{aligned}$$

This expression is nonpositive because of Hypothesis 7.1 ii) and because $q(s, \bar{p}, \bar{q}) \in q(t, \bar{p}, \bar{q}) + \mathcal{D}$ for $s \geq t \geq 0$. We thus have, for all $t \geq 0$:

$$|\dot{z}(t)|^2 \leq L(t, \bar{p}, \bar{q}) \leq L(0, \bar{p}, \bar{q}) \leq |\dot{z}(0)|^2 + \mathbf{H}\mathcal{V}(\bar{q})z(0) \cdot z(0) \leq |y|^2 + \mathbf{H}\mathcal{V}(q_1)x \cdot x,$$

and so

$$\left|\Pi' R(t, 0, \bar{p}, \bar{q})\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right| = |\dot{z}(t)| \leq |y| + \|\mathbf{H}\mathcal{V}(q_1)\|^{1/2} |x|.$$

The other inequality comes from the last one and from $|z(t)| \leq |x| + \int_0^t |\dot{z}(s)| ds$.

◇

Proposition 7.3 *Suppose that Hypotheses 1.1, 4.1, 4.2 and 7.1 hold. Then the asymptotic velocity p_∞ is a C^1 function of the initial data.*

Proof. Let (\bar{p}_0, \bar{q}_0) , $\gamma > 0$, $t_1 \in \mathbb{R}$, U as in the proof of Proposition 6.3. Then, for all $(\bar{p}, \bar{q}) \in U$ and $t \geq t_1$ we have

$$\| \mathbf{HV}(q(t, \bar{p}, \bar{q})) \| \leq h_1(\gamma(t - t_1)).$$

On the other hand, from Lemma 7.2 and formula (6.4), we get that, again for all $(\bar{p}, \bar{q}) \in U$, $t \geq t_1$:

$$|Dq(t, \bar{p}, \bar{q})| \leq a_1 + a_2(t - t_1),$$

where

$$a_1 := \sup_{(\bar{p}, \bar{q}) \in U} |Dq(t_1, \bar{p}, \bar{q})|, \quad a_2 := \sup_{(\bar{p}, \bar{q}) \in U} \left(|Dp(t_1, \bar{p}, \bar{q})| + \| \mathbf{HV}(q_1) \|^{1/2} |Dq(t_1, \bar{p}, \bar{q})| \right).$$

We can finally write, for all $(\bar{p}, \bar{q}) \in U$, $t \geq t_1$:

$$| - \mathbf{HV}(q(t, \bar{p}, \bar{q})) Dq(t, \bar{p}, \bar{q}) | \leq (a_1 + a_2(t - t_1)) h_1(\gamma(t - t_1))$$

and we are done. ◇

Also Proposition 6.4 remains true in the modified hypotheses.

Proposition 7.4 *In the hypotheses of Proposition 7.3, for each $\mu > 0$, and each $\gamma > 0$ there exists $d_0 \geq 0$ such that*

$$\left(\begin{array}{l} \bar{p} \in \mathcal{D}^\circ, \quad \text{dist}(\bar{p}, \partial\mathcal{D}) \geq \gamma, \\ \bar{q} \in q_1 + \mathcal{D}, \quad \text{dist}(\bar{q}, q_1 + \partial\mathcal{D}) \geq d_0 \end{array} \right) \Rightarrow \left\| Dp_\infty(\bar{p}, \bar{q}) - D\Pi(\bar{p}, \bar{q}) \right\| \leq \mu.$$

Proof. Fix $\mu > 0$ and $\gamma > 0$. Let $\bar{p} \in \mathcal{D}^\circ$, $\text{dist}(\bar{p}, \partial\mathcal{D}) \geq \gamma$, $\bar{q} \in q_1 + \mathcal{D}$ and $\text{dist}(\bar{q}, q_1 + \partial\mathcal{D}) \geq d_0$. Then, for all $t \geq 0$

$$\| \mathbf{HV}(q(t, \bar{p}, \bar{q})) \| \leq h_1(\gamma t + d_0).$$

Moreover, from Lemma 7.2, we get

$$\begin{aligned} |Dq(t, \bar{p}, \bar{q})| &= \left| \Pi R(t, 0, \bar{p}, \bar{q}) \begin{pmatrix} D\Pi' \\ D\Pi \end{pmatrix} \right| \leq |D\Pi'| + t \left(|D\Pi'| + \| \mathbf{HV}(q_1) \|^{1/2} |D\Pi| \right) := \\ &:= b_1 + b_2 t. \end{aligned}$$

In conclusion,

$$\begin{aligned} |Dp_\infty(\bar{p}, \bar{q}) - D\Pi(\bar{p}, \bar{q})| &= \left| \int_0^{+\infty} - \mathbf{HV}(q(t, \bar{p}, \bar{q})) Dq(t, \bar{p}, \bar{q}) dt \right| \leq \\ &\leq \int_0^{+\infty} (b_1 + b_2 t) h_1(\gamma t + d_0) dt. \end{aligned}$$

It is clear that we can choose d_0 so large that the last integral is as small as needed. ◇

Proposition 7.5 *In the hypotheses of Proposition 5.3 and either 6.4 or 7.6, the mapping p_∞ is surjective from $\mathbb{R}^n \times \mathbb{R}^n$ onto \mathcal{D}° .*

Proof. From Proposition 4.3, the image of p_∞ is contained in \mathcal{D}° . To prove the reverse inclusion, let $\bar{p}_0 \in \mathcal{D}^\circ$. We can solve the equation $p_\infty(\bar{p}, \bar{q}) = \bar{p}_0$ via the contraction principle. Let $\gamma := \text{dist}(\bar{p}_0, \partial\mathcal{D}) > 0$. From Proposition 5.3 and either 6.4 or 7.6, there exists $\bar{q} \in \mathbb{R}^n$ such that

$$|p_\infty(\bar{p}, \bar{q}) - \bar{p}| \leq \frac{\gamma}{2}, \quad \|Dp_\infty(\bar{p}, \bar{q}) - \text{D}\Pi\| \leq \frac{1}{2}$$

for all $\bar{p} \in \mathcal{D}$ such that $\text{dist}(\bar{p}, \partial\mathcal{D}) \geq \gamma/2$. Now the mapping $\bar{p} \mapsto \bar{p}_0 + \bar{p} - p_\infty(\bar{p}, \bar{q})$ is a contraction of the closed ball $\{\bar{p} \in \mathbb{R}^n : |\bar{p} - \bar{p}_0| \leq \gamma/2\}$ into itself. The corresponding fixed point \bar{p} solves $p_\infty(\bar{p}, \bar{q}) = \bar{p}_0$. Actually, we could make it without differentiability, in the mere hypotheses of Propositions 5.2 and 5.3, if we were willing to conjure up Brouwer's fixed point theorem.

