Collective Almost Synchronization in Complex Networks

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This work introduces the phenomenon of Collective Almost Synchronization (CAS), which describes a universal way of how patterns can appear in complex networks even for small coupling strengths. The CAS phenomenon appears due to the existence of an approximately constant local mean field and is characterized by having nodes with trajectories evolving around periodic stable orbits. Common notion based on statistical knowledge would lead one to interpret the appearance of a local constant mean field as a consequence of the fact that the behavior of each node is not correlated to the behaviors of the others. Contrary to this common notion, we show that various well known weaker forms of synchronization (almost, time-lag, phase synchronization, and generalized synchronization) appear as a result of the onset of an almost constant local mean field. If the memory is formed in a brain by minimising the coupling strength among neurons and maximising the number of possible patterns, then the CAS phenomenon is a plausible explanation for it.

Spontaneous emergence of collective behavior is common in nature [1–3]. It is a natural phenomenon characterized by a group of individuals that are connected in a network by following a dynamical trajectory that is different from the dynamics of their own. Since the work of Kuramoto [4], the spontaneous emergence of collective behavior in networks of phase oscillators with full connected nodes or with nodes connected by some special topologies [5] is analytically well understood. Kuramoto considered a fully connected network of an infinite number of phase oscillators. If θ_i is the variable describing the phase of an oscillator i in the network, and $\overline{\theta}$ represents the mean field defined as $\overline{\theta} = \frac{1}{N} \sum_{i=1}^{N} \theta_i$, collective behavior appears in the network because every node becomes coupled to the mean field. Peculiar characteristics of this collective behavior is that not only $\theta_i \neq \overline{\theta}$ but also nodes evolve in a way that cannot be described by the evolution of only one individual node, when isolated from the network.

In contrast to collective behavior, another widely studied behavior of a network is when all nodes behave equally, and their evolution can be described by an individual node when isolated from the network. This state is known as complete synchronization [6]. If x_i represents the state variables of an arbitrary node i of the network and x_j of another node j, and $\overline{x}$ represents the mean field of a network, complete synchronization appears when $x_i = x_j = \overline{x}$, for all time. The main mechanisms responsible for the onset of complete synchronization in dynamical networks were clarified in [7–9]. In networks whose nodes are coupled by non-linear functions, such as those that depend on time-delays [9] or those that describe how neurons chemically connect [10], the evolution of the synchronous nodes might be different from the evolution of an individual node, when isolated from

the network. However, when complete synchronization is achieved in such networks, $x_i = x_j = \overline{x}$.

In natural networks as biological, social, metabolic, neural networks, etc, [11], the number of nodes is often large but finite; the network is not fully connected and heterogeneous. The later means that each node has a different dynamical description or the coupling strengths are not all equal for every pair of nodes, and one will not find two nodes, say it x_i and x_j, that have equal trajectories. For such heterogeneous networks, as in [12, 13], found in natural networks and in experiments [14], one expects to find other weaker forms of synchronous behavior, such as practical synchronization [15], phase synchronization [14], time-lag synchronization [16], and generalized synchronization [17].

We report a phenomenon that may appear in complex networks "far away" from coupling strengths that typically produce complete synchronization or these weaker forms of synchronization. However, the reported phenomenon can be characterized by the same conditions used to verify the existence of these weaker forms of synchronization. We call it Collective Almost Synchronization (CAS). It is a consequence of the appearance of an approximately constant local mean field and is characterized by having nodes with trajectories evolving around stable periodic orbits, denoted by $\boldsymbol{\Xi}_{p_i}(t)$, and regarded as a CAS pattern. The appearance of an almost constant mean field is associated with a regime of weak interaction (weak coupling strength) in which nodes behave independently [18, 19]. In such conditions, even weaker forms of synchronization are ruled out to exist. But, contrary to common notion based on basic statistical arguments, we show that actually it is the existence of an approximately constant local mean field that paves the way for weaker forms of synchronization (such as almost, time-

lag, phase, or generalized synchronization) to occur in complex networks.

