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RIEMANN-HILBERT PROBLEMS WITH CANONICAL NORMALIZATION AND FAMILIES OF COMMUTING OPERATORS.

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Abstract. We start with a Riemann-Hilbert Problems (RHP) with canonical normalization whose sewing functions depends on several additional variables. Using Zakharov-Shabat theorem we are able to construct a family of ordinary differential operators for which the solution of the RHP is a common fundamental analytic solution. This family of operators obviously commute. Thus we are able to construct new classes of integrable nonlinear evolution equations.

1. Introduction

The development of the soliton theory revealed an important class of NLEE (nonlinear evolution equations) that describe special types of wave-wave interactions [1, 29, 16, 4, 23, 19, 30] which play important role in various fields in physics.

A formal approach to the integrable equations started by Gel'fand and Dickey [3, 17] and developed actively later on (see e.g. [17] and the references therein) is well known. It allows one to construct the Lax representations for important classes of NLEE such as the dispersionless KP hierarchy but it disregards the spectral properties of the Lax operators.

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The topic quickly attracted mathematicians from spectral theory, dynamical systems, Lie algebras, Hamiltonian dynamics, differential geometry, see [29, 25, 34, 35, 5, 16, 4] and the numerous references therein. It attracted also a number of physicists because they found important applications of these NLEE in fluid mechanics, nonlinear optics, superconductivity, plasma physics etc. As a result many different approaches for investigating the soliton equations and constructing their Lax representations, soliton solutions, integrals of motion, Hamiltonian hierarchies etc. were developed, see [36, 25, 2, 29, 23, 35, 21]. Of course, it is not possible in a short paper to list all important references that cover the broad topics mentioned above.

The inverse scattering method has been applied to many physically important multidimensional evolution equations including the N-wave equation, Davey-Stewartson, Kadomtsev-Petviashvilli etc. [33, 21, 22, 34, 35, 36, 28]. They have been treated by nonlocal generalizations of the Riemann-Hilbert problem and by the $\bar{\partial}$-method.

In the present paper we propose an alternative approach to the same class of equations using as a starting point the Riemann-Hilbert problem (RHP) [38, 39, 34, 35, 29, 37]; the importance of the canonical normalization of RHP was noticed in [10, 6]. Our aim is to show that this allows one to construct rings of commuting operators and in addition gives a tool to study their spectral properties.

In Section 2 below we start with some preliminaries concerning the RHP. In Section 3 we use the solutions of the RHP to construct family of jets of order k, in Section 4 we list their simplest reductions. In the last two Sections we demonstrate how this construction can be used to solve NLEE in two and higher dimensional space-times. In Section 5 we use jets of order 1 to reproduce well known results about the 3-wave equations in two- and three-dimensional space-times. We also demonstrate the integrability of N-wave type equations in higher dimensional space-times. In Section 6 we use jets of order 2 which allows us to construct new types of integrable N-wave interactions whose interaction terms contain quadratic and cubic nonlinearities, as well as x-derivatives. These equations also allow integrable extensions to three-dimensional space-time. The last Section contains discussion and conclusions.

2. RHP with canonical normalization

Let us formulate the RHP:

$$\xi^+(\vec{x},t,\lambda) = \xi^-(\vec{x},t,\lambda)G(\vec{x},t,\lambda), \qquad \lambda^k \in \mathbb{R}, \qquad \lim_{\lambda\to\infty} \xi^+(\vec{x},t,\lambda) = \mathbb{1}, \tag{1}$$

where $\xi^{\pm}(\vec{x},t,\lambda)$ take values in the simple Lie group $\mathfrak{G}$ with Lie algebra $\mathfrak{g}$. $\xi^{+}(\vec{x},t,\lambda)$ (resp. $\xi^{-}(\vec{x},t,\lambda)$) is an analytic functions of λ for $\operatorname{Im}\lambda^k > 0$ (resp. for $\operatorname{Im}\lambda^k < 0$). For simplicity we consider particular type of dependence of the sewing function $G(\vec{x},t,\lambda)$ on the auxiliary variables:

$$i\frac{\partial G}{\partial x_s} - \lambda^k[J_s, G(\vec{x},t,\lambda)] = 0, \qquad i\frac{\partial G}{\partial t} - \lambda^k[K, G(\vec{x},t,\lambda)] = 0. \tag{2}$$

where $k \geq 1$ is a fixed integer and J_s are linearly independent elements of the Cartan subalgebra $J_s \in \mathfrak{h} \subset \mathfrak{g}$.