◇

8. Higher Order Differentiability

Let us denote by D_1, D_2 the partial derivative operators with respect to any two components of (\bar{p}, \bar{q}) , by $D_{1,2}$ the second order derivative $D_1 D_2$, and by $D^3 \mathcal{V}(q)$ the third differential of the potential \mathcal{V} , regarded as a bilinear operator from $\mathbb{R}^n \times \mathbb{R}^n$ into \mathbb{R}^n , with the norm

$$\|D^3 \mathcal{V}(q)\| := \sup_{|x| \leq 1, |y| \leq 1} |D^3 \mathcal{V}(q)(x, y)|.$$

To get the second order differentiability of the asymptotic velocity p_∞ , what we need is (local) uniform integrability of

$$\int_0^{+\infty} \left(- \underbrace{D^3 \mathcal{V}(q(t, \bar{p}, \bar{q})) \left(D_1 q(t, \bar{p}, \bar{q}), D_2 q(t, \bar{p}, \bar{q}) \right)}_{:= r(t, \bar{p}, \bar{q})} - \text{HV}(q(t, \bar{p}, \bar{q})) D_{1,2} q(t, \bar{p}, \bar{q}) \right) dt. \quad (8.1)$$

To this purpose we must obtain an estimate on the growth of $z(t) = D_{1,2} q(t, \bar{p}, \bar{q})$, which is a solution of the non-homogeneous linear differential equation:

$$\ddot{z}(t) = - \text{HV}(q(t, \bar{p}, \bar{q})) z(t) - r(t, \bar{p}, \bar{q}),$$

or, in first order system form:

$$\frac{d}{dt} \begin{pmatrix} z(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -\text{HV}(q(t, \bar{p}, \bar{q})) & 0 \end{pmatrix} \begin{pmatrix} z(t) \\ \dot{z}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ -r(t, \bar{p}, \bar{q}) \end{pmatrix}.$$

Remind the evolution operator R introduced in Section 6. The function $z(t)$ can be expressed via R with the classical method of variation of the constants:

$$\begin{pmatrix} z(t) \\ \dot{z}(t) \end{pmatrix} = \int_0^t R(t, s, \bar{p}, \bar{q}) \begin{pmatrix} 0 \\ -r(s, \bar{p}, \bar{q}) \end{pmatrix} ds \quad (8.2)$$

(note that in our case $z(0) = \dot{z}(0) = 0$).

We already have two sets of hypotheses that give an estimate of the evolution operator. All we are left to do is to give bounds on $r(t, \bar{p}, \bar{q})$.

In the setting of Section 6, $D_1 q(t, \bar{p}, \bar{q})$ and $D_2 q(t, \bar{p}, \bar{q})$ grow less than exponentially as $t \rightarrow +\infty$. If we assume that $\|D^3 \mathcal{V}(q(t, \bar{p}, \bar{q}))\|$ decreases exponentially, then our scheme seems to work out.

Hypothesis 8.1 \mathcal{V} is a C^3 function and there exist $q_2 \in \mathbb{R}^n$, $A_2 \geq 0$, $\lambda_2 > 0$ such that

$$q \in q_2 + \mathcal{D} \quad \Rightarrow \quad \|D^3 \mathcal{V}(q)\| \leq A_2 \exp\left(-\lambda_2 \text{dist}(q, q_2 + \partial \mathcal{D})\right).$$

We can safely assume that q_2 coincides with the q_1 of Hypothesis 6.1.

In the frame of Section 7, $D_1q(t, \bar{p}, \bar{q})$ and $D_2q(t, \bar{p}, \bar{q})$ grow linearly as $t \rightarrow +\infty$. Therefore the following assumption seems appropriate.

Hypothesis 8.2 \mathcal{V} is a C^3 function and there exist $q_2 \in \mathbb{R}^n$ and a weakly decreasing function $h_2: \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $\int_0^{+\infty} x^2 h_2(x) dx < +\infty$ and

$$q \in q_2 + \mathcal{D} \quad \Rightarrow \quad \|D^3\mathcal{V}(q)\| \leq h_2\left(\text{dist}(q, q_2 + \partial\mathcal{D})\right).$$

Here, too, we can assume q_2 to coincide with the q_1 of Hypothesis 7.1.

Proposition 8.3 Suppose that Hypotheses 1.1, 4.1, 4.2, 6.1 and 8.1 hold. Then the asymptotic velocity p_∞ is a C^2 function of the initial data.

Proof. Let (\bar{p}_0, \bar{q}_0) , $\gamma > 0$, U , $\varepsilon > 0$, $t_1 \leq t_0$ be as in the proof of Proposition 6.3. We showed there that

$$|D_i q(t, \bar{p}, \bar{q})| \leq a_1 e^{\varepsilon(t-t_0)} + a_2, \quad i = 1, 2$$

for all $t \geq t_0$ and all $(\bar{p}, \bar{q}) \in U$. This, together with Hypothesis 8.1 shows the uniform integrability of the first half of the integral (8.1):

$$|r(t, \bar{p}, \bar{q})| \leq A_2 e^{-\lambda_2 \gamma(t-t_1)} (a_1 e^{\varepsilon(t-t_0)} + a_2) \leq a_3 e^{(\varepsilon - \lambda_2 \gamma)t}.$$

