Integrable equations and classical S-matrix

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Abstract
We study amplitudes of five-wave interactions for evolution Hamiltonian equations differ from the KdV equation by the form of dispersion law. We find that five-wave amplitude is canceled for all three known equations (KdV, Benjamin-Ono and equation of intermediate waves) and for two new equations which are natural generalizations of mentioned above.

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1 Introduction

At least three important Hamiltonian evolutionary equations appearing in the theory of ocean waves are completely integrable (see, for instance [1]). They are:

1. KdV equation

$$u_t = u_{xxx} + uu_x \tag{1.1}$$

2. Benjamin-Ono equation

$$u_t = \hat{I}(u_{xx}) + uu_x \tag{1.2}$$

Here $\hat{I}$ is the Hilbert transform.

3. Intermediate wave equation

$$u_t = \hat{F}(u) + uu_x \tag{1.3}$$

Here $\hat{F}$ is a pseudo-differential operator with symbol

$$F(k) = ak^2 \coth bk - ck. \tag{1.4}$$

In the limit $b \to \infty,\ a = 1,\ c = 0$ equation (1.4) tends to the Benjamin-Ono equation. In the limit $b \to 0,\ a = \frac{3}{b},\ c = \frac{3}{b^2}$ equation (1.4) goes to the KdV equation.

In this article we address the following question: could one find other integrable equations of the type (1.3)? We presume that the discussed equations are Hamiltonian and admit the Gardner Poisson structure

$$u_t = \frac{\partial}{\partial x}\frac{\delta H}{\delta u}$$

or, in terms of Fourier transforms

$$u(k)_t = ik\frac{\delta H}{\delta u(k)}, \quad u(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{-ikx}dx$$

Here $H = H_2 + H_3$ where

$$H_2 = \frac{1}{2}\int_{-\infty}^{\infty}\frac{F(k)}{k}u(k)u(-k)dk, \quad H_3 = \frac{1}{6}u^3 = \frac{1}{6}\int_{-\infty}^{\infty}u(k_1)u(k_2)u(k_3)\delta(k_1+k_2+k_3)dk_1dk_2dk_3$$

Thus we assume that $F(k)$ is an odd function, $F(-k) = -F(k)$. For KdV we have $F(k) = -k^3$ and for Benjamin-Ono equation $F(k) = -|k|k$.

In this article we classify all integrable equations of the form (1.3). The answer is the following: there is only one extra equation given by

$$F(k) = ak^2 \cot bk - ck. \tag{1.5}$$

In the limit $b \to 0,\ a = -\frac{3}{b},\ c = \frac{3}{b^2}$ we get $F(k) = -k^3$. The dispersion relation (1.5) has singularities at $k_n = \frac{\pi n}{b}$.

We show also that the following $(1+2)$-dimensional equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}\hat{L}\Big(\frac{\partial}{\partial y}\Big)u + uu_x \tag{1.6}$$

where $u = u(x,y,t)$ and $\hat{L}(p) = \epsilon\frac{e^{\epsilon p}+1}{e^{\epsilon p}+1}$ is integrable. In the limit $\epsilon \to 0$ this equation is well-known Khokhlov-Zabolotskaya equation (see for instance [7]).

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}\Big(\frac{\partial}{\partial y}\Big)^{-1}u + uu_x.$$

This equation is the dispersionless limit of both KP1 and KP2 equations.

Note that equation 1.3 has the following universal conservation laws for arbitrary function $F(k)$:

$$I_0 = \int_{-\infty}^{\infty} u dx,\ I_1 = \int_{-\infty}^{\infty} u^2 dx,\ I_3 = H.$$

The question is for which functions $F(k)$ there exists at least one additional conservation law given by a power series in u starting from the quadratic term

$$I_3 = I^{(2)} + I^{(3)} + ...$$

where

$$I^{(2)} = \int_{-\infty}^{\infty} g^{(2)}(k)u(k)u(-k)dk, \quad I^{(3)} = \int_{-\infty}^{\infty} g^{(3)}(k_1,k_2)u(k_1)u(k_2)u(-k_1-k_2)dk_1dk_2, ...$$

and $g^{(2)}(k) = g^{(2)}(-k)$ is a real function different from a linear combination $c_1k + c_2F(k)$?

Existence of this conservation law is not a proof of integrability, while nonexistence is a clear manifestation of non-integrability. Thus to accomplish our task we must prove integrability of all new equations separately. This will be done in other publication.

2 Scattering matrix

Following Zakharov and Shulman [2]-[6] one can introduce a so-called formal scattering matrix for the equation (1.3) with arbitrary $F(k)$. Let us write this equation in Fourier components

$$u(k)_t = iF(k)u(k) + ik\int_{-\infty}^{\infty} u(k_1)u(k_2)\delta(k-k_1-k_2)dk_1dk_2 \tag{2.7}$$

Than we introduce $c(k)$ by $u(k) = c(k)e^{iF(k)t}$, assume that $c(k) \to c^-(k)$ when $t \to -\infty$ and rewrite equation (2.7) in Picard form as follow

$$c(k) = c^-(k) + ik\lim_{\epsilon\to 0}\int_{-\infty}^{t}\int_{-\infty}^{\infty} c(k_1,\tau)c(k_2,\tau)e^{i(F(k_1)+F(k_2)-F(k))\tau-\epsilon|\tau|}\delta(k-k_1-k_2)dk_1dk_2d\tau \tag{2.8}$$

Than we solve equation (2.8) by iterations, send $t \to \infty$ and than $\epsilon \to 0$. Let $c(k,t) \to c^+(k)$ when $t \to \infty$. We end up with c^+ expressed through $c-$ in terms of the so-called formal scattering matrix S:

$$c^+(k) = Sc^-(k) = c^-(k) + \tag{2.9}$$

$$\sum_{n=2}^{\infty} \int_{-\infty}^{\infty} S(k, k_1, ..., k_n)\delta(F(k) - F(k_1) - ... - F(k_n))\delta(k - k_1 - ... - k_n)c^-(k_1)...c^-(k_n)dk_1...dk_n$$

Functions $S(k, k_1, ..., k_n)$ are called amplitudes of wave scattering of order $n+1$. Arguments of delta-functions in (2.9) are called resonance conditions and equation

$$S(k, k_1, ..., k_n) = 0$$

where $k, k_1, ..., k_n$ are subject to resonance conditions is called $n+1$-wave equation. The first resonance condition

$$F(k) = F(k_1) + F(k_2), \quad k = k_1 + k_2$$

has only trivial solutions like $k = 0,\ k_2 = -k_1$ or $k_2 = 0,\ k = k_1$, thus three-wave equation is not significant. In the same way four-wave resonance conditions

$$F(k) = F(k_1) + F(k_2) + F(k_3), \quad k = k_1 + k_2 + k_3$$

have only trivial solutions like $k = k_1,\ k_3 = -k_2$. Hence the first nonlinear resonance process is five-waves interaction governed by resonance conditions

$$k = k_1 + k_2 + k_3 + k_4, \quad F(k) = F(k_1) + F(k_2) + F(k_3) + F(k_4) \tag{2.10}$$

Suppose that $k > 0$. At least one wave vector in the right hand side of (2.10) must be negative. Let $k_4 < 0$. We replace $k_4 \to -k_4$ and $k \to k_5$ and rewrite equations (2.10) as follow

$$k_4 + k_5 = k_1 + k_2 + k_3, \quad F(k_4) + F(k_5) = F(k_1) + F(k_2) + F(k_3) \tag{2.11}$$

All wave vectors in (2.11) are positive. Moreover, we assume them ordered as follow

$$k_2 > k_4 > k_5 > k_3 > k_1$$

Under this assumption the five-wave amplitude is

$$S(k_1, k_2, k_3, k_4, k_5) = F_{12}(F_{45} + G_{53} + G_{43}) + F_{13}(F_{45} + G_{23} + G_{24}) +$$

$$G_{51}(F_{23} + G_{43} + G_{24}) + G_{41}(F_{23} + G_{53} + G_{25}) + F_{45}F_{23} + G_{24}G_{53} + G_{25}G_{43}$$

Here

$$F_{ij} = \frac{k_i + k_j}{F(k_i + k_j) - F(k_i) - F(k_j)}, \quad G_{ij} = \frac{k_i - k_j}{F(k_i - k_j) - F(k_i) + F(k_j)}$$

for $i \neq j = 1, ..., 5$.

The necessary condition for integrability is cancellation of five-wave amplitude on the resonance manifold (2.11).

3 Cancellation of five-wave amplitude for known integrable systems

Let $F(k) = k^3$ (this is KdV case). Then

$$F_{ij} = \frac{1}{3k_i k_j}, \ G_{ij} = -\frac{1}{3k_i k_j}.$$

After simple calculation we obtain

$$S_{12345} = \frac{1}{9k_1k_2k_3k_4k_5}(k_4 + k_5 - k_1 - k_2 - k_3) = 0. \tag{3.12}$$

Notice that for cancellation of five-waves amplitude in this case we do not use the frequency resonance condition in (2.11).

As far as all $k_i > 0$ one can set $F(k) = k^2$ for the Benjamin-Ono case. Then

$$F_{ij} = \frac{k_i + k_j}{2k_i k_j}, \quad G_{ij} = -\frac{1}{2k_j},$$

Now

$$S_{12345} = \frac{k_1k_4k_3 + k_1k_5k_3 - k_1k_4k_5 + k_2k_4k_3 + k_2k_5k_3 - k_2k_4k_5 - k_5k_3k_4 + k_2k_1k_4 + k_2k_1k_5}{2k_1k_2k_3k_4k_5}.$$

One can check that it can be written in the form

$$\frac{k_4 + k_5}{4k_1k_2k_3k_4k_5}(k_4^2 + k_5^2 - k_1^2 - k_2^2 - k_3^2) + \frac{(k_1 + k_2 + k_3)(k_4 + k_5) + k_4^2 + k_5^2}{4k_1k_2k_3k_4k_5}(k_1 + k_2 + k_3 - k_4 - k_5). \tag{3.13}$$

Thus the cancellation by virtue of (2.11) is obvious.

To check cancellation for the generic dispersion relation (1.4) one notice first that S_{12345} is invariant with respect to transformation $F(k) \to F(\alpha k) + \beta k$ where $\alpha \neq 0$, β are arbitrary constants. Moreover, one can replace $F(k)$ by $F(k, p) = k^{2\frac{1+e^p}{1-e^p}}$ because exponent is not an algebraic function. Resonance conditions now read:

$$p_4 + p_5 = p_1 + p_2 + p_3, \quad k_4 + k_5 = k_1 + k_2 + k_3$$

and

$$F(k_4, p_4) + F(k_5, p_5) = F(k_1, p_1) + F(k_2, p_2) + F(k_3, p_3).$$

Now five-waves amplitude depends on ten variables

$$S_{12345} = S(k_1, ..., k_5, p_1, ..., p_5).$$

Checking cancellation of this amplitude by virtue of resonance conditions took approximately ten minutes for computer algebra system Maple. Explicit representation of S_{12345} in the form similar to (3.12) and (3.13) is so cumbercome that we do not present it here.

4 Solving the functional equation

Consider the five-waves equation $S(k_1, ..., k_5) = 0$ as a functional equation for function $f(k) = F(k)$. This functional equation reads:

$$F(k_1,k_2)(F(k_4,k_5)+G(k_5,k_3)+G(k_4,k_3))+F(k_1,k_3)(F(k_4,k_5)+G(k_2,k_5)+G(k_2,k_4))+$$
$$G(k_5,k_1)(F(k_2,k_3)+G(k_4,k_3)+G(k_2,k_4))+G(k_4,k_1)(F(k_2,k_3)+G(k_5,k_3)+G(k_2,k_5))+ \quad (4.14)$$
$$F(k_4,k_5)F(k_2,k_3)+G(k_2,k_4)G(k_5,k_3)+G(k_2,k_5)G(k_4,k_3)=0$$

where $F(x,y) = \frac{x+y}{f(x+y)-f(x)-f(y)}$, $G(x,y) = \frac{x-y}{f(x-y)-f(x)+f(y)}$ and $k_1, ..., k_5$ satisfy the following constrains:

$$k_1 + k_2 + k_3 = k_4 + k_5, \tag{4.15}$$
$$f(k_1) + f(k_2) + f(k_3) = f(k_4) + f(k_5). \tag{4.16}$$

Note that if $f(k)$ is a solution of this functional equation, then $f_1(k) = af(bk) + ck$ is also a solution for arbitrary constants $a, b \neq 0,\ c$. We say that such solutions are equivalent.

Proposition. Any solution of the functional equation (4.14) analytic near zero and such that $f(0) = 0$ is equivalent to one of the following: $f_1(k) = k^2,\ f_2(k) = k^3,\ f_3(k) = k^2\frac{e^k+1}{e^k-1}$.

Remark. Here we suppose that $a, b \neq 0$ in our equivalence relation are complex numbers. If we restrict ourself to real numbers, then there exist one more non-equivalent solution $f_4(k) = k^2 \cot(k)$.

Proof. Set $k_2 = k_4 + u,\ k_5 = k_3 + v$, then the constrain (4.15) is equivalent to $k_1 = v - u$. Expanding the constrain (4.16) near $u = v = 0$ we obtain

$$v = \frac{f'(k_4)}{f'(k_3)}u + o(u). \tag{4.17}$$

Expanding (4.14) near $u = v = 0$ and substituting (4.17) we obtain in the first non-trivial term:

$$-(-f(k_4)+f(k_4-k_3)+f(k_3))(f(k_4)-f(k_4+k_3)+f(k_3))(-4k_3f(k_3)+2k_3f(k_4+k_3)-k_4f(k_4-k_3)-$$
$$k_4f(k_4+k_3)-2k_3f(k_4-k_3)+2k_4f(k_4))f'(k_4)^2+$$
$$(-f(k_4)+f(k_4-k_3)+f(k_3))(f(k_4)-f(k_4+k_3)+f(k_3))$$
$$(-2k_3f(k_3)+k_3f(k_4+k_3)-2k_4f(k_4-k_3)-2k_4f(k_4+k_3)-k_3f(k_4-k_3)+4k_4f(k_4))f'(k_3)^2+$$
$$k_4(-f(k_4)+f(k_4-k_3)+f(k_3))(f(k_4)-f(k_4+k_3)+f(k_3))(2k_4f(k_4)+k_3f(k_4+k_3)-k_3f(k_4-k_3)-$$
$$2k_3f(k_3)-k_4f(k_4-k_3)-k_4f(k_4+k_3))f'(k_3)f''(k_4)-$$
$$k_3(-f(k_4)+f(k_4-k_3)+f(k_3))(f(k_4)-f(k_4+k_3)+f(k_3))$$

$$(2k_4f(k_4)+k_3f(k_4+k_3)-k_3f(k_4-k_3)-2k_3f(k_3)-k_4f(k_4-k_3)-k_4f(k_4+k_3))f''(k_3)f'(k_4)$$
$$+(-2k_3^2f(k_4)^2-k_3^2f(k_4-k_3)^2-k_3^2f(k_4+k_3)^2+2k_4^2f(k_4)f(k_4+k_3)-$$
$$4k_3k_4f(k_4-k_3)f(k_3)+4k_3k_4f(k_4)f(k_4-k_3)-2k_4k_3f(k_4-k_3)^2-k_4^2f(k_4+k_3)^2-$$
$$4k_3k_4f(k_4)f(k_4+k_3)+2k_4^2f(k_3)f(k_4+k_3)+2k_3^2f(k_4+k_3)f(k_3)+8k_4k_3f(k_4)f(k_3)+$$
$$2k_3^2f(k_4)f(k_4+k_3)+2k_3k_4f(k_4+k_3)^2+2k_3^2f(k_4-k_3)f(k_4)-4k_3k_4f(k_4+k_3)f(k_3)-k_4^2f(k_4-k_3)^2-$$
$$2k_4^2f(k_4)^2-2k_3^2f(k_4-k_3)f(k_3)-2k_4^2f(k_3)f(k_4-k_3)-$$
$$2k_3^2f(k_3)^2-2k_4^2f(k_3)^2+2k_4^2f(k_4)f(k_4-k_3))f'(k_4)^2f'(k_3)-$$
$$(-f(k_4)+f(k_4-k_3)+f(k_3))(f(k_4)-f(k_4+k_3)+f(k_3)) \tag{4.18}$$
$$(2k_3f(k_3)-k_3f(k_4+k_3)-k_4f(k_4-k_3)-k_4f(k_4+k_3)+k_3f(k_4-k_3)+2k_4f(k_4))f'(k_3)f'(k_4)+$$
$$(k_4^2f(k_4+k_3)^2+2k_4k_3f(k_4-k_3)^2+4k_3k_4f(k_4-k_3)f(k_3)-2k_4^2f(k_4)f(k_4-k_3)+$$
$$k_4^2f(k_4-k_3)^2+k_3^2f(k_4+k_3)^2+2k_3^2f(k_4)^2+2k_4^2f(k_3)f(k_4-k_3)+2k_4^2f(k_4)^2+k_3^2f(k_4-k_3)^2+$$
$$4k_3k_4f(k_4+k_3)f(k_3)-2k_3k_4f(k_4+k_3)^2-2k_3^2f(k_4-k_3)f(k_4)+2k_4^2f(k_3)^2-2k_4^2f(k_3)f(k_4+k_3)+$$
$$4k_3k_4f(k_4)f(k_4+k_3)-8k_4k_3f(k_4)f(k_3)+2k_3^2f(k_3)^2+2k_3^2f(k_4-k_3)f(k_3)-$$
$$2k_4^2f(k_4)f(k_4+k_3)-4k_3k_4f(k_4)f(k_4-k_3)-2k_3^2f(k_4)f(k_4+k_3)-2k_3^2f(k_4+k_3)f(k_3))f'(k_3)^2f'(k_4)-$$
$$k_4(-f(k_4)+f(k_4-k_3)+f(k_3))^2(f(k_4)-f(k_4+k_3)+f(k_3))^2f''(k_4)+$$
$$k_3(-f(k_4)+f(k_4-k_3)+f(k_3))^2(k_4+k_3)f'(k_4+k_3)f'(k_4)^2-$$
$$k_4(-f(k_4)+f(k_4-k_3)+f(k_3))^2(k_4+k_3)f'(k_4+k_3)f'(k_3)^2+$$
$$(-f(k_4)+f(k_4-k_3)+f(k_3))^2(k_4-k_3)(k_4+k_3)f'(k_4+k_3)f'(k_3)f'(k_4)-$$
$$k_3(f(k_4)-f(k_4+k_3)+f(k_3))^2(k_4-k_3)f'(k_4-k_3)f'(k_4)^2-$$
$$k_4(f(k_4)-f(k_4+k_3)+f(k_3))^2(k_4-k_3)f'(k_4-k_3)f'(k_3)^2+$$
$$k_3(-f(k_4)+f(k_4-k_3)+f(k_3))^2(f(k_4)-f(k_4+k_3)+f(k_3))^2f''(k_3)+$$
$$2(-f(k_4)+f(k_4-k_3)+f(k_3))^2(f(k_4)-f(k_4+k_3)+f(k_3))^2f'(k_4)-$$
$$2(-f(k_4)+f(k_4-k_3)+f(k_3))^2(f(k_4)-f(k_4+k_3)+f(k_3))^2f'(k_3)+$$
$$(f(k_4)-f(k_4+k_3)+f(k_3))^2(k_4-k_3)(k_4+k_3)f'(k_4-k_3)f'(k_3)f'(k_4)=0$$

Assume without loss of generality $f(k)=a_2k^2+a_3k^3+...$ Expanding (4.18) near $k_3=k_4=0$ we get $a_2a_3=0$. We have different cases:

Case 1. Let $a_2\neq 0$, then $a_3=0$. Without loss of generality we assume $f(k)=k^2+a_4k^4+a_5k^5...$ Expanding (4.18) near $k_3=0$ we obtain under this assumption in the first non-trivial term:

$$-5f'(k_4)^2+8k_4f''(k_4)f'(k_4)+f'(k_4)k_4^2f'''(k_4)-3k_4^2f''(k_4)^2=0.$$

The only solution of this 3rd-order ODE of the form $f(k)=k^2+a_4k^4+a_5k^5...$ is $f(k)=k^2$.

Case 2. Let $a_3 \neq 0$, then $a_2 = 0$. Without loss of generality we assume $f(k) = k^3 + a_4 k^4 + a_5 k^5 ...$ Expanding (4.18) near $k_3 = 0$ we obtain under this assumption in the first non-trivial term:

$$-4f'(k_4)^2 f'''(k_4) + 12f'(k_4)^2 - k_4 f'(k_4)^2 f''''(k_4) - 2f'(k_4)k_4^2 f'''(k_4)+$$

$$4f'(k_4)k_4 f''(k_4)f'''(k_4) - 18k_4 f''(k_4)f'(k_4) + 6f'(k_4)f''(k_4)^2 - 3f''(k_4)^3 k_4 + 6k_4^2 f''(k_4)^2 = 0.$$

Any solution of this 4th-order ODE of the form $f(k) = k^3 + a_4 k^4 + a_5 k^5 ...$ is equivalent to either $f(k) = k^3$ or $f(k) = k^2 \frac{e^k+1}{e^k-1}$ (or $f(k) = k^2 \cot(k)$ if our group of equivalence is real rather then complex).

Note that in the case $a_2 = a_3 = 0$ there are not non-trivial solutions.

5 New equations

Our results show that if we set

$$F(k) = ak^3 \cot bk - ck$$

where $a,\ b,\ c$ are constants, the five-wave amplitude is also zero. The corresponding equation hardly has any physical importance because $F(k) = \infty$ at $bk = \pi n$. Anyway, in the limit $b \to 0,\ a = \frac{3}{b},\ c = \frac{3}{b^2}$ we get $F * k) = k^3$ and the equations goes to the KdV. Another equation is more interesting. Let $u = u(x, y, t)$ be a function in two spacial coordinated $x,\ y$ variables and F is given by

$$F(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = a\frac{\partial^2}{\partial x^2} \coth a\frac{\partial}{\partial y}.$$

The five-wave amplitude is zero in this case. This equation might be useful in applications. In the limit $a \to 0$ we have $F \to \frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial x}\right)^{-1}$ and our equation degenerates to the following form

$$u_t = \partial_x^2 \partial_y^{-1} u + uu_x.$$

This equation can be compared with Khokhov-Zabolotskaya equation (see, for instance, [7])

$$u_t = \partial_y^2 \partial_x^{-1} u + uu_x.$$

But these equations are not equivalent.

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