ON THE COMPLETE INTEGRABILITY OF A ONE GENERALIZED RIEMANN TYPE HYDRODYNAMIC SYSTEM

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Abstract. The complete integrability of a generalized Riemann type hydrodynamic system is studied by means of symplectic and differential-algebraic tools. A compatible pair of polynomial Poissonian structures, Lax type representation and related infinite hierarchy of conservation laws are constructed.

1. Introduction

We are interested in studying the complete integrability of the following dispersionless Riemann type hydrodynamic flow

$$D_t^{N-1}u = \bar{z}_x^2, \qquad D_t\bar{z} = 0 \tag{1.1}$$

on a 2π-periodic functional manifold $\bar{M}^N \subset C^{(\infty)}(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^N)$, where $N \in \mathbb{N}$ is an arbitrary natural number, a vector $(u, D_tu, D_t^2u, ..., D_t^{N-1}u, \bar{z})^\intercal \in \bar{M}^N$, differentiations $D_x := \partial/\partial x$, $D_t := \partial/\partial t + u\partial/\partial x$ satisfy the Lie-algebraic commutator relationship

$$[D_x, D_t] = u_xD_x. \tag{1.2}$$

and $t \in \mathbb{R}$ is an evolution parameter. The system can be considered as a slight generalization of the dispersionless Riemann type hydrodynamic system suggested recently by M. Pavlov and D. Holm in the form

$$D_t^{N-1}u = \bar{z}, \qquad D_t\bar{z} = 0 \tag{1.3}$$

for $N \in \mathbb{N}$ and extensively studied in [1, 3, 4, 5, 2, 6], where it was stated that it is a Lax type integrable bi-Hamiltonian flow on the manifold $\bar{M}^N$, possessing an infinite hierarchy of commuting to each other already dispersive and also Lax type integrable Hamiltonian flows.

For the case $N = 2$ it is well known [8, 10] that the Riemann type hydrodynamic system (1.1) is a smooth Lax type integrable bi-Hamiltonian flow on the 2π-periodic functional manifold $\bar{M}^2$, whose Lax type representation is given by the following compatible linear system of equations:

$$D_xf = \begin{pmatrix} \bar{z}_x & 0 \\ -\lambda[u + u_x/(2\bar{z}_x)] & -\bar{z}_{xx}/(2\bar{z}_x) \end{pmatrix} f, \qquad D_tf = \begin{pmatrix} 0 & 0 \\ -\lambda\bar{z}_x & u_x) \end{pmatrix} f, \tag{1.4}$$

where $f \in C^{(\infty)}(\mathbb{R}^2;\mathbb{R}^2)$ and $\lambda \in \mathbb{R}$ is an arbitrary spectral parameter.

The present work is devoted to studying the Lax type integrability of the Riemann type hydrodynamic system (1.1) at $N = 3$ on a 2π-periodic functional manifold $\bar{M}^3 \subset C^{(\infty)}(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^3)$ for a vector $(u, v, \bar{z})^\intercal \in \bar{M}^3$ in the following extended form:

$$D_tu = v, \qquad D_tv = \bar{z}_x^2, \qquad D_t\bar{z} = 0. \tag{1.5}$$

The flow (1.5) can be equivalently rewritten as a one on 2π-periodic functional manifold $M^3 \subset C^{(\infty)}(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^3)$ for a vector $(u, v, z)^\intercal \in M^3$ as

$$D_tu = v, \qquad D_tv = z, \qquad D_tz = -2zu_x, \tag{1.6}$$

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where, for further convenience, we have done the following change of variables: $z := \bar{z}_x^2$. We will also use below the next form of the flow (1.6):

$$\left.\begin{array}{r} du/dt = v - uu_x \\ dv/dt = z - uv_x \\ dz/dt = -2u_x z - uz_x \end{array}\right\} := K[u,v,z], \tag{1.7}$$

defining a standard smooth dynamical system on the infinite-dimensional functional manifold M^3, where $K : M^3 \to T(M^3)$ is the corresponding smooth vector field on M^3.