We must now estimate $z(t)$:

$$\begin{pmatrix} z(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} D_{1,2}q(t_0, \bar{p}, \bar{q}) \\ D_{1,2}p(t_0, \bar{p}, \bar{q}) \end{pmatrix} + \int_{t_0}^t R(t, s, \bar{p}, \bar{q}) \begin{pmatrix} 0 \\ -r(s, \bar{p}, \bar{q}) \end{pmatrix} ds. \quad (8.3)$$

We have

$$R(t, s, \bar{p}, \bar{q}) = R(t-s, 0, p(s, \bar{p}, \bar{q}), q(s, \bar{p}, \bar{q})). \quad (8.4)$$

From Lemma 6.2 we get a constant a_4 such that

$$\left| \Pi R(t, s, \bar{p}, \bar{q}) \begin{pmatrix} 0 \\ y \end{pmatrix} \right| \leq a_4 |y| e^{\varepsilon(t-s)}$$

for all $(\bar{p}, \bar{q}) \in U$, $t_0 \leq s \leq t$. Hence

$$\left| \int_{t_0}^t \Pi R(t, s, \bar{p}, \bar{q}) \begin{pmatrix} 0 \\ -r(s, \bar{p}, \bar{q}) \end{pmatrix} ds \right| \leq a_3 a_4 \int_{t_0}^t e^{\varepsilon(t-s)} e^{(\varepsilon - \lambda_2 \gamma)s} ds \leq a_5 e^{\varepsilon t}.$$

Estimating $|D_{1,2}q(t_0, \bar{p}, \bar{q})|$ on U by a constant, we can write $|z(t)| \leq a_6 e^{\varepsilon t}$. The last step is:

$$|-\text{HV}(q(t, \bar{p}, \bar{q})) D_{1,2}q(t, \bar{p}, \bar{q})| \leq A_2 e^{-\lambda_2 \gamma(t-t_1)} a_6 e^{\varepsilon t},$$

for all $(\bar{p}, \bar{q}) \in U$, $t \geq t_0$, and the proof is complete. \diamond

Proposition 8.4 *Suppose that Hypotheses 1.1, 4.1, 4.2, 7.1 and 8.2 hold. Then the asymptotic velocity p_∞ is a C^2 function of the initial data.*

Proof. Let (\bar{p}_0, \bar{q}_0) , U , γ , t_1 as in the proof of Proposition 6.3. In the proof of Proposition 7.3 we saw that

$$|D_i q(t, \bar{p}, \bar{q})| \leq a_1 + a_2(t - t_1), \quad i = 1, 2$$

for all $(\bar{p}, \bar{q}) \in U$, $t \geq t_1$. This, together with Hypothesis 8.2, gives

$$|r(t, \bar{p}, \bar{q})| \leq (a_1 + a_2(t - t_1)) h_2(\gamma(t - t_1)),$$

and half of the job is done. Using again formulas (8.3) and (8.4) (with t_1 instead of t_0), together with Lemma 7.2, we get

$$\begin{aligned} \left| \int_{t_1}^t \Pi' R(t, s, \bar{p}, \bar{q}) \begin{pmatrix} 0 \\ -r(s, \bar{p}, \bar{q}) \end{pmatrix} ds \right| &\leq \int_{t_1}^t |r(s, \bar{p}, \bar{q})| ds \leq \\ &\leq \int_{t_1}^{+\infty} (a_1 + a_2(s - t_1)) h_2(\gamma(s - t_1)) ds := a_3, \\ |\dot{z}(t)| &\leq a_4 + a_3 := a_5, \quad |z(t)| \leq a_6 + a_5(t - t_1). \end{aligned}$$

Finally:

$$|-\text{HV}(q(t, \bar{p}, \bar{q})) D_{1,2} q(t, \bar{p}, \bar{q})| \leq (a_6 + a_5(t - t_1)) h_2(\gamma(t - t_1))$$

for all $(\bar{p}, \bar{q}) \in U$, $t \geq t_1$. The proof is complete. \diamond

The hypotheses that guarantee higher order derivatives of p_∞ are now easy to guess. Denote by $D^m \mathcal{V}$ the m -th differential of \mathcal{V} , viewed as a multilinear operator form $(\mathbb{R}^n)^{m-1}$ into \mathbb{R}^n , endowed with the norm

$$\|D^m \mathcal{V}(q)\| := \sup \left\{ |D^m \mathcal{V}(q)(x^{(1)}, \dots, x^{(m-1)})| : x^{(i)} \in \mathbb{R}^n, |x^{(i)}| \leq 1 \right\}.$$

Hypothesis 8.5

H_m) \mathcal{V} is a C^{m+1} function and there exist $\lambda_m > 0$, $A_m > 0$ and $q_m \in \mathbb{R}^n$ such that

$$q \in q_m + \mathcal{D} \quad \Rightarrow \quad \|D^{m+1} \mathcal{V}(q)\| \leq A_m \exp\left(-\lambda_m \text{dist}(q, q_m + \partial \mathcal{D})\right);$$

H'_m) \mathcal{V} is a C^{m+1} function and there exist $q_m \in \mathbb{R}^n$ and a weakly decreasing function $h_m: \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $\int_0^{+\infty} x^m h_m(x) dx < +\infty$ and

$$q \in q_m + \mathcal{D} \quad \Rightarrow \quad \|D^{m+1} \mathcal{V}(q)\| \leq h_m\left(\text{dist}(q, q_m + \partial \mathcal{D})\right).$$

The following proposition can be proved by induction on m , with essentially the same reasoning used in Propositions 8.3 and 8.4.

Proposition 8.6 *The asymptotic velocity p_∞ is a C^m function of the initial data, $m \geq 2$, if we assume Hypotheses 1.1, 4.1, 4.2, and either i) or ii) of the following:*

- i) Hypothesis 6.1 plus H_2, H_3, \dots, H_m of Hypothesis 8.5;
- ii) Hypothesis 7.1 plus H'_2, H'_3, \dots, H'_m of Hypothesis 8.5.