Denote all the d variables of a node i by $\mathbf{x}_i$, then we define that this node presents CAS if the following inequality

$$|\mathbf{x}_i(t) - \mathbf{\Xi}_{p_i}(t - \tau_i)| < \epsilon_i \qquad (1)$$

is satisfied for *most of the time*. The double vertical bar | | represents that we are taking the absolute difference between vector components appearing inside the bars ($L1$ norm). ϵ_i is a small quantity, not arbitrarily small, but reasonably smaller than the envelop of the oscillations of the variables $\mathbf{x}_i(t)$. $\mathbf{\Xi}_{p_i}(t)$ is the d-dimensional CAS pattern. It is determined by the effective coupling strength p_i, a quantity that measures the influence on the node i of the nodes that are connected to it, and the expected value of the local mean field at the node i, denoted by $\mathbf{C}_i$. The local mean field, denoted by $\overline{\mathbf{x}}_i$, is defined only by the nodes that are connected to the node i. The CAS pattern is the solution of a simplified set of equations describing the network when $\overline{\mathbf{x}}_i = \mathbf{C}_i$. According to Eq. (1), if a node in the network presents the CAS pattern, its trajectory stays intermittently close to the CAS pattern but with a time-lag between the trajectories of the node and of the CAS pattern. This property of the CAS phenomenon shares similarities with the way complete synchronization appears in networks of nodes coupled under time-delay functions [9]. In such networks, nodes become completely synchronous to a solution of the network that is different from the solution of an isolated node of the network. Additionally, the trajectory of the nodes present a time-lag to this solution.

The CAS phenomenon inherits the three main characteristics of a collective behavior: (a) the variables of a node i ($\mathbf{x}_i$) differ from both the mean field $\overline{\mathbf{x}}$ and the local mean field $\overline{\mathbf{x}}_i$; (b) if the local mean fields of a group of nodes and their effective coupling are either equal or approximately equal, that causes all the nodes in this group to follow the same or similar behaviors; (c) there can exist an infinitely large number of different behaviors (CAS patterns).

If the CAS phenomenon is present in a network, other weaker forms of synchronization can be detected. This link is fundamental when making measurements to detect the CAS phenomenon.

In Ref. [15], the phenomenon of almost synchronization is introduced, when a master and a slave in a master-slave system of coupled oscillators have equal phases but their amplitudes can be different. If a node i presents the CAS phenomenon [satisfying Eq. (1)] and $\tau_i = 0$ in Eq. (1), then the node i is almost synchronous to the pattern $\mathbf{\Xi}_{p_i}$.

Time-lag synchronization [16] is a phenomenon that describes two identical signals, but whose variables have a time-lag with respect to each other, i.e. $\mathbf{x}_i(t) = \mathbf{x}_j(t - \tau)$. In practice, however, an equality between $\mathbf{x}_i(t)$ and $\mathbf{x}_j(t - \tau)$ should not be expected to be typically found,

but rather

$$\mathbf{x}_i(t) \cong \mathbf{x}_j(t - \tau), \qquad (2)$$

meaning that there is not a constant τ that can be found such that $\mathbf{x}_i(t) = \mathbf{x}_j(t - \tau)$. Another suitable way of writing Eq. (2) is by $|\mathbf{x}_i(t) - \mathbf{x}_j(t - \tau)| \leq \gamma$. If two nodes i and j that present the CAS phenomenon, have the same CAS pattern, and $\tau_i \neq \tau_j \neq 0$, then

$$|\mathbf{x}_i(t) - \mathbf{x}_j(t - \tau_{ij})| \leq \epsilon_{ij} \qquad (3)$$

or alternatively $\mathbf{x}_i(t) \cong \mathbf{x}_j(t - \tau_{ij})$, for most of the time, τ_{ij} representing the time-lag between $\mathbf{x}_i$ and $\mathbf{x}_j$. This means that almost time-lag synchronization occurs for two nodes that present the CAS phenomenon and that are almost locked to the same CAS pattern. Even though nodes that have equal or similar local mean field (which usually happens for nodes that have equal or similar degrees) become synchronous with the same CAS pattern (a stable periodic orbit), the value of their trajectories at a given time might be different, since their trajectories reach the neighborhood of their CAS patterns in different places of the orbit. As a consequence, we expect that two nodes that exhibit the same CAS should present between themselves a time-lag synchronous behavior. For some small amounts of time, the difference $|\mathbf{x}_i(t) - \mathbf{x}_j(t - \tau_{ij})|$ can be large, since $\tau_i \neq \tau_j$ and $\epsilon_i \neq \epsilon_j$, in Eq. (1). The closer $\overline{\mathbf{x}}_i$ and $\overline{\mathbf{x}}_j$ are to $\mathbf{C}_i$, the smaller is ϵ_{ij} in Eq. (3).