Below, based on the symplectic gradient-holonomic and differential algebraic tools, we will prove the following main proposition.

Proposition 1.1. *The Riemann type hydrodynamic flow (1.7) is a bi-Hamiltonian dynamical system on the functional manifold M^3 with respect to two compatible Poissonian structures ϑ, η : $T^*(M^3) \to T(M^3)$*

$$\vartheta := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2z^{1/2}D_x z^{1/2} \end{pmatrix}, \eta := \begin{pmatrix} \partial^{-1} & u_x\partial^{-1} & 0 \\ \partial^{-1}u_x & v_x\partial^{-1} + \partial^{-1}v_x & \partial^{-1}z_x - 2z \\ 0 & z_x\partial^{-1} + 2z & 0 \end{pmatrix}, \tag{1.8}$$

possessing an infinite hierarchy of commuting to each other conservation laws and a non-autonomous Lax type representation in the form

$$\begin{aligned} D_t f &= \begin{pmatrix} 0 & 0 & 0 \\ -\lambda & 0 & 0 \\ 0 & -\lambda z_x & u_x \end{pmatrix} f, \\ D_x f &= \begin{pmatrix} \lambda^2 u\sqrt{z} & \lambda v\sqrt{z} & z \\ -\lambda^3 tu\sqrt{z} & -\lambda^2 tv\sqrt{z} & -\lambda tz \\ \lambda^4(tuv - u^2) - & -\lambda v_x/\sqrt{z} + & \lambda^2\sqrt{z}(u - tv) - \\ -\lambda^2 u_x/\sqrt{z} & +\lambda^3(tv^2 - uv) & -z_x/2z \end{pmatrix} f, \end{aligned} \tag{1.9}$$

where $\lambda \in \mathbb{R}$ is an arbitrary spectral parameter and $f \in C^{(\infty)}(\mathbb{R}^2; \mathbb{R}^3)$.

2. The symplectic gradient-holonomic integrability analysis

2.1. Poissonian structure analysis on the functional manifold M^3.

Based on the symplectic gradient-holonomic approach [8, 10, 11] to studying the integrability of smooth nonlinear dynamical systems on functional manifolds, one can find a set of conservation laws for (1.7), if to construct some solutions $\varphi := \varphi[u,v,z] \in T^*(M^3)$ to the following functional Lax gradient equation:

$$d\varphi/dt + K'^{,*}\varphi = grad\mathcal{L}, \tag{2.1}$$

where $\varphi' = \varphi'^{,*}$, $\mathcal{L} \in D(M^3)$ is a suitable Lagrangian functional and a linear operator $K'^{,*}$: $T^*(M^3) \to T^*(M^3)$ is, adjoint with respect to the standard convolution $(\cdot,\cdot)$ on $T^*(M^3) \times T(M^3)$, the Frechet-derivative of a nonlinear mapping $K : M^3 \to T(M^3)$:

$$K'^{,*} = \begin{pmatrix} uD_x & -v_x & z_x + 2zD_x \\ 1 & u_x + uD_x & 0 \\ 0 & 1 & -u_x + uD_x \end{pmatrix}. \tag{2.2}$$

The Lax gradient equation (2.1) can be, owing to (1.3), rewritten as

$$D_t\varphi + k[u,v,z]\varphi = grad\ \mathcal{L}, \tag{2.3}$$

where the matrix operator

$$k[u,v,z] := \begin{pmatrix} 0 & -v_x & z_x + 2zD_x \\ 1 & u_x & 0 \\ 0 & 1 & -u_x \end{pmatrix}. \tag{2.4}$$