9. Complete Integrability and Persistence

It is time to exploit the regularity theory developed so far to achieve our main goal: the integrability of the system

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p}, \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q}, \quad \mathcal{H}(p, q) := \frac{1}{2}|p|^2 + \mathcal{V}(q). \quad (9.1)$$

We introduce the notation X_f to mean the Hamiltonian vector field defined by some smooth $f: \mathbb{R}^{2n} \rightarrow \mathbb{R}$. In particular $X_{\mathcal{H}}$ is the vector field in (9.1). We will say that X_f is *complete* if all the solutions of the Hamilton equations $\dot{q} = \partial f / \partial p$, $\dot{p} = -\partial f / \partial q$ are global, that is, defined on the whole of \mathbb{R} .

We are going to prove that the components of the asymptotic velocity are n first integrals independent and in involution. Moreover we can include the Hamiltonian \mathcal{H} into a set of n globally independent first integrals in involution $\mathcal{F}_1, \dots, \mathcal{F}_n$, whose associated vector fields $X_{\mathcal{F}_1}, \dots, X_{\mathcal{F}_n}$ are complete. The \mathcal{F}_i , $2 \leq i \leq n$, will be obtained from p_∞ through a linear transformation. We will use the fact that, in our hypotheses, $\frac{1}{2}|p_\infty|^2$ is just \mathcal{H} , up to an immaterial additive constant (Corollary 4.6).

Theorem 9.1 (Complete Integrability) *Assume the hypotheses of Proposition either 8.3 or 8.4. Then the n components*

$$p_{\infty,1}, \dots, p_{\infty,n}$$

of the asymptotic velocity are independent C^2 integrals of motion and they are (pairwise) in involution. This means that, for all $(\bar{p}, \bar{q}) \in \mathbb{R}^n \times \mathbb{R}^n$, the gradients

$$\nabla p_{\infty,1}(\bar{p}, \bar{q}), \nabla p_{\infty,2}(\bar{p}, \bar{q}), \dots, \nabla p_{\infty,n}(\bar{p}, \bar{q}) \quad (9.2)$$

are linearly independent and the Poisson brackets vanish identically:

$$\{p_{\infty,i}, p_{\infty,j}\}(\bar{p}, \bar{q}) = 0.$$

Hence the system (9.1) is integrable by quadratures.

Furthermore, there exists an orthogonal transformation $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that the functions $\mathcal{F}_1, \dots, \mathcal{F}_n$ defined as

$$\begin{aligned} \mathcal{F}_1 &:= \mathcal{H} \\ \mathcal{F}_i &:= (Ap_\infty)_i \quad \text{for } i = 2, 3, \dots, n, \end{aligned} \quad (9.3)$$

are independent C^2 integrals of motion, (pairwise) in involution and the Hamiltonian vector fields $X_{\mathcal{F}_i}$ ($1 \leq i \leq n$) are complete.

In particular, the system (9.1) is completely integrable.

Proof. Claiming that the gradients in (9.2) are independent is the same as saying that the Jacobian matrix Dp_∞ (p_∞ thought of as column vector) has maximum rank. Denote by $\{\Phi^t\}_{t \in \mathbb{R}}$ the flow of the system (1.1), i.e., $\Phi^t(\bar{p}, \bar{q}) = (p(t, \bar{p}, \bar{q}), q(t, \bar{p}, \bar{q}))$. Since p_∞ is an integral of motion,

$$p_\infty \circ \Phi^t = p_\infty \quad \forall t \in \mathbb{R}.$$

By differentiating at (\bar{p}, \bar{q}) we have

$$Dp_\infty(\Phi^t(\bar{p}, \bar{q})) D\Phi^t(\bar{p}, \bar{q}) = Dp_\infty(\bar{p}, \bar{q}).$$

From this we have that

$$\text{rank } Dp_\infty$$

is a constant of motion, because $D\Phi^t(\bar{p}, \bar{q})$ is invertible.

From Poisson's Theorem (see e.g. [1], Section 40) we know that the Poisson brackets

$$\{p_{\infty, i}, p_{\infty, j}\}$$

are also constants of motion.

We now use Proposition either 6.5 or 7.6. Along any trajectory, the velocity $p(t, \bar{p}, \bar{q})$ eventually enters \mathcal{D}° and keeps from its boundary a distance larger than $\gamma > 0$. Moreover, $q(t, \bar{p}, \bar{q})$ enters all the sets of the form $q_0 + \mathcal{D}$ (Proposition 4.4). Hence, along any trajectory the derivatives of p_∞ tend to the derivatives of the projection Π and we can compute:

$$\begin{aligned} \text{rank } Dp_\infty(\bar{p}, \bar{q}) &= \text{rank } Dp_\infty(\Phi^t(\bar{p}, \bar{q})) = \lim_{t \rightarrow +\infty} \text{rank } Dp_\infty(\Phi^t(\bar{p}, \bar{q})) = \\ &= \text{rank } D\Pi = n \end{aligned}$$

(the set of the $n \times 2n$ matrices with maximum rank is open in \mathbb{R}^{2n^2}) and

$$\begin{aligned} \{p_{\infty, i}, p_{\infty, j}\}(\bar{p}, \bar{q}) &= \{p_{\infty, i}, p_{\infty, j}\}(\Phi^t(\bar{p}, \bar{q})) = \lim_{t \rightarrow +\infty} \{p_{\infty, i}, p_{\infty, j}\}(\Phi^t(\bar{p}, \bar{q})) = \\ &= \{\Pi_i, \Pi_j\} = 0 \end{aligned}$$

Now, let us prove the second part of the theorem. We can use the Remark 2.6 to construct an orthogonal transformation $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (we will write A also for the associated matrix) such that the first component $(Aw)_1$ is strictly positive for all $w \in \mathcal{D}^\circ$. Hence, from Proposition 4.3 we have that

$$(Ap_\infty(\bar{p}, \bar{q}))_1 > 0 \quad \forall (\bar{p}, \bar{q}). \quad (9.4)$$

Define $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n$ as in (9.3). Proposition 4.5 implies that $\mathcal{F}_1 = \mathcal{H} = \frac{1}{2}|p_\infty|^2 + \inf \mathcal{V}$. Let $D\mathcal{F}$ be the Jacobian matrix of $\mathcal{F} := (\mathcal{F}_1, \dots, \mathcal{F}_n)^T$ (the T means transposition of matrices). Proving that $\mathcal{F}_1, \dots, \mathcal{F}_n$ are independent is equivalent to proving that the rank of $D\mathcal{F}$ is n . Since \mathcal{F} is a first integral of motion, the rank of $D\mathcal{F}$ does not change along each trajectory.