The canonical normalization of the RHP means that we can introduce the asymptotic expansion

$$\xi^{\pm}(\vec{x},t,\lambda) = \exp Q(\vec{x},t,\lambda), \qquad Q(\vec{x},t,\lambda) = \sum_{k=1}^{\infty} Q_k(\vec{x},t)\lambda^{-k}. \tag{3}$$

Since $\xi^{\pm}(\vec{x},t,\lambda)$ are group elements then all $Q_k(\vec{x},t) \in \mathfrak{g}$. However,

$$\mathcal{J}_s(\vec{x},t,\lambda) = \xi^{\pm}(\vec{x},t,\lambda)J_s\hat{\xi}^{\pm}(\vec{x},t,\lambda), \qquad \mathcal{K}(\vec{x},t,\lambda) = \xi^{\pm}(\vec{x},t,\lambda)K\hat{\xi}^{\pm}(\vec{x},t,\lambda), \tag{4}$$

belong to the algebra $\mathfrak{g}$ for any J and K from $\mathfrak{g}$. If in addition K also belongs to the Cartan subalgebra $\mathfrak{h}$, then

$$[\mathcal{J}_s(\vec{x},t,\lambda), \mathcal{K}(\vec{x},t,\lambda)] = 0. \tag{5}$$

An important tool in our considerations plays the well known Zakharov-Shabat theorem [38, 39] formulated below

Theorem 1. *Let $\xi^{\pm}(x,t,\lambda)$ be solutions to the RHP (1) whose sewing function depends on the auxiliary variables $\vec{x}$ and t via eq. (2). Then $\xi^{\pm}(x,t,\lambda)$ are fundamental solutions of the following set of differential operators:*

$$\begin{aligned} L_s\xi^{\pm} &\equiv i\frac{\partial \xi^{\pm}}{\partial x_s} + U_s(\vec{x},t,\lambda)\xi^{\pm}(\vec{x},t,\lambda) - \lambda^k[J_s, \xi^{\pm}(\vec{x},t,\lambda)] = 0, \\ M\xi^{\pm} &\equiv i\frac{\partial \xi^{\pm}}{\partial t} + V(\vec{x},t,\lambda)\xi^{\pm}(\vec{x},t,\lambda) - \lambda^k[K, \xi^{\pm}(\vec{x},t,\lambda)] = 0. \end{aligned} \tag{6}$$

P r o o f. The proof follows the lines of [38, 39]. We introduce the functions:

$$\begin{aligned} g_s^{\pm}(\vec{x},t,\lambda) &= i\frac{\partial \xi^{\pm}}{\partial x_s}\hat{\xi}^{\pm}(\vec{x},t,\lambda) + \lambda^k\xi^{\pm}(\vec{x},t,\lambda)J_s\hat{\xi}^{\pm}(\vec{x},t,\lambda), \\ g^{\pm}(\vec{x},t,\lambda) &= i\frac{\partial \xi^{\pm}}{\partial t}\hat{\xi}^{\pm}(\vec{x},t,\lambda) + \lambda^k\xi^{\pm}(\vec{x},t,\lambda)K\hat{\xi}^{\pm}(\vec{x},t,\lambda), \end{aligned} \tag{7}$$

and using (2) prove that

$$g_s^+(\vec{x},t,\lambda) = g_s^-(\vec{x},t,\lambda), \qquad g^+(\vec{x},t,\lambda) = g^-(\vec{x},t,\lambda), \tag{8}$$

which means that these functions are analytic functions of λ in the whole complex λ-plane. Next we find that:

$$\lim_{\lambda\to\infty} g_s^+(\vec{x},t,\lambda) = \lambda^k J_s, \qquad \lim_{\lambda\to\infty} g^+(\vec{x},t,\lambda) = \lambda^k K. \tag{9}$$

and make use of Liouville theorem to get

$$\begin{aligned} g_s^+(\vec{x},t,\lambda) &= g_s^-(\vec{x},t,\lambda) = \lambda^k J_s - \sum_{l=1}^{k} U_{s;l}(\vec{x},t)\lambda^{k-l}, \\ g^+(\vec{x},t,\lambda) &= g^-(\vec{x},t,\lambda) = \lambda^k K - \sum_{l=1}^{k} V_l(\vec{x},t)\lambda^{k-l}. \end{aligned} \tag{10}$$

We shall see below that the coefficients $U_{s;l}(\vec{x},t)$ and $V_l(\vec{x},t)$ can be expressed in terms of the asymptotic coefficients Q_s in eq. (3). $\square$

Lemma 1. *The set of operators L_s and M commute, i.e. the following set of equations hold:*

$$\begin{aligned} & i\frac{\partial U_s}{\partial x_j} - i\frac{\partial U_j}{\partial x_s} + [U_s(\vec{x},t,\lambda) - \lambda^k J_s, U_j(\vec{x},t,\lambda) - \lambda^k J_j] = 0, \\ & i\frac{\partial U_s}{\partial t} - i\frac{\partial V}{\partial x_s} + [U_s(\vec{x},t,\lambda) - \lambda^k J_s, V(\vec{x},t,\lambda) - \lambda^k K] = 0. \end{aligned} \tag{11}$$

where

$$U_s(\vec{x},t,\lambda) = \sum_{l=1}^{k} U_{s;l}(\vec{x},t)\lambda^{k-l}, \qquad V(\vec{x},t,\lambda) = \sum_{l=1}^{k} V_l(\vec{x},t)\lambda^{k-l}. \tag{12}$$

P r o o f. The set of the operators L_s and M (6) have a common FAS, i.e. they all must commute. The eqs. (11) are an immediate consequence of (6). $\square$

3. Jets of order k

In what follows we will consider the jets of order k of $\mathcal{J}(x,\lambda)$ and $\mathcal{K}(x,\lambda)$, see (5). We introduce them by:

$$\begin{aligned} \mathcal{J}_s(\vec{x},t,\lambda) &\equiv \left(\lambda^k \xi^\pm(\vec{x},t,\lambda) J_l \hat{\xi}^\pm(\vec{x},t,\lambda)\right)_+ = \lambda^k J_s - U_s(\vec{x},t,\lambda), \\ \mathcal{K}(\vec{x},t,\lambda) &\equiv \left(\lambda^k \xi^\pm(\vec{x},t,\lambda) K \hat{\xi}^\pm(\vec{x},t,\lambda)\right)_+ = \lambda^k K - V(\vec{x},t,\lambda). \end{aligned} \tag{13}$$

The subscript + used above means that we insert the asymptotic expansions of $\xi^{\pm}$ and their inverse (3) and cut off the terms with negative powers of λ.

Obviously $U_s(x) \in \mathfrak{g}$ can be expressed in terms of $Q_s(x)$. In doing this we take into account (5) and obtain [18]

$$\mathcal{J}_s(\vec{x}, t, \lambda) = J_s + \sum_{k=1}^{\infty} \frac{1}{k!} \mathrm{ad}_Q^k J_s, \qquad \mathcal{K}(\vec{x}, t, \lambda) = K + \sum_{k=1}^{\infty} \frac{1}{k!} \mathrm{ad}_Q^k K, \tag{14}$$

and therefore for $U_{s;l}$ we get:

$$\begin{aligned} U_{s;1}(\vec{x}, t) &= -\mathrm{ad}_{Q_1} J_s, \qquad U_{s;2}(\vec{x}, t) = -\mathrm{ad}_{Q_2} J_s - \frac{1}{2} \mathrm{ad}_{Q_1}^2 J_s \\ U_{s;3}(\vec{x}, t) &= -\mathrm{ad}_{Q_3} J_s - \frac{1}{2} \left(\mathrm{ad}_{Q_2} \mathrm{ad}_{Q_1} + \mathrm{ad}_{Q_1} \mathrm{ad}_{Q_2} \right) J_s - \frac{1}{6} \mathrm{ad}_{Q_1}^3 J_s \\ &\ \vdots \\ U_{s;k}(\vec{x}, t) &= -\mathrm{ad}_{Q_k} J_s - \frac{1}{2} \sum_{s+p=k} \mathrm{ad}_{Q_s} \mathrm{ad}_{Q_p} J_s \\ &\qquad - \frac{1}{6} \sum_{s+p+r=k} \mathrm{ad}_{Q_s} \mathrm{ad}_{Q_p} \mathrm{ad}_{Q_r} J_s - \cdots - \frac{1}{k!} \mathrm{ad}_{Q_1}^k J_s, \end{aligned} \tag{15}$$

and similar expressions for $V_l(\vec{x}, t)$ with J_s replaced by K.

4. Reductions of polynomial bundles

An important tool to construct new integrable NLEE is based on Mikhailov's group of reductions [26]. Below we will use mainly $\mathbb{Z}_2$ and $\mathbb{Z}_N$ with $N > 2$ reduction groups. The basic $\mathbb{Z}_2$-examples are as follows:

$$\begin{aligned} &\text{a)} \quad A\xi^{+,\dagger}(x, t, \epsilon\lambda^*)\hat{A} = \hat{\xi}^-(x, t, \lambda), \qquad AQ^{\dagger}(x, t, \epsilon\lambda^*)\hat{A} = -Q(x, t, \lambda), \\ &\text{b)} \quad B\xi^{+,*}(x, t, \epsilon\lambda^*)\hat{B} = \xi^-(x, t, \lambda), \qquad BQ^*(x, t, \epsilon\lambda^*)\hat{B} = Q(x, t, \lambda), \\ &\text{c)} \quad C\xi^{+,T}(x, t, -\lambda)\hat{C} = \hat{\xi}^-(x, t, \lambda), \qquad CQ^{\dagger}(x, t, -\lambda)\hat{C} = -Q(x, t, \lambda), \end{aligned} \tag{16}$$

where $\epsilon^2 = 1$ and A, B and C are elements of the group $\mathfrak{G}$ such that $A^2 = B^2 = C^2 = \mathbb{1}$. As for the $\mathbb{Z}_N$-reductions we may have:

$$D\xi^{\pm}(x, t, \omega\lambda)\hat{D} = \xi^{\pm}(x, t, \lambda), \qquad DQ(x, t, \omega\lambda)\hat{D} = Q(x, t, \lambda), \tag{17}$$

where $\omega^N = 1$ and $D^N = \mathbb{1}$.

These relations allow us to introduce algebraic relations between the matrix elements of $Q(x, t, \lambda)$ which will be automatically compatible with the NLEE. The classes of inequivalent reductions of the N-wave equations related to the low-rank simple Lie algebras are given in [8, 9, 11, 12, 13, 15].

5. On N-wave equations ($k=1$) in 2 and more dimensions

The integrability of the N-wave equations has been well known for several decades now, [25, 33, 36, 29, 14, 23, 22]. Their Lax representation involves two Lax operators linear in λ which are particular case of (6) with $k=1$:

$$\begin{aligned} L\xi^{\pm} \equiv & i\frac{\partial \xi^{\pm}}{\partial x} + [J, Q(x,t)]\xi^{\pm}(\vec{x},t,\lambda) - \lambda[J, \xi^{\pm}(\vec{x},t,\lambda)] = 0, \\ M\xi^{\pm} \equiv & i\frac{\partial \xi^{\pm}}{\partial t} + [K, Q(x,t)]\xi^{\pm}(\vec{x},t,\lambda) - \lambda[K, \xi^{\pm}(\vec{x},t,\lambda)] = 0. \end{aligned} \tag{18}$$

The corresponding equations take the form:

$$i\left[J, \frac{\partial Q}{\partial t}\right] - i\left[K, \frac{\partial Q}{\partial x}\right] - [[J,Q],[K,Q(x,t)]] = 0 \tag{19}$$

In fact the construction of the FAS for the operator L (18) [38, 39, 36, 29] was the important step forward, that demonstrated the importance of the RHP for solving integrable equations.

The most important and nontrivial example of such NLEE is the 3-wave equations in two-dimensional space-time [36, 29]. The most important and nontrivial case corresponds to $\mathfrak{g} \simeq sl(3)$

$$Q(x,t) = \left(\begin{array}{ccc} 0 & u_1 & u_3 \\ -v_1 & 0 & u_2 \\ -v_3 & -v_2 & 0 \end{array}\right), \qquad \begin{array}{c} J = \operatorname{diag}(a_1, a_2, a_3), \\ K = \operatorname{diag}(b_1, b_2, b_3), \end{array} \tag{20}$$

with $\operatorname{tr} J = \operatorname{tr} K = 0$ and $a_1 > a_2 > a_3$. We also impose the reduction (16a) with $A = \operatorname{diag}(1, \epsilon_1, \epsilon_2)$ where $\epsilon_1^2 = \epsilon_2^2 = 1$. Then the 3-wave equations take the form:

$$\begin{aligned} &\frac{\partial u_1}{\partial t} - \frac{a_1 - a_2}{b_1 - b_2}\frac{\partial u_1}{\partial x} + \kappa\epsilon_1\epsilon_2 u_2^* u_3 = 0, \\ &\frac{\partial u_2}{\partial t} - \frac{a_2 - a_3}{b_2 - b_3}\frac{\partial u_2}{\partial x} + \kappa\epsilon_1 u_1^* u_3 = 0, \\ &\frac{\partial u_3}{\partial t} - \frac{a_1 - a_3}{b_1 - b_3}\frac{\partial u_3}{\partial x} + \kappa\epsilon_2 u_1^* u_2^* = 0, \end{aligned} \tag{21}$$

where

$$\kappa = a_1(b_2 - b_3) - a_2(b_1 - b_3) + a_3(b_1 - b_2). \tag{22}$$

Depending on the choice of the reduction and on interrelations between the group velocities the 3-wave interactions may describe qualitatively different processes: soliton decay and explosive soliton instability [36, 29].

In the case of 3-dimensional space-time we consider Q of the form (20), but now let u_j and v_j be functions of $x_1 = x$, $x_2 = y$ and t. Let also $J_1 = J$ and $J_2 = I = \operatorname{diag}(c_1, c_2.c_3)$. Now the corresponding solution of the RHP $\xi^{\pm}(x, y, t, \lambda)$ will be FAS not only of L and M above, but also of

$$P\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial y} + [I, Q(x,t)]\xi^{+}(\vec{x}, t, \lambda) - \lambda[I, \xi^{+}(\vec{x}, t, \lambda)] = 0, \tag{23}$$

and all these three operators will mutually commute, i.e. along with $[L, M] = 0$ we will have also $[L, P] = 0$ and $[P, M] = 0$. As a result $Q(x, y, t)$ will satisfy two more NLEE of the form (24). Obviously it will satisfy also

$$\begin{aligned} 2\frac{\partial u_1}{\partial t} - \frac{a_1 - a_2}{b_1 - b_2}\frac{\partial u_1}{\partial x} - \frac{a_1 - a_2}{c_1 - c_2}\frac{\partial u_1}{\partial y} + (\kappa_1 + \kappa_2)\epsilon_1\epsilon_2 u_2^* u_3 = 0,\\ 2\frac{\partial u_2}{\partial t} - \frac{a_1 - a_3}{b_1 - b_3}\frac{\partial u_2}{\partial x} - \frac{a_1 - a_3}{c_1 - c_3}\frac{\partial u_2}{\partial y} + (\kappa_1 + \kappa_2)\epsilon_1 u_1^* u_3 = 0,\\ 2\frac{\partial u_3}{\partial t} - \frac{a_2 - a_3}{b_2 - b_3}\frac{\partial u_3}{\partial x} - \frac{a_2 - a_3}{c_2 - c_3}\frac{\partial u_3}{\partial y} + (\kappa_1 + \kappa_2)\epsilon_2 u_1^* u_2^* = 0. \end{aligned} \tag{24}$$

which is linear combination of the three equations mentioned above. Here $\kappa_1 = \kappa$ (see eq. (22) and

$$\kappa_2 = a_1(c_2 - c_3) - a_2(c_1 - c_3) + a_3(c_1 - c_2). \tag{25}$$

These three wave equations are related to the real forms of the algebra $sl(3)$ which has rank 2. Therefore, trying to add more auxiliary variables to the solution of the RHP will not be effective since only two elements of all $x_s J_s$ will be linearly independent.

For N-wave equations related to Lie algebras $\mathfrak{g}$ of higher rank r we can add up to r auxiliary variables. The corresponding PDE takes the form:

$$r\frac{\partial Q}{\partial t} - \sum_{s=1}^{r} (\operatorname{ad}_{J_s}^{-1}\operatorname{ad}_J)\frac{\partial Q}{\partial x_s} - i\sum_{s=1}^{r} \operatorname{ad}_{J_s}^{-1}\,[[J, Q], [J_s, Q(\vec{x}, t)]] = 0 \tag{26}$$

where Q is an $n \times n$ off-diagonal matrix depending on $r + 1$ variables. We remind that if $J = \operatorname{diag}(a_1, \ldots, a_n)$ then

$$(\operatorname{ad}_J Q)_{jk} \equiv ([J, Q])_{jk} = (a_j - a_k)Q_{jk}, \qquad (\operatorname{ad}_J^{-1} Q)_{jk} = \frac{1}{a_j - a_k}Q_{jk},$$

and similarly for the other J_s. The coefficient r multiplying the t-derivative can be removed by rescaling of t.

Again we can use additional reductions of the type (16). More details about these equations will be given elsewhere.

6. New N-wave equations ($k = 2$) in 2 and more dimensions

Here we shall give examples of new types of N-wave equations. Let us choose again $\mathfrak{g} = sl(3)$. The general form of the potentials is given by

$$
Q_1(\vec{x},t) = \left(\begin{array}{ccc} 0 & u_1 & u_3 \\ -v_1 & 0 & u_2 \\ -v_3 & -v_2 & 0 \end{array}\right), \qquad Q_2(\vec{x},t) = \left(\begin{array}{ccc} q_{11} & w_1 & w_3 \\ -z_1 & q_{22} & w_2 \\ -z_3 & -z_2 & q_{33} \end{array}\right), \tag{27}
$$

We also fix up $k = 2$. Then the Lax pair becomes

$$
\begin{aligned}
L\xi^{\pm} &\equiv i\frac{\partial \xi^{\pm}}{\partial x} + U(x,t,\lambda)\xi^{\pm}(x,t,\lambda) - \lambda^2]J, \xi^{\pm}(x,t,\lambda)] = 0, \\
M\xi^{\pm} &\equiv i\frac{\partial \xi^{\pm}}{\partial t} + V(x,t,\lambda)\xi^{\pm}(x,t,\lambda) - \lambda^2]K, \xi^{\pm}(x,t,\lambda)] = 0,
\end{aligned} \tag{28}
$$

where using eq. (15)

$$
\begin{aligned}
U &\equiv U_2 + \lambda U_1 = \left([J, Q_2(x)] - \frac{1}{2}[[J, Q_1], Q_1(x)]\right) + \lambda [J, Q_1], \\
V &\equiv V_2 + \lambda V_1 = \left([K, Q_2(x)] - \frac{1}{2}[[K, Q_1], Q_1(x)]\right) + \lambda [K, Q_1].
\end{aligned}
$$

Note, that this Lax pair is independent of the diagonal elements of Q_2.

If we retain the generic potentials (27) the Lax pair above will provide us with a set of 6 new complicated equations for the 6 independent functions u_j and w_j. To make the things more simple we impose a $\mathbb{Z}_2$-reduction of the form (16a) with $A = \mathrm{diag}\,(1, \epsilon, 1)$, $\epsilon^2 = 1$. Thus Q_1 and Q_2 get reduced into:

$$
Q_1 = \left(\begin{array}{ccc} 0 & u_1 & 0 \\ \epsilon u_1^* & 0 & u_2 \\ 0 & \epsilon u_2^* & 0 \end{array}\right), \qquad Q_2 = \left(\begin{array}{ccc} 0 & 0 & w_3 \\ 0 & 0 & 0 \\ w_3^* & 0 & 0 \end{array}\right), \tag{29}
$$

and J and K are as in (20). Now L and M involve only 3 independent functions. Skipping the details we get a new type of integrable 3-wave equations:

$$
\begin{aligned}
& i(a_1 - a_2)\frac{\partial u_1}{\partial t} - i(b_1 - b_2)\frac{\partial u_1}{\partial x} + \epsilon\kappa u_2^* u_3 + \epsilon \frac{\kappa(a_1 - a_2)}{(a_1 - a_3)} u_1 |u_2|^2 = 0, \\
& i(a_2 - a_3)\frac{\partial u_2}{\partial t} - i(b_2 - b_3)\frac{\partial u_2}{\partial x} + \epsilon\kappa u_1^* u_3 - \epsilon \frac{\kappa(a_2 - a_3)}{(a_1 - a_3)} |u_1|^2 u_2 = 0, \\
& i(a_1 - a_3)\frac{\partial u_3}{\partial t} - i(b_1 - b_3)\frac{\partial u_3}{\partial x} - \frac{i\kappa}{a_1 - a_3}\frac{\partial (u_1 u_2)}{\partial x} \\
& \quad + \epsilon\kappa \left(\frac{a_1 - a_2}{a_1 - a_3}|u_1|^2 + \frac{a_2 - a_3}{a_1 - a_3}|u_2|^2\right) u_1 u_2 + \epsilon\kappa u_3 (|u_1|^2 - |u_2|^2) = 0,
\end{aligned} \tag{30}
$$

where the interaction constant κ is given by (22) and:

$$u_3 = w_3 + \frac{2a_2 - a_1 - a_3}{2(a_1 - a_3)} u_1 u_2. \tag{31}$$

The diagonal terms in the Lax representation are λ-independent. Two of them read:

$$\begin{aligned} & i(a_1 - a_2)\frac{\partial |u_1|^2}{\partial t} - i(b_1 - b_2)\frac{\partial |u_1|^2}{\partial x} - \epsilon\kappa(u_1 u_2 u_3^* - u_1^* u_2^* u_3) = 0, \\ & i(a_2 - a_3)\frac{\partial |u_2|^2}{\partial t} - i(b_2 - b_3)\frac{\partial |u_2|^2}{\partial x} - \epsilon\kappa(u_1 u_2 u_3^* - u_1^* u_2^* u_3) = 0, \end{aligned} \tag{32}$$

These relations are satisfied identically as a consequence of the NLEE (30). The third one also vanishes since $\operatorname{tr}[L, M] = 0$.

Let us now consider the case when the sewing function G of the RHP depends on 3 variables: t, $x_1 = x$ and $x_2 = y$ with $J_1 = J$ and $J_2 = I = \operatorname{diag}(c_1, c_2, c_3)$. For $k = 2$ we obtain a set of three ordinary differential operators: L, M (28) and

$$\begin{aligned} P\xi^{\pm} &\equiv i\frac{\partial \xi^{\pm}}{\partial y} + W(x,y,t,\lambda)\xi^{\pm}(x,y,t,\lambda) - \lambda^2]I, \xi^{\pm}(x,y,t,\lambda)] = 0, \\ W &\equiv W_2 + \lambda W_1 \\ &= \left([I, Q_2(x,y,t)] - \frac{1}{2}[[I, Q_1], Q_1(x,y,t)] \right) + \lambda [I, Q_1(x,y,t)], \end{aligned} \tag{33}$$

commuting identically with respect to λ. It is obvious that $[L, P] = 0$ if

$$\begin{aligned} & i(a_1 - a_2)\frac{\partial u_1}{\partial t} - i(c_1 - c_2)\frac{\partial u_1}{\partial y} + \epsilon\kappa_2 u_2^* u_3 + \epsilon\frac{\kappa_2(a_1 - a_2)}{(a_1 - a_3)} u_1 |u_2|^2 = 0, \\ & i(a_2 - a_3)\frac{\partial u_2}{\partial t} - i(c_2 - c_3)\frac{\partial u_2}{\partial y} + \epsilon\kappa_2 u_1^* u_3 - \epsilon\frac{\kappa_2(a_2 - a_3)}{(a_1 - a_3)} |u_1|^2 u_2 = 0, \\ & i(a_1 - a_3)\frac{\partial u_3}{\partial t} - i(c_1 - c_3)\frac{\partial u_3}{\partial y} - \frac{i\kappa_2}{a_1 - a_3}\frac{\partial (u_1 u_2)}{\partial y} \\ & \quad + \epsilon\kappa_2 \left(\frac{a_1 - a_2}{a_1 - a_3}|u_1|^2 + \frac{a_2 - a_3}{a_1 - a_3}|u_2|^2 \right) u_1 u_2 + \epsilon\kappa_2 u_3(|u_1|^2 - |u_2|^2) = 0, \end{aligned} \tag{34}$$

where κ_2 is given by eq. (25). It is not difficult to write down the third new 3-wave equation which is a consequence of the commutation $[M, P] = 0$.

Since the three operators L, M, and N mutually commute, u_1, u_2 and u_3 as functions of x, y and t should satisfy *simultaneously* the three NLEE of the type (30). Therefore they should satisfy also *any* NLEE which is obtained as, say linear combination of the above:

$$
\begin{aligned}
&2i\frac{\partial u_1}{\partial t} - i(\vec{v}_{(1)}\cdot\nabla)u_1 + \epsilon(\kappa_1+\kappa_2)\left(\frac{u_2^* u_3}{a_1-a_2} + \frac{u_1|u_2|^2}{(a_1-a_3)}\right) = 0,\\
&2i\frac{\partial u_2}{\partial t} - i(\vec{v}_{(2)}\cdot\nabla)u_2 + \epsilon(\kappa_1+\kappa_2)\left(\frac{u_1^* u_3}{a_1-a_3} - \frac{u_2|u_1|^2}{(a_1-a_3)}\right) = 0,\\
&2i\frac{\partial u_3}{\partial t} - i(\vec{v}_{(3)}\cdot\nabla)u_3 - i\frac{(\vec{\kappa}\cdot\nabla)(u_1u_2)}{(a_1-a_3)^2} + \frac{\epsilon(\kappa_1+\kappa_2)}{a_1-a_3}(|u_1|^2-|u_2|^2)u_3\\
&\quad + \frac{\epsilon(\kappa_1+\kappa_2)}{(a_1-a_3)^2}\left((a_1-a_2)|u_1|^2 + (a_2-a_3)|u_2|^2\right)u_1u_2 = 0.
\end{aligned} \tag{35}
$$

Here $\nabla = (\partial_x, \partial_y)^T$, the characteristic velocities $\vec{v}_{(j)}$, $j=1,2,3$ and $\vec{\kappa}$ are two-component vectors given by:

$$
\begin{aligned}
\vec{v}_{(1)} &= \frac{1}{a_1-a_2}\begin{pmatrix} b_1-b_2 \\ c_1-c_2 \end{pmatrix}, \qquad & \vec{v}_{(2)} &= \frac{1}{a_2-a_3}\begin{pmatrix} b_2-b_3 \\ c_2-c_3 \end{pmatrix},\\
\vec{v}_{(3)} &= \frac{1}{a_1-a_3}\begin{pmatrix} b_1-b_3 \\ c_1-c_3 \end{pmatrix}, \qquad & \vec{\kappa} &= \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix},
\end{aligned} \tag{36}
$$

and $\kappa_1 = \kappa$, see eq. (22).

7. Discussion and conclusions

We have proposed a method for constructing families of commuting operators. Applied to jets of order 1 with $\mathfrak{g} \simeq sl(3)$ this method reproduces the well known results for the 3-wave equations in two- and three-dimensional space-times. It is shown that N-wave equations related to Lie algebras of rank r allow integrable extensions to $r+1$-dimensional space-times. Below we briefly discuss some open problems and generalizations.

Using jets of order 2 gives us the simplest nontrivial examples for new types of integrable 3-wave equation whose interaction terms contain quadratic and cubic nonlinearities, as well as x-derivatives. These equations also allow integrable extensions to three-dimensional space-time.

It is not difficult to obtain many other new integrable 3- and N-wave equations. Indeed, one can choose: i) higher rank simple Lie algebras; ii) different types of grading; iii) different power k of the polynomials $U(\vec{x},t,\lambda)$ and $V(\vec{x},t,\lambda)$ and iv) different reductions of U and V.

These new NLEE must be Hamiltonian. It is natural to view the jets $U(\vec{x},t,\lambda)$ as elements of more complicated co-adjoint orbits of the relevant Kac-Moody algebra, generated by the chosen grading of $\mathfrak{f}$, see [24, 31, 32].

By construction, the method allows treating multi-dimensional NLEE. In the examples above we used the algebra $sl(3)$ and demonstrated integrable 3-wave equations in $2+1$-dimensional space-time. If we want to study new types of integrable N-wave models in $r+1$ space-time dimensions we have to consider Lie algebras of rank r and accordingly larger values for N.

The method allows one also to apply Zakharov-Shabat dressing method [38, 39, 27, 20] for constructing their explicit (N-soliton) solutions. Instead of solving the inverse scattering problem for L we would rather deal with a Riemann-Hilbert problem with canonical normalization. For polynomials of order k the contour on which the RHP is defined consists of k straight lines $l_k\colon\ \arg\lambda = \pi i/k$ passing through the origin. Of course, it may necessary to use dressing factors with more specific λ-dependence.

This approach can be used also to analyze the NLEE derived by Gel'fand-Dickey approach [3, 17]. It would provide the possibility to systematically construct the spectral decompositions that linearize the relevant NLEE [7, 16]. Still more challenging is to study the soliton interactions of the new N-wave equations.

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