The first vector elements

$$\begin{aligned} \varphi_\vartheta[u,v,z] &= (z - uv_x, -v + uu_x, u), \mathcal{L}_\vartheta = 0 \\ \varphi_\eta[u,v,z] &= (v_x, -u_x, -1)^\intercal, \mathcal{L}_\eta = 0, \\ \varphi_0[u,v,z] &= (-(u_x z^{-1/2})_x, (z^{-1/2})_x, (v_x/2 - u_x^2/4)z^{-3/2})^\intercal, \mathcal{L}_0 = 0, \end{aligned} \tag{2.5}$$

as can be easily checked, exactly solve the functional equation (2.3). Having applied the standard Volterra homotopy formula

$$H := \int_0^1 d\mu(\varphi[\mu u, \mu v, \mu z], (u,v,z)^\intercal), \tag{2.6}$$

one finds the next conservation laws for (1.3):

$$\begin{aligned} H_\eta &= \frac{1}{2}\int_0^{2\pi} dx(2uz - v^2 - u^2 v_x), \\ H_\vartheta &:= \int_0^{2\pi} dx(uv_x/2 - vu_x/2 - z), \quad H_0 := \frac{1}{2}\int_0^{2\pi} dx(u_x^2 - 2v_x)z^{-1/2}. \end{aligned} \tag{2.7}$$

It is now easy enough, making use of the conservation laws (2.7), to construct a Poissonian structure $\vartheta : T^*(M^3) \to T(M^3)$ for dynamical system (1.7). If to represent

$$\begin{aligned} H_\vartheta &= \int_0^{2\pi} dx(uv_x/2 - vu_x/2 - z) := (\psi_\vartheta, (u_x, v_x, z_x)^\intercal), \\ \psi_\vartheta &:= (-v/2, u/2, z^{-1/2}D_x^{-1}z^{1/2}/2)^\intercal, \end{aligned} \tag{2.8}$$

then one obtains that the vector $\psi_\vartheta \in T^*(M^3)$ satisfies the Lax gradient equation (2.3):

$$D_t\psi_\vartheta + k[u,v,z]\psi_\vartheta = grad\ \mathcal{L}_\vartheta, \tag{2.9}$$

where the Lagrangian function $\mathcal{L}_\vartheta = (\psi_\vartheta, K) - H_\vartheta$. Thus, based on the inverse co-symplectic functional expression

$$\vartheta^{-1} := \psi'_\vartheta - \psi'^{,*}_\vartheta = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & z^{-1/2}D_x^{-1}z^{-1/2}/2 \end{pmatrix} \tag{2.10}$$

one easily obtains the linear co-symplectic operator on the manifold M^3 :

$$\vartheta := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2z^{1/2}D_x z^{1/2} \end{pmatrix} \tag{2.11}$$

being the corresponding Poissonian operator for the Riemann type dynamical system (1.2). It is also important to observe that the dynamical system (1.2) is a Hamiltonian flow on the functional manifold M^3 with respect to the Poissonian structure (2.11):

$$K[u,v,z] = -\vartheta\ grad\ H_\eta. \tag{2.12}$$

2.2. **Poissonian structure analysis on the functional manifold $\bar{M}^3$.** Below we will construct, for convenience, other Poissonian structures for dynamical system (1.5) on the manifold $\bar{M}^3$, rewritten in the following equivalent form:

$$\left.\begin{aligned} du/dt = v - uu_x \\ dv/dt = \bar{z}_x^2 - uv_x \\ d\bar{z}/dt = 0. \end{aligned}\right\} := \bar{K}[u,v,\bar{z}], \tag{2.13}$$

where $\bar{K} : \bar{M}^3 \to T(\bar{M}^3)$ is the corresponding vector field on $\bar{M}^3$. To proceed with, we need to obtain additional solutions to the related Lax gradient equation (2.3) on the functional manifold $\bar{M}^3$

$$D_t\bar{\psi} + \bar{k}[u,v,z]\bar{\psi} = grad\ \bar{\mathcal{L}}, \tag{2.14}$$