From Proposition either 6.5 or 7.6 we know that along all trajectories the matrix Dp_∞ tends to the matrix $D\Pi = (I_n, 0)$, so that $D(Ap_\infty) \rightarrow D(A\Pi) = (A, 0) = (I_n, 0) \begin{pmatrix} A \\ 0 \end{pmatrix}$, and $D\mathcal{F}_1$ tends to $(p_\infty^T, 0) = ((A^T Ap_\infty)^T, 0) = ((Ap_\infty)^T, 0) \begin{pmatrix} A \\ 0 \end{pmatrix}$. Thus we can write

$$\begin{aligned} \text{rank } D\mathcal{F}(\bar{p}, \bar{q}) &= \lim_{t \rightarrow +\infty} \text{rank } D\mathcal{F}(\Phi^t(\bar{p}, \bar{q})) = \\ &= \text{rank} \left(\begin{array}{cccc|ccc} (Ap_\infty)_1 & & \dots & & (Ap_\infty)_n & & & & \\ 0 & 1 & 0 & \dots & 0 & & & & \\ 0 & 0 & 1 & \dots & 0 & & & & \\ \vdots & \vdots & & \ddots & \vdots & & & & \\ 0 & 0 & 0 & \dots & 1 & & & & \end{array} \begin{array}{c} \vdots \\ \dots \\ 0 \\ \dots \\ \vdots \end{array} \right) \begin{pmatrix} A \\ 0 \end{pmatrix} = n \end{aligned}$$

because of (9.4). So $\mathcal{F}_1, \dots, \mathcal{F}_n$ are independent.

We have

$$\{\mathcal{H}, \mathcal{F}_i\} = \{\mathcal{H}, (Ap_\infty)_i\} = \sum_{j=1}^n A_{ij} \{\mathcal{H}, p_{\infty,j}\} = 0$$

because the Poisson brackets are bilinear. Therefore, $\mathcal{F}_1, \dots, \mathcal{F}_n$ are pairwise in involution since $p_{\infty,1}, \dots, p_{\infty,n}$ are.

The last result in particular says that \mathcal{H} is a first integral of each $X_{\mathcal{F}_i}$. This implies the completeness of each vector field $X_{\mathcal{F}_i}$ as we can see by the same argument which gave the completeness of $X_{\mathcal{H}}$ in Section 1.

◇

We remark that we could have included the Hamiltonian \mathcal{H} into a set of n *locally* independent integrals of motion, extracted from p_∞ , in the following way. We could have simply chosen a nonvanishing component of p_∞ (there is always one, locally, because $p_\infty \in \mathcal{D}^\circ$) and replaced it with \mathcal{H} . This would give independence generally only in a neighbourhood of each level set of p_∞ , unless we have been lucky or deft from the very beginning in the choice of the orthonormal reference basis of \mathbb{R}^n .

It is legitimate to ask what happens to the integrability of our system if we perturb the potential \mathcal{V} in a compact set K :

$$H(p, q) := \frac{1}{2}|p|^2 + \mathcal{V}(q) + f(q), \quad (9.5)$$

with f a smooth function vanishing outside K . The new potential $\mathcal{V} + f$ does not need to be a cone potential, or, even if it is, it may not have the same cone. Think for example to the case when the cone \mathcal{C} of \mathcal{V} has empty interior. The global hypotheses 4.1 and 4.2 are very sensitive to the cones \mathcal{C} and \mathcal{D} , and there is no hope to verify them for the new cone. Nevertheless, we will prove that if $|\nabla f|$ is sufficiently small, then all the trajectories of the new system eventually quit K for good, the potential \mathcal{V} leads henceforth undisturbed, and all our conclusions about regularity and integrability remain true.

Theorem 9.2 (Persistence) *Suppose that \mathcal{V} verifies the hypotheses of Theorem 9.1. Let $K \subset \mathbb{R}^n$ be compact. Then there exists an $\varepsilon > 0$ with the following property. If $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is a C^3 function with support in K and*

$$\sup |\nabla f| < \varepsilon, \quad (9.6)$$

then the Hamiltonian system whose Hamiltonian H is given by Equation (9.5) is completely integrable. Namely, all trajectories have asymptotic velocities for which all the claims in Theorem 9.1 apply.

Proof. We still denote by \mathcal{C} and \mathcal{D} the cones associated with \mathcal{V} . Let $v \in \mathcal{C} \setminus \{0\}$. Since K is compact and \mathcal{D} has nonempty interior, there exist $q', q'' \in \mathbb{R}^n$, $q'' \in q' + \mathcal{D}$, such that

$$K \subset q' + \mathcal{D}, \quad q \cdot v \leq q'' \cdot v \quad \forall q \in K.$$

Define ε by

$$\varepsilon := \inf \{ -\nabla \mathcal{V}(q) \cdot v : q \in q' + \mathcal{D}, q \cdot v \leq q'' \cdot v \}.$$

From Hypothesis 4.2, ε is positive. Suppose that f is a C^3 function with support in K and verifies (9.6). Denote by $(p(t, \bar{p}, \bar{q}), q(t, \bar{p}, \bar{q}))$ the trajectories corresponding to the new Hamiltonian (9.5). We claim that

$$\forall (\bar{p}, \bar{q}) \quad \exists t_0 \in \mathbb{R} \quad \text{such that} \quad \forall t \geq t_0 \quad q(t, \bar{p}, \bar{q}) \cdot v > q'' \cdot v. \quad (9.7)$$

In fact, there certainly exists $\tilde{q} \in \mathbb{R}^n$ such that $q(t, \bar{p}, \bar{q}) \in \tilde{q} + \mathcal{D}$ for all $t \in \mathbb{R}$ because Hypothesis 4.1 still holds for $\mathcal{V} + f$ and \mathcal{D} . We can safely assume $q' \in \tilde{q} + \mathcal{D}$. Let

$$\varepsilon' := \inf \{ -\nabla \mathcal{V}(q) \cdot v : q \in \tilde{q} + \mathcal{D}, q \cdot v \leq q'' \cdot v \}.$$

We can write

$$\left(q \in \tilde{q} + \mathcal{D} \quad \text{and} \quad q \cdot v \leq q'' \cdot v \right) \quad \Rightarrow \quad -\nabla(\mathcal{V} + f)(q) \cdot v \geq \min\{\varepsilon - \inf |\nabla f|, \varepsilon'\} > 0.$$

Now the same reasoning as in the proof of Proposition 4.3 leads to (9.7).

To conclude, we only need to note that the t_0 in (9.7) can be chosen locally independent of (\bar{p}, \bar{q}) , and that in all the hypotheses on \mathcal{V} after Section 4 we can always suppose $q_0, q_1 \dots$ to belong to $q'' + \mathcal{D}$.

◇

10. Examples

We are going to present a class of examples for which our assumptions for complete C^2 integrability hold, in the convex case. Conditions for C^k integrability, $2 \leq k \leq +\infty$, are easily added.

Hypotheses 10.1 For $\alpha = 1, \dots, N$, let f_α be a C^3 real function, defined either on all of \mathbb{R} or on the interval $]0, +\infty[$, such that, for all α ,

$$\inf f_\alpha = 0, \quad \sup f_\alpha = +\infty. \quad (10.1)$$

$$f'_\alpha(x) < 0 \quad \forall x, \quad (10.2)$$

$$f''_\alpha(x) > 0 \quad \forall x \geq 1, \quad (10.3)$$

$$f'''_\alpha \text{ is weakly decreasing on } [1, +\infty[, \quad f'''_\alpha(x) < 0 \quad \forall x \geq 1. \quad (10.4)$$

Let $v_1, \dots, v_N \in \mathbb{R}^n \setminus \{0\}$ be such that

$$v_\alpha \cdot v_\beta \geq 0 \quad \forall \alpha, \beta. \quad (10.5)$$

Define the following potential \mathcal{V} :

$$\mathcal{V}(q) := \sum_{\alpha=1}^N f_\alpha(q \cdot v_\alpha). \quad (10.6)$$

either on \mathbb{R}^n or in the set $\{w \in \mathbb{R}^n : w \cdot v_\alpha > 0 \quad \forall \alpha = 1, \dots, N\}$. The gradient, Hessian and third differential of \mathcal{V} are given by

$$\nabla \mathcal{V}(q) = \sum_{\alpha=1}^N f'_\alpha(q \cdot v_\alpha) v_\alpha, \quad (10.7)$$

$$\text{H}\mathcal{V}(q) = \sum_{\alpha=1}^N f''_\alpha(q \cdot v_\alpha) v_\alpha \otimes v_\alpha \quad (10.8)$$

$$\text{D}^3 \mathcal{V}(q) = \sum_{\alpha=1}^N f'''_\alpha(q \cdot v_\alpha) v_\alpha \otimes v_\alpha \otimes v_\alpha, \quad (10.9)$$

where \otimes indicates the tensor product: $(v_\alpha \otimes v_\alpha)_{i,j} := v_{\alpha,i} v_{\alpha,j}$, etc.

From (10.1), (10.2), (10.5) and (10.7) it is clear that Hypothesis 1.1 is satisfied for \mathcal{V} . Also the other requirements for integrability are met, and the proof will be made in several steps, culminating in the statement of Proposition 10.3 and its corollaries.

It will be convenient to have from the start a $q_0 \in \mathbb{R}^n$ such that

$$q_0 \cdot v_\alpha \geq 1 \quad \forall \alpha \quad (10.10)$$

(for example, $q_0 := \rho \sum_{\beta=1}^N v_\beta$, for ρ large). Recall Proposition 2.5 and note that, for all $q \in q_0 + \mathcal{D}$, $\alpha_0 = 1, \dots, N$

$$q \cdot v_{\alpha_0} = (q - q_0) \cdot v_{\alpha_0} + q_0 \cdot v_{\alpha_0} \geq \min_{\beta} |v_\beta| \operatorname{dist}(q, q_0 + \partial\mathcal{D}) + 1. \quad (10.11)$$

Lemma 10.2 *The closure of the convex cone \mathcal{C} generated by the forces $-\nabla\mathcal{V}$ coincides with the convex cone generated by the v_α :*

$$\bar{\mathcal{C}} = \left\{ \sum_{\alpha=1}^N c_\alpha v_\alpha : c_\alpha \geq 0 \right\},$$

and the dual \mathcal{D} of \mathcal{C} is given by $\{w : w \cdot v_\alpha \geq 0 \forall \alpha\}$.

Proof. Denote by $\bar{\mathcal{C}}$ the cone generated by v_1, \dots, v_N and by D its dual. Of course $\bar{\mathcal{C}} \subset \bar{\mathcal{C}}$, so that $D \subset \mathcal{D}$. Take $w \in \partial D$. We are done if we show that $w \in \partial\mathcal{D}$. Consider the following two sets of indices: $I_1 := \{\alpha : w \cdot v_\alpha > 0\}$, $I_2 := \{\beta : w \cdot v_\beta = 0\}$. From Proposition 2.5 and the fact that $w \in \partial D$ we deduce that $I_2 \neq \emptyset$. Evaluate now $-\nabla\mathcal{V}$ along the line $q_0 + \tau w$ (q_0 given by (10.10)), for $\tau \rightarrow +\infty$; the terms $f'_\alpha((q_0 + \tau w) \cdot v_\alpha)$ vanish for $\alpha \in I_1$, so that

$$\mathcal{C} \ni -\nabla\mathcal{V}(q_0 + \tau w) \rightarrow - \sum_{\beta \in I_2} f'_\beta(q_0 \cdot v_\beta) v_\beta := \tilde{v} \in \bar{\mathcal{C}}.$$

We have $\tilde{v} \neq 0$ because $-f'_\alpha > 0$ and the v_β lie in the interior of a half-space (the cone $\bar{\mathcal{C}}$ is proper). From the definition of I_2 we see that $w \cdot \tilde{v} = 0$ and finally $w \in \partial\mathcal{D}$ because of Proposition 2.4. The formula for \mathcal{D} is an easy consequence. ◇

Lemma 10.3 *Let $g: [0, +\infty[\rightarrow \mathbb{R}$ be a nonnegative, weakly decreasing function such that*

$$\int_0^{+\infty} x^m g(x) dx < +\infty$$

for some integer $m \geq 0$. Then $\lim_{x \rightarrow +\infty} x^{m+1} g(x) = 0$.

Proof. Suppose there exists $x_i \nearrow +\infty$ such that $x_i^{m+1}g(x_i) \geq \varepsilon > 0$ for all $i \geq 1$. Then

$$x_{i-1} < x \leq x_i \quad \Rightarrow \quad x^m g(x) \geq x^m g(x_i) \geq \varepsilon \frac{x^m}{x_i^{m+1}}.$$

Define $h: [0, +\infty[\rightarrow \mathbb{R}$ as

$$h(x) := \begin{cases} 0 & \text{if } 0 \leq x \leq x_1, \\ \varepsilon x^m / x_i^{m+1} & \text{if } x_{i-1} < x \leq x_i, i \geq 2. \end{cases}$$

Then $h(x) \leq x^m g(x)$ and

$$\int_0^{+\infty} h(x) dx = \sum_{i=2}^{+\infty} \frac{\varepsilon}{x_i^{m+1}} \cdot \frac{x_i^{m+1} - x_{i-1}^{m+1}}{m+1} = \frac{\varepsilon}{m+1} \sum_{i=2}^{+\infty} \left(1 - \frac{x_{i-1}^{m+1}}{x_i^{m+1}}\right).$$

We can safely assume, for instance, that $x_i \geq 2x_{i-1}$ for all $i \geq 2$, and this yields h , and hence $x^m g(x)$ too, to have infinite integral, against the hypothesis. \diamond

Lemma 10.4 Let $f \in C^M([0, +\infty[)$, $M \geq 1$, and suppose that, for all $x \geq 0$, $m = 0, \dots, M$,

$$f^{(m)}(x) \begin{cases} > 0 & \text{if } m \text{ is even,} \\ < 0 & \text{if } m \text{ is odd.} \end{cases}$$

Then $\int_0^{+\infty} x^{m-1} |f^{(m)}(x)| dx < +\infty$ for all $m = 1, \dots, M$.

Proof. It is certainly true if $M = 1$. Suppose it is true for $M - 1$ and write

$$\int_0^{+\infty} x^{M-1} f^{(M)}(x) dx = x^{M-1} f^{(M-1)}(x) \Big|_{x=0}^{+\infty} - (M-1) \int_0^{+\infty} x^{M-2} f^{(M-1)}(x) dx.$$

The last integral converges for the induction hypothesis. The term $x^{M-1} f^{(M-1)}(x)$ is infinitesimal as $x \rightarrow +\infty$ again because $x^{M-2} f^{(M-1)}(x)$ is integrable, with the help of Lemma 10.3. \diamond

Verification of 4.1 Let $M \geq 0$. From 10.1 and 10.2 we get $x_M \in \mathbb{R}$ such that

$$f_\alpha(x) \geq M \quad \forall x \leq x_M \quad \forall \alpha = 1, \dots, N.$$

Let $q_M \in \mathbb{R}^n$ such that

$$q_M \cdot v_\alpha \leq x_M \quad \forall \alpha$$

(for example, $q_M = \theta \sum_\alpha v_\alpha$ for θ negatively large). Then, for all $q \in \mathbb{R}^n \setminus (q_M + \mathcal{D})$ there exists α_M such that $q_M \cdot v_{\alpha_M} \leq x_M$ and hence

$$\mathcal{V}(q) \geq f_{\alpha_M}(q \cdot v_{\alpha_M}) \geq f_{\alpha_M}(x_M) \geq M.$$

\diamond

Verification of 4.2 Let q' belong to the domain of \mathcal{V} and $q'' \in q' + \mathcal{D}$. Let $v \in \bar{\mathcal{C}} \setminus \{0\}$. Up to a reordering of indices, we can write

$$v = \sum_{\beta=1}^{N'} c_{\beta} v_{\beta} \in \bar{\mathcal{C}} \quad \text{with } 1 \leq N' \leq N \text{ and } c_{\beta} > 0. \quad (10.12)$$

By (10.7), (10.12), (10.2) and (10.5) we have

$$\begin{aligned} -\nabla \mathcal{V}(q) \cdot v &= - \sum_{\alpha=1}^N f'_{\alpha}(q \cdot v_{\alpha}) v_{\alpha} \cdot v = \sum_{\alpha=1}^N \sum_{\beta=1}^{N'} c_{\beta} (-f'_{\alpha}(q \cdot v_{\alpha})) v_{\alpha} \cdot v_{\beta} \geq \\ &\geq - \sum_{\beta=1}^{N'} c_{\beta} f'_{\beta}(q \cdot v_{\beta}) |v_{\beta}|^2. \end{aligned}$$

Moreover

$$q \in q' + \mathcal{D} \quad \Rightarrow \quad \forall \beta \quad q' \cdot v_{\beta} \leq q \cdot v_{\beta}.$$

On the other hand

$$q \cdot v \leq q'' \cdot v \quad \Rightarrow \quad \exists \beta_0 \text{ such that } q \cdot v_{\beta_0} \leq q'' \cdot v_{\beta_0}.$$

Hence

$$\left(q \in q' + \mathcal{D} \text{ and } q \cdot v \leq q'' \cdot v \right) \quad \Rightarrow \quad \exists \beta_0 \text{ such that } q' \cdot v_{\beta_0} \leq q \cdot v_{\beta_0} \leq q'' \cdot v_{\beta_0}.$$

For each $\beta = 1, \dots, N'$, define

$$\varepsilon_{\beta} := \min\{-c_{\beta} f'_{\beta}(x) : q' \cdot v_{\beta} \leq x \leq q'' \cdot v_{\beta}\} > 0.$$

We can conclude that

$$\left(q \in q' + \mathcal{D} \text{ and } q \cdot v \leq q'' \cdot v \right) \quad \Rightarrow \quad -\nabla \mathcal{V}(q) \cdot v \geq \min_{\beta} \varepsilon_{\beta} > 0.$$

◇

Verification of 5.1 Recall q_0 from formula (10.9). Since $x \mapsto |f'_{\alpha}(x)|$ is weakly decreasing on $[1, +\infty[$ and from (10.11), we can compute, for all $q \in q_0 + \mathcal{D}$,

$$|\nabla \mathcal{V}(q)| \leq \sum_{\alpha=1}^N |f'_{\alpha}(q \cdot v_{\alpha})| |v_{\alpha}| \leq h_0 \left(\text{dist}(q, q_0 + \partial \mathcal{D}) \right),$$

where

$$h_0(x) := \sum_{\alpha=1}^N |v_{\alpha}| \left| f'_{\alpha} \left(\min_{\beta} |v_{\beta}| x + 1 \right) \right|,$$

and h_0 is weakly decreasing and integrable on $[0, +\infty[$ because each $|f'_{\alpha}|$ is weakly decreasing and integrable on $[1, +\infty[$.

◇

Verification of 7.1 i). From Formula (10.3) we see that each f_α is convex on $[1, +\infty[$. So the potential \mathcal{V} is convex on $q_0 + \mathcal{D}$, q_0 given by (10.10), because it is sum of convex functions. ◇

Verification of 7.1 ii). From Formula (10.4) we see that f_α'' is a weakly decreasing function on $[1, +\infty[$. If $q', q'' \in q_0 + \mathcal{D}$, $q'' \in q' + \mathcal{D}$, then we have $q'' \cdot v_\alpha \geq q' \cdot v_\alpha \geq q_0 \cdot v_\alpha \geq 1$ and

$$f_\alpha''(q'' \cdot v_\alpha) \leq f_\alpha''(q' \cdot v_\alpha).$$

Hence, from (10.8),

$$\begin{aligned} \text{HV}(q'')z \cdot z &= \sum_{\alpha=1}^N f_\alpha''(q'' \cdot v_\alpha)(v_\alpha \cdot z)^2 \leq \\ &\leq \sum_{\alpha=1}^N f_\alpha''(q' \cdot v_\alpha)(v_\alpha \cdot z)^2 = \text{HV}(q')z \cdot z. \end{aligned}$$

◇

Verification of 7.1 iii). We have, from Formula (10.8) and (10.11), for all $q \in q_0 + \mathcal{D}$,

$$\|\text{HV}(q)\| \leq \sum_{\alpha=1}^N |f_\alpha''(q \cdot v_\alpha)| |v_\alpha|^2 \leq h_1\left(\text{dist}(q, q_0 + \partial\mathcal{D})\right),$$

where

$$h_1(x) := \sum_{\alpha=1}^N |v_\alpha|^2 |f_\alpha''(\min_{\beta} |v_\beta|x + 1)|,$$

and h_1 is weakly decreasing on $[0, +\infty[$ and $\int_0^{+\infty} x|h_1(x)| dx < +\infty$ because of Lemma 10.4. ◇

Verification of 8.2 From (10.9) and (10.4) we have, for all $q \in q_0 + \mathcal{D}$,

$$\|\text{D}^3\mathcal{V}(q)\| \leq h_2\left(\text{dist}(q, q_0 + \partial\mathcal{D})\right),$$

where

$$h_2(x) := \sum_{\alpha=1}^N |v_\alpha|^3 |f_\alpha'''(\min_{\beta} |v_\beta|x + 1)|$$

and, as usual, h_2 is weakly decreasing on $[0, +\infty[$ and $\int_0^{+\infty} x^2 h_2(x) dx < +\infty$. ◇

Proposition 10.5 *Suppose that the functions f_1, \dots, f_N and the vectors v_1, \dots, v_N satisfy the Hypotheses 10.1. Let the potential \mathcal{V} be defined by Formula (10.6). Then Theorem 9.2 applies, so that the Hamiltonian system*

$$\dot{q} = p, \quad \dot{p} = -\nabla \mathcal{V}(q)$$

is C^2 -completely integrable.

If we assume, moreover, that the f_α are C^{k+1} , $2 < k \leq +\infty$, and that for all $4 \leq m \leq k+1$, $\alpha = 1, \dots, N$, $x \geq 1$ we have

$$f_\alpha^{(m)}(x) \begin{cases} > 0 & \text{if } m \text{ is even} \\ < 0 & \text{if } m \text{ is odd,} \end{cases}$$

(and that $|f^{(k+1)}|$ be weakly decreasing if $k < +\infty$), then p_∞ is C^k .

Corollary 10.6 *Let $v_1, \dots, v_N \in \mathbb{R}^n \setminus \{0\}$ be such that $v_\alpha \cdot v_\beta \geq 0$ for all α, β . Let $r > 0$ and define the potential*

$$\mathcal{V}(q) := \sum_{\alpha=1}^N \frac{1}{(q \cdot v_\alpha)^r}$$

on the set $\mathcal{D}^\circ = \{q \in \mathbb{R}^n : q \cdot v_\alpha > 0 \forall \alpha\}$. Then the associated Hamiltonian system is C^∞ -completely integrable.

Corollary 10.7 *Let $v_1, \dots, v_N \in \mathbb{R}^n \setminus \{0\}$ be such that $v_\alpha \cdot v_\beta \geq 0$ for all α, β , and let $c_\alpha > 0$. Define the potential*

$$\mathcal{V}(q) := \sum_{\alpha=1}^N c_\alpha e^{-q \cdot v_\alpha}$$

on \mathbb{R}^n . Then the associated Hamiltonian system is C^∞ -completely integrable.

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