Phase synchronization [14] is a phenomenon where the phase difference, denoted by $\Delta\phi_{ij}$, between the phases of two signals (or nodes in a network), $\phi_i(t)$ and $\phi_j(t)$, remains bounded for all time

$$\Delta\phi_{ij} = \left| \phi_i(i) - \frac{p}{q}\phi_j(t) \right| \leq S. \qquad (4)$$

In Ref. [14] $S = 2\pi$ and p and q are two rational numbers. If p and q are irrational numbers and S is a reasonably small constant, then phase synchronization can be referred as to irrational phase synchronization [20]. The value of S is calculated in order to encompass oscillatory systems that possess either a time varying time-scale or a variable time-lag. Simply make the constant S to represent the growth of the phase in the faster time scale during one period of the slower time scale. Phase synchronization between two coupled chaotic oscillators was explained as being the result of a state where the two oscillators have all their unstable periodic orbits phase-locked [14]. Nodes that present the CAS phenomenon have unstable periodic orbits that are locked to the stable periodic orbits described by $\mathbf{\Xi}_i(t)$. If $\mathbf{\Xi}_i(t)$ has a period P_i and the phase of this CAS pattern changes $D\phi_i$ within one period, so the angular frequency is $\omega_i = D\phi_i/P_i$. If $\mathbf{\Xi}_j(t)$ has a period P_j and the phase of its CAS patter changes $D\phi_j$ within one period, so the angular frequency is $\omega_j = D\phi_j/P_j$. Then, the CAS patterns of these nodes are phase synchronous by a ratio of $\frac{p}{q} = \omega_i/\omega_j$. Since the trajectories of these nodes are locked to these patterns, the nodes are phase synchronous by this same ra-

tio, which can be rational or irrational. Assume additionally that, as one changes the coupling strengths between the nodes, the expected value $\mathbf{C}_i$ of the local mean field of a group of nodes remains the same. As a consequence, as one changes the coupling strengths, both the CAS pattern and the ratio $\frac{p}{q} = \frac{p_j D \phi_i}{p_i D \phi_j}$ remain unaltered, and the observed phase synchronization between nodes in this group is stable under parameter alterations.

Consider a network of N nodes with nodes connected diffusively (more general networks are treated in the Supplementary Information) described by

$$\dot{\mathbf{x}}_i = \mathbf{F}_i(\mathbf{x}_i) + \sigma \sum_{j=1}^{N} \mathbf{A}_{ij} \mathbf{E}(\mathbf{x}_j - \mathbf{x}_i), \qquad (5)$$

where $\mathbf{x}_i \in \Re^d$ is a d-dimensional vector describing the state variables of the node i, $\mathbf{F}_i$ represents the dynamical system of the node i, and $\mathbf{A}_{ij}$ is the adjacent matrix. If $A_{ij} = 1$, then, the node j is connected to the node i. $\mathbf{E}$ is the coupling function The degree of a node can be calculated by $k_i = \sum_{j=1}^{N} A_{ij}$.

The CAS phenomenon appears when the local mean field of a node i, $\overline{\mathbf{x}}_i(t) = 1/k_i \sum_j A_{ij} \mathbf{x}_j$, is approximately constant and $\overline{\mathbf{x}}_i(t) \cong \mathbf{C}_i$. Then, the equations for the network can be described by

$$\dot{\mathbf{x}}_i = \mathbf{F}_i(\mathbf{x}_i) - p_i E(\mathbf{x}_i) + p_i E(\mathbf{C}_i) + \delta_i, \qquad (6)$$

where $p_i = \sigma k_i$ and the residual term is $\delta_i = p_i(\overline{\mathbf{x}}_i(t) - \mathbf{C}_i)$. The CAS pattern of the node i (a stable periodic orbit) is calculated in the variables that produce a finite bounded local average field. If all components of $\mathbf{x}_i$ are bounded, then the CAS pattern is given by a solution of

$$\dot{\mathbf{\Xi}}_{p_i} = F_i(\mathbf{\Xi}_{p_i}) - p_i E(\mathbf{\Xi}_{p_i}) + p_i E(\mathbf{C}_i). \qquad (7)$$

which is just the same set of equations (6) without the residual term. So, if $\overline{\mathbf{x}}_i(t) = \mathbf{C}_i$, the residual term $\delta_i = 0$, and if Eq. (7) has no positive Lyapunov exponents ($\mathbf{\Xi}_{p_i}$ is a stable periodic orbit), then the node x_i describes a stable periodic orbit. If $\overline{\mathbf{x}}_i(t) - \mathbf{C}_i$ is larger than zero but $\mathbf{\Xi}_{p_i}$ is a stable periodic orbit, then the node x_i describes a perturbed version of $\mathbf{\Xi}_{p_i}$. The closer $\overline{\mathbf{x}}_i$ is to $\mathbf{C}_i$, the larger the time that Eq. (1) is satisfied at a given time. The more stable the periodic orbit is [the larger the largest negative Lyapunov exponents of Eq. (7)], the longer Eq. (1) is satisfied at a given time.

If the network has unbounded state variables (as it is the case of Kuramoto networks [4]), the CAS pattern is the periodic orbit of period T_i defined in the velocity space such that $\dot{\mathbf{\Xi}}_{p_i}(t) = \dot{\mathbf{\Xi}}_{p_i}(t + T_i)$.

Notice that whereas Eqs. (5) and (6) represent a Nd-dimensional system, Eq. (7) has only dimension d.

The existence of this approximately constant local mean field is a consequence of the Central Limit Theorem, applied to variables with correlation (for more details, see Supplementary Information). The expected value of the local mean field can be calculated by

$$\mathbf{C}_i =_{\lim t \to \infty} \frac{1}{t} \int \overline{\mathbf{x}}_i(t) dt, \qquad (8)$$

where in practice we consider t to be large, but finite. The larger the degree of a node, the higher is the probability for the local mean field to be close to an expected value and smaller its variance. If the probability to find a certain value for the local mean field of the node i does not depend on the higher order moments of $\overline{\mathbf{x}}_i(t)$, then this probability tends to be Gaussian for sufficiently large k_i. As a consequence, the variance μ^2 of the local mean field is proportional to k_i^{-1}.

There are two criteria for the node i to present the CAS phenomenon:

Criterion 1: The Central Limit Theorem can be applied, i.e., $\mu_i^2 \propto k_i^{-1}$. Therefore, the larger the degree of a node, the smaller the variation of the local mean field $\overline{\mathbf{x}}_i(t)$ about its expected value $\mathbf{C}_i$.

Criterion 2: The CAS pattern $\mathbf{\Xi}_i(t)$ describes a stable periodic orbit. The node trajectory can be considered to be a perturbed version of its CAS pattern. The more stable the faster trajectories of nodes come to the neighborhood of the periodic orbits (CAS patterns), and the longer they stay around them.

Whenever the Central Limit Theorem applies, the random variables involved are independent. But, the Central Limit Theorem can also be applied to variables with correlation. If nodes that present the CAS phenomenon are locked to the same CAS pattern, their trajectories still arrive to the CAS pattern at different "random" times, allowing for the Central Limit Theorem to be applied. But the time-lag between two nodes (τ_{ij}) is approximately constant, since the CAS pattern has a well defined period, and the trajectories of these nodes are locked into it. The local mean field measured in a node i remains unaltered as one changes the coupling strength either when the network has an infinite number of nodes (e.g. Kuramoto networks) or the nodes have a symmetric natural measured (See Secs. C, D, and E of Supplementary Information). However, as we show in the following example, the local mean field remains unaltered even when the network has only a finite number of nodes and it has a natural measure with no special symmetrical properties.

As an example to illustration how the CAS phenomenon appears in a complex network, we consider a scaling-free network formed by, say, $N = 1000$ Hindmarsh-Rose neurons, with neurons coupled electrically. The network is described by

$$\begin{aligned}
\dot{x}_i &= y_i + 3x_i^2 - x_i^3 - z_i + I + \sigma \sum_{j=1}^{N} A_{ij}(x_j - x_i) \\
\dot{y}_i &= 1 - 5x_i^2 - y_i \qquad\qquad\qquad\qquad (9) \\
\dot{z}_i &= -rz_i + 4r(x_i + 1.618)
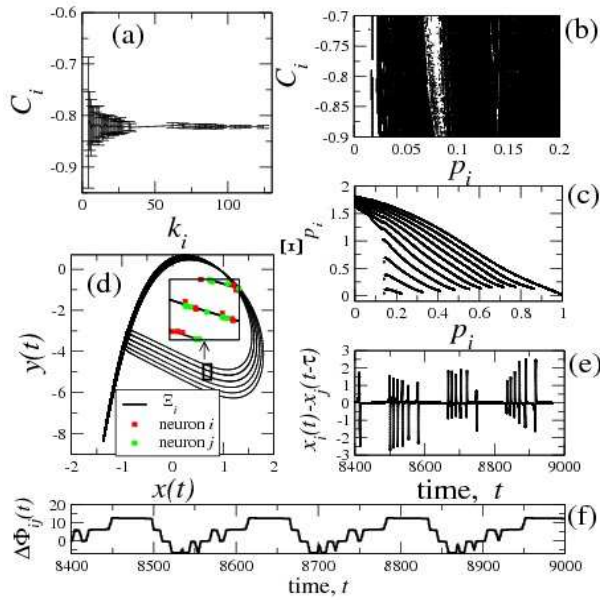\end{aligned}$$

FIG. 1: [Color online] (a) Expected value of the local mean field of the node i against the node degree k_i. The error bar indicates the variance (μ_i^2) of $\overline{x}_i$. (b) Black points indicate the value of C_i and p_i for Eq. (10) to present a stable periodic orbit (no positive Lyapunov exponents). The maximal values of the periodic orbits obtained from Eq. (10) is shown in the bifurcation diagram in (c) considering $C_i = -0.82$ and $\sigma = 0.001$. (d) The CAS pattern for a neuron i with degree $k_i=25$ (with $\sigma = 0.001$ and $C = -0.82$). In the inset, the same CAS pattern of the neuron i and some sampled points of the trajectory for the neuron i and another neuron j with degree $k_j = 25$. (e) The difference between the first coordinates of the trajectories of neurons i and j, with a time-lag of $\tau_{ij} = 34.2$. (f) Phase difference between the phases of the trajectories for neurons i and j.

where I=3.25 and r=0.005. The first coordinate of the equations that describe the CAS pattern is given by

$$\dot{\Xi}_{x_i} = \Xi_{y_i} + 3\Xi_{x_i}^2 - \Xi_{x_i}^3 - \Xi_{z_i} + I_i - p_i\Xi_{x_i} + p_iC_i. \quad (10)$$

The others are given by $\dot{\Xi}_{y_i} = 1 - \Xi_{x_i}^2 - \Xi_{y_i}$, $\dot{\Xi}_{z_i} = -r\Xi_{z_i} + 4r(\Xi_{x_i} + 1.618)$. In this network, we have numerically verified that **criterion 1** is satisfied for neurons that have degrees $k \geq 10$ if $\sigma \leq \sigma^*$, with $\sigma^* \cong 0.001$. In Fig. 1(a), we show the expected value C_i of the local mean field of the first coordinate x_i of a neuron i with respect to the neuron degree (indicated in the horizontal axis), for $\sigma = 0.001$. The error bar indicates the variance of C_i which fits to $\propto k_i^{-1.0071}$. In (b), we show a parameter space to demonstrate that the CAS phenomenon is a robust and stable phenomenon. Numerical integration of Eqs. (9) for $p_i \in [0.001, 1]$ produces $C_i \in [-0.9, 0.7]$. We integrate Eq. (10) by using $C_i \in [-0.9, 0.7]$ and $p_i \in [0, 0.2]$, to show that the CAS pattern is stable for most of the values. So, variations in C_i of a network caused by changes in a parameter do not modify the stability of the CAS pattern calculated by Eq. (10). For $\sigma = 0.001$, Eqs. (9) yields many nodes for which

$\overline{x}_i \cong -0.82$. So, to calculate the CAS pattern for these nodes, we use $C_i = -0.82$ and $\sigma = 0.001$ in Eqs. (10). The CAS pattern obtained, as we vary p_i, is shown in the bifurcation diagram in (c), by plotting the local maximal points of the CAS patterns. **Criterion 2** is satisfied for most of the range of values of p_i that produces a stable periodic CAS pattern. A neuron that has a degree k_i is locked to the CAS pattern calculated by integrating Eqs. (10) using $k_i\sigma = p_i$ and the measured expected value for the local mean field, C_i. In (d), we show the periodic orbit corresponding to a CAS pattern associated to a neuron i with degree $k_i = 25$ (for σ=0.001) and in the inset the sampled points of the trajectories of this same neuron i and of another neuron j that has not only equal degree (k_j=25), but it feels also a local mean field of $C_j \cong -0.82$. In (e), we show that these two neurons have a typical time-lag synchronous behavior. In (f), we observe $p/q = 1$ phase synchronization between these two neurons for a long time, considering that the phase difference remains bounded by $S = 6 \times 2\pi$ as defined in Eq. (4), where the number 6 is the number of spikings within one period of the slower time-scale. In order to verify Eq. (4) for all time, we need to choose a ratio that is approximately equal to 1 ($p/q \cong 1$), but not exactly 1 to account for slight differences in the local mean field of these two neurons. Since C_i depends on σ for networks that have neurons possessing a finite degree, we do not expect to observe a stable phase synchronization in this network. Small changes in σ may cause small changes in the ratio p/q. Notice however that Eq. (4) might be satisfied for a very long time, for $p/q = 1$. If neurons are locked to different CAS patterns (and therefore have different local mean field), Eqs. (1) and (4) are both satisfied, but phase synchronization will not be 1:1, but with a ratio of p/q (see Sec. E in Supplementary Information for an example).

If neurons in this scaling-free network become completely synchronous, it is necessary that $\sigma(N) \geq 2\sigma^{CS}(N = 2)/|\lambda_2|$ (Ref. [7]). $\sigma^{CS}(N = 2) \cong 0.5$ represents the value of the coupling strength when two bidirectionally coupled neurons become completely synchronous. $\lambda_2 = -2.06$ is the largest non-positive eigenvalue of the Laplacian matrix defined as $A_{ij} - \text{diag}(k_i)$. So, $\sigma^{CS}(N) \geq 1/2.06 \cong 0.5$. The CAS phenomenon appears when $\sigma^{CAS}(N = 1000) \leq 0.001$, a coupling strength 500 times smaller than the one which produces complete synchronization. Similar conclusions would be obtained when one considers networks of different sizes, with nodes having the same dynamical descriptions and same connecting topology.

Concluding, in this work we introduce the phenomenon of Collective Almost Synchronization (CAS), a phenomenon that is characterized by having nodes possessing approximately constant local mean fields. The appearance of an approximately constant mean field is a consequence of a regime of weak interaction between the nodes responsible to place the node trajectory around stable periodic orbits. A network has the CAS phe-

nomenon if the Central Limit Theorem can be applied, and it exists an approximately constant mean field. In other words, the CAS is invariant to changes in the value of the expected value of the local mean field, that might appear due to parameter alterations (e.g. coupling strength). If the expected value of the local field changes, but the Central limit Theorem can still be applied, nodes of the network will present the CAS phenomenon and the observed weak forms of synchronization among the nodes might (or not) be preserved. As examples of how common this phenomenon could be, we have asserted its appearance in a large networks of chaotic maps (see supplementary information), Hindmarsh-Rose neurons, and Kuramoto oscillators (see supplementary information). In the Supplementary Information, we also discuss that the CAS phenomenon is a possible source of coherent motion in systems that are models for the appearance of collective motion in social, economical, and animal behaviour.

I. SUPPLEMENTARY INFORMATION

A. CAS and generalized synchronization

Generalized synchronization [17, 21] is a common behavior in complex networks [22–24], and should be expected to be found typically. This phenomenon is defined as $x_i = \Phi(y_i)$, where Φ is considered to be a continuous function. As explained in Refs. [17, 21], generalized synchronization appears due to the existence of a low-dimensional synchronous manifold, often a very complicated and unknown manifold.

Recent works [12, 13, 25, 26] have reported that nodes in the network that are highly connected become synchronous. As shown in ref. [23], that is a manifestation of generalized synchronization [17, 21] in complex networks. For a fixed coupling strength among the nodes with heterogeneous degree distributions and for the usual diffusively coupling configuration one should expect that the set of hub nodes (highly connected nodes) provides a skeleton about which synchronization is developed. Reference [27] demonstrates how ubiquitous generalized synchronization is in complex networks. It is shown that a necessary condition for its appearance in oscillators coupled in a driven-response (master-slave) configuration is that the modified dynamics of the response system presents a stable periodic behavior. The modified dynamics is a set of equations constructed by considering only the variables of the response system. In a complex network, a modified dynamics of a node is just a system of equations that contains only variables of that node.

An important contribution to understand why generalized synchronization is a ubiquitous property in complex network is given by the numerical work of Ref. [23] and the theoretical work of Ref. [24]. In Refs. [23, 24] the ideas of Ref. [27] are extended to complex networks. In particular, the work of Ref. [24] shows that gener-

alized synchronization occurs whenever there is at least one node whose modified dynamics is periodic. All the nodes that have a stable and periodic modified dynamics become synchronous in the generalized sense with the nodes that have a chaotic modified dynamics. The general theorem presented in Ref. [24] is a powerful tool for the understanding of weak