where the matrix operator

$$\bar{k}[u,v,\bar{z}] := \begin{pmatrix} 0 & -v_x & -\bar{z}_x \\ 1 & u_x & 0 \\ 0 & -2\partial\ \bar{z}_x & u_x \end{pmatrix}, \tag{2.15}$$

and which we rewrite in the following componentwise form:

$$
\begin{aligned}
D_t\bar{\psi}^{(1)} &= v_x\bar{\psi}^{(2)} + \bar{z}_x\bar{\psi}^{(3)} + \delta\bar{\mathcal{L}}/\delta u, \\
D_t\bar{\psi}^{(2)} &= -\bar{\psi}^{(1)} - u_x\bar{\psi}^{(2)} + \delta\bar{\mathcal{L}}/\delta v, \\
D_t\bar{\psi}^{(3)} &= 2(\bar{z}_x\bar{\psi}^{(2)})_x - u_x\bar{\psi}^{(3)} + \delta\bar{\mathcal{L}}/\delta\bar{z},
\end{aligned} \tag{2.16}
$$

where a vector $\bar{\psi} := (\bar{\psi}^{(1)}, \bar{\psi}^{(2)}, \bar{\psi}^{(3)})^{\intercal} \in T^*(\bar{M}^3)$. As a simple consequence of (2.16) one obtains the following system of differential relationships:

$$
\begin{gathered}
D_t^3\tilde{\psi}^{(2)} = -2\bar{z}_x^2\tilde{\psi}_x^{(2)} + D_t^2\partial^{-1}(\delta\bar{\mathcal{L}}/\delta v) - \\
-\partial^{-1} < grad\ \bar{\mathcal{L}}, (u_x, v_x, \bar{z}_x)^{\intercal} >, \\
D_t\tilde{\psi}^{(2)} = -\tilde{\psi}^{(1)} + \partial^{-1}(\delta\bar{\mathcal{L}}/\delta v), \\
D_t\tilde{\psi}^{(3)} = 2\bar{z}_x\tilde{\psi}_x^{(2)} + \partial^{-1}(\delta\bar{\mathcal{L}}/\delta\bar{z}),
\end{gathered} \tag{2.17}
$$

where we have put, by definition, $(\bar{\psi}^{(1)}, \bar{\psi}^{(2)}, \bar{\psi}^{(3)})^{\intercal} := (\tilde{\psi}_x^{(1)}, \tilde{\psi}_x^{(2)}, \tilde{\psi}_x^{(3)})^{\intercal}$. The latter make it possible, having solved the first equation of system (2.17), to solve recurrently its next two equations. Namely, it is easy to observe that the following three vector elements

$$
\begin{gathered}
\tilde{\psi}_0 = (-v, u, -2\bar{z}_x)^{\intercal}, \quad \bar{\mathcal{L}}_0 = 0; \\
\tilde{\psi}_\theta = (-u_x/\bar{z}_x, 1/\bar{z}_x, (u_x^2 - 2v_x)/(2\bar{z}_x^2))^{\intercal}, \bar{\mathcal{L}}_\theta = 0; \\
\tilde{\psi}_\eta = (u/2, 0, \partial^{-1}[(2v_x - u_x^2)/(2\bar{z}_x)], \bar{\mathcal{L}}_\eta = (D_x\tilde{\psi}_\eta, \bar{K}) - H_\vartheta,
\end{gathered} \tag{2.18}
$$

solve the system (2.17). The first two elements of (2.18) give rise to the Volterra symmetric vectors $\bar{\psi}_0 = D_x\tilde{\psi}_0, \bar{\psi}_\theta = D_x\tilde{\psi}_\theta \in T^*(\bar{M}^3) : \bar{\psi}_0' = \bar{\psi}_0^{\prime,*}, \bar{\psi}_\theta' = \bar{\psi}_\theta^{\prime,*}$ entailing the trivial conservation laws $(\bar{\psi}_0, \bar{K}) = 0 = (\bar{\psi}_\theta, \bar{K})$. The third element of (2.18) gives rise to the Volterra not symmetric vector $\bar{\psi}_\eta := D_x\tilde{\psi}_\eta : \bar{\psi}_\eta' \neq \bar{\psi}_\eta^{\prime,*}$, entailing the next inverse co-symplectic functional expression:

$$
\bar{\eta}^{-1} := \bar{\psi}_\eta' - \bar{\psi}_\eta^{\prime,*} = \left(\begin{array}{ccc}
\partial & 0 & -\partial\frac{u_x}{\bar{z}_x} \\
0 & 0 & \partial\frac{1}{\bar{z}_x} \\
-\frac{u_x}{\bar{z}_x}\partial & \frac{1}{\bar{z}_x}\partial & \begin{array}{c}\frac{u_x}{2\bar{z}_x}\partial\frac{u_x}{\bar{z}_x} - \\ -\frac{v_x}{\bar{z}_x}\partial\frac{1}{\bar{z}_x} - \frac{1}{\bar{z}_x}\partial\frac{v_x}{\bar{z}_x}\end{array}
\end{array}\right). \tag{2.19}
$$

Respectively, the Poissonian operator $\bar{\eta} : T^*(\bar{M}^3) \to T(\bar{M}^3)$ equals

$$
\bar{\eta} = \left(\begin{array}{ccc}
\partial^{-1} & u_x\partial^{-1} & 0 \\
\partial^{-1}u_x & v_x\partial^{-1} + \partial^{-1}v_x & \partial^{-1}\bar{z}_x \\
0 & \bar{z}_x\partial^{-1} & 0
\end{array}\right), \tag{2.20}
$$

subject to which the following Hamiltonian representation

$$
\bar{K}[u, v, \bar{z}] = -\bar{\eta}\ grad\ H_\eta|_{z=z_x^2} \tag{2.21}
$$

on the manifold $\bar{M}^3$ holds.

2.3. **Hamiltonian integrability analysis.** Turn now back to the integrability analysis of dynamical system (1.7) on the functional manifold M^3. It is easy to recalculate the form of Poissonian operator (2.20) on the manifold $\bar{M}^3$ to that acting on the manifold M^3 :

$$
\eta := \left(\begin{array}{ccc}
\partial^{-1} & u_x\partial^{-1} & 0 \\
\partial^{-1}u_x & v_x\partial^{-1} + \partial^{-1}v_x & \partial^{-1}\bar{z}_x \\
0 & \bar{z}_x\partial^{-1} & 0
\end{array}\right), \tag{2.22}
$$

subject to which the Hamiltonian representation (2.21) becomes, respectively,

$$
K[u, v, z] = -\eta\ grad\ H_\vartheta\ . \tag{2.23}
$$

As next important point we have checked that the Poissonian operators (2.11) and (2.22) are compatible [9, 8, 10, 7] on the manifold $\bar{M}^3$, that is the operator pencil $(\vartheta + \lambda\eta) : T^*(M^3) \to T(M^3)$ is also Poissonian for arbitrary $\lambda \in \mathbb{R}$. As a consequence, any operator of the form

$$
\vartheta_n := \vartheta(\vartheta^{-1}\eta)^n \tag{2.24}
$$

for all $n \in \mathbb{Z}$ is Poissonian on the manifold M^3. Based now on the homotopy formula (2.6) and recursion property of the Poissonian pair (2.12) and (2.22), it is easy to construct a related infinite hierarchy of commuting to each other conservation laws

$$\begin{aligned} \gamma_j &= \int_0^1 d\mu(grad\ \gamma_j[\mu u, \mu v, \mu z], (u, v, z)^\intercal), \\ grad\ \gamma_j[u, v, z] &:= \Lambda^j grad\ H_\eta, \end{aligned} \tag{2.25}$$

for the dynamical system (1.7), where $j \in \mathbb{Z}_+$ and $\Lambda := \vartheta^{-1}\eta : T^*(M^3) \to T^*(M^3)$ is the corresponding recursion operator, satisfying the so called associated Lax type commutator relationship

$$d\Lambda/dt = [\Lambda, K^{\prime,*}]. \tag{2.26}$$

Thus, one can formulate the following proposition.

Proposition 2.1. *The Riemann type hydrodynamic system (1.7) is a bi-Hamiltonian dynamical system on the functional manifold M^3 with respect to two compatible Poissonian structures $\vartheta, \eta : T^*(M^3) \to T(M^3)$*

$$\vartheta := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2z^{1/2}D_x z^{1/2} \end{pmatrix}, \eta := \begin{pmatrix} \partial^{-1} & u_x\partial^{-1} & 0 \\ \partial^{-1}u_x & v_x\partial^{-1} + \partial^{-1}v_x & \partial^{-1}z_x - 2z \\ 0 & z_x\partial^{-1} + 2z & 0 \end{pmatrix} \tag{2.27}$$

and possessing an infinite hierarchy of commuting to each other conservation laws (2.25).

Concerning the existence of an additional infinite and parametrically $\mathbb{R} \ni \lambda$-ordered hierarchy of conservation laws for dynamical system (1.2) we note also that the following already dispersive nonlinear dynamical system

$$\left.\begin{aligned} du/d\tau &= -(z^{-1/2})_x \\ dv/d\tau &= -(u_x z^{-1/2})_x \\ dz/d\tau &= z^{1/2}(\tfrac{u_x^2 - 2v_x}{2z})_x \end{aligned}\right\} = -\vartheta\ grad\ H_0[u, v, z] := \tilde{K}[u, v, z] \tag{2.28}$$

allows by means of solving the corresponding Lax equation

$$d\tilde{\varphi}/dt + \tilde{K}^{\prime,*}\tilde{\varphi} = 0 \tag{2.29}$$

for an element $\tilde{\varphi} \in T^*(M^3)$ in a suitably chosen asymptotic form to construct an infinite ordered hierarchy of conservation laws for (1.2), on which we will not stop here. The latter and the existence of an infinite and parametrically $\mathbb{R} \ni \lambda$-ordered hierarchy of conservation laws for the Riemann type dynamical system (1.2) strongly motivates us that it is a completely integrable by Lax nonlinear dynamical system on the functional manifold M^3. It will be stated in the next Section by means of new differential-algebraic tools, devised recently in [6, 1, 2].

3. Differential-algebraic integrability analysis: the case $N = 3$

Consider a polynomial differential ring $\mathcal{K}\{u\} \subset \mathcal{K} := \mathbb{R}\{\{x, t\}\}$ generated by a fixed functional variable $u \in \mathbb{R}\{\{x, t\}\}$ and invariant with respect to two differentiations $D_x := \partial/\partial x$ and $D_t := \partial/\partial t + u\partial/\partial x$, satisfying the Lie-algebraic commutator relationship (1.2)

$$[D_x, D_t] = u_x D_x \tag{3.1}$$

jointly with constraint (1.6) in the differential-algebraic functional form

$$D_t^3 u = -2D_t^2 u D_x u.$$

Since the Lax type representation for the dynamical system (1.7) can be interpreted [1, 10] as the existence of a finite-dimensional invariant ideal $\mathcal{I}\{u\} \subset \mathcal{K}\{u\}$, realizing the corresponding finite-dimensional representation of the the Lie-algebraic commutator relationship (3.1), this ideal can be constructed as

$$\mathcal{I}\{u\} := \{\lambda^2 u f_1 + \lambda v f_2 + z^{1/2} f_3 \in \mathcal{K}\{u\} : f_j \in \mathcal{K}, j \in \overline{1,3}, \lambda \in \mathbb{R}\}, \tag{3.2}$$

where $v = D_t u, z = D_t^2 u$ and $\lambda \in \mathbb{R}$ is an arbitrary real parameter. To find finite-dimensional representations of the D_x- and D_t-differentiations, it is necessary [1] first to find the D_t-invariant kernel $\ker D_t \subset \mathcal{I}\{u\}$ and next to check its invariance with respect to the D_x-differentiation. One can obtain easily that the

$$\ker D_t = \{f \in \mathcal{K}^3 : D_t f = q(\lambda) f, \quad \lambda \in \mathbb{R}\}, \tag{3.3}$$

where the matrix $q(\lambda) := q[u, v, z; \lambda] \in End\ \mathcal{K}\{u\}^3$ is given as

$$q(\lambda) = \begin{pmatrix} 0 & 0 & 0 \\ -\lambda & 0 & 0 \\ 0 & -\lambda z_x & u_x \end{pmatrix}. \tag{3.4}$$

Now to find the corresponding representation of the D_x-differentiation in the space $\mathcal{K}^3$, it is enough to find such a matrix $l(\lambda) := l[u, v, z; \lambda] \in End\ \mathcal{K}\{u\}^3$ that

$$D_x f = l(\lambda) f \tag{3.5}$$

for $f \in \mathcal{K}\{u\}^3$ and the related ideal

$$\mathcal{R}\{u\} := \{< g, f >_{\mathcal{K}^3} : f \in \ker D_t \subset \mathcal{K}^3\{u\},\ g \in \mathcal{K}\{u\}^3\} \tag{3.6}$$

is D_x-invariant with respect to the differentiation (3.5). This invariance condition allows to construct by means of simple enough calculations the following matrix

$$l(\lambda) = \begin{pmatrix} \lambda^2 u\sqrt{z} & \lambda v\sqrt{z} & z \\ -\lambda^3 t u\sqrt{z} & -\lambda^2 t v\sqrt{z} & -\lambda t z \\ \lambda^4(tuv - u^2) - & -\lambda v_x/\sqrt{z} + & \lambda^2\sqrt{z}(u - tv) - \\ -\lambda^2 u_x/\sqrt{z} & +\lambda^3(tv^2 - uv) & -z_x/2z \end{pmatrix}. \tag{3.7}$$

Remark 3.1. It is easy enough to make similar to above differential-algebraic calculations for the case $N = 2$ and obtain that the corresponding Riemann type hydrodynamic system

$$D_t u = \bar{z}_x^2, \qquad D_t \bar{z} = 0 \tag{3.8}$$

on the functional manifold $\bar{M}^2$ possesses the following matrix Lax type representation:

$$D_t f = \begin{pmatrix} 0 & 0 \\ -\lambda \bar{z}_x & u_x \end{pmatrix}, \qquad D_x f = \begin{pmatrix} \bar{z}_x & 0 \\ -\lambda(u + u_x/(2\bar{z}_x) & -\bar{z}_{xx}/(2\bar{z}_x) \end{pmatrix} f, \tag{3.9}$$

where $\lambda \in \mathbb{R}$ is an arbitrary spectral parameter and $f \in C^{(\infty)}(\mathbb{R}^2; \mathbb{R}^2)$.

The obtained above results we will formulate as our main proposition subject to the Lax type integrability of the Riemann type hydrodynamic system (1.7) at $N = 3$.

Proposition 3.2. *The Riemann type hydrodynamic flow (1.7) is a Lax type integrable bi-Hamiltonian dynamical system on the functional manifold M^3 with respect to two compatible Poissonian structures*

$$\vartheta := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2z^{1/2} D_x z^{1/2} \end{pmatrix}, \eta := \begin{pmatrix} \partial^{-1} & u_x \partial^{-1} & 0 \\ \partial^{-1} u_x & v_x \partial^{-1} + \partial^{-1} v_x & \partial^{-1} z_x - 2z \\ 0 & z_x \partial^{-1} + 2z & 0 \end{pmatrix}, \tag{3.10}$$

possessing an infinite hierarchy commuting to each other conservation laws and a non-autonomous Lax type representation in the following matrix form:

$$\begin{aligned} D_t f &= \begin{pmatrix} 0 & 0 & 0 \\ -\lambda & 0 & 0 \\ 0 & -\lambda z_x & u_x \end{pmatrix} f, \\ D_x f &= \begin{pmatrix} \lambda^2 u\sqrt{z} & \lambda v\sqrt{z} & z \\ -\lambda^3 t u\sqrt{z} & -\lambda^2 t v\sqrt{z} & -\lambda t z \\ \lambda^4(tuv - u^2) - & -\lambda v_x/\sqrt{z} + & \lambda^2\sqrt{z}(u - tv) - \\ -\lambda^2 u_x/\sqrt{z} & +\lambda^3(tv^2 - uv) & -z_x/2z \end{pmatrix} f, \end{aligned} \tag{3.11}$$

where $\lambda \in \mathbb{R}$ is an arbitrary spectral parameter and $f \in C^{(\infty)}(\mathbb{R}^2; \mathbb{R}^3)$.

The matrices (3.7) are, as seen, of nonstandard form depending explicitly on the temporal evolution parameter $t \in \mathbb{R}$. Nonetheless, the matrices (3.4) and (3.7) satisfy for all $\lambda \in \mathbb{R}$ the well known Zakharov-Shabat compatibility condition

$$D_t l(\lambda) = [q(\lambda), l(\lambda)] + D_x l(\lambda) - u_x l(\lambda), \tag{3.12}$$

following from the Lax type relationships (3.3) and (3.5)

$$D_t f = q(\lambda)f, \qquad D_x f = l(\lambda)f \tag{3.13}$$

and the commutator condition (3.1). Moreover, taking into account that the dynamical system (1.7) possesses a compatible Poissonian pair (2.11) and (2.22) depending only on the variables $(u, v, z)^\intercal \in M^3$ and not depending on the temporal variable $t \in \mathbb{R}$, one can certainly assume that it also possesses a standard autonomous Lax type representation which one can to search by means of a suitable gauge type transformation of (3.13), what we plan to do in a separate work.

4. Conclusion

A new nonlinear Riemann type hydrodynamic equation (1.1) at $N = 2$ and 3 proves to be a very interesting example of a Lax type integrable dynamical system, whose integrability prerequisites, such as compatible Poissonian structures, infinite hierarchy of conservation laws and related Lax type representation were constructed by means of both the symplectic gradient-holonomic approach [8, 10, 11] and new differential-algebraic tools devised recently [1, 4] for studying integrability of a special infinite hierarchy of Riemann type hydrodynamic systems. It is also evident that the dynamical system (1.1) is a Lax type integrable bi-Hamiltonian flow for arbitrary integers $N \in \mathbb{N}$, that can be easily stated by means of the differential-algebraic approach, devised and successfully applied in this work for the case $N = 2$ and 3.

As a most learnable lesson from the present work one can infer the following statement: if *a priori* given nonlinear dynamical system is within the symplectic gradient-holonomic tools suspected to be Lax type integrable, then its Lax type representation, if it exists, can be successfully found by means of a suitably constructed invariant differential ideal $\mathcal{I}\{u\}$ of the ring $\mathcal{K}\{u\}$ within the differential-algebraic approach, mentioned above and applied in this work. Thereby, it would be important to test these differential-algebraic tools applying them to other Lax type integrable nonlinear dynamical systems and to single out those algebraic structures responsible for the existence of a related finite-dimensional matrix representation for the basic D_x- and D_t-differentiations in a vector space $\mathcal{K}^p$ for some finite $p \in \mathbb{Z}_+$. As a particular differential-algebraic problem of interest, concerning these matrix representations, one can conceive the one consisting in effective construction of functional generators of the corresponding invariant finite-dimensional ideals $\mathcal{I}\{u\} \subset \mathcal{K}\{u\}$ under given differential-algebraic constraints imposed on the D_x- and D_t-differentiations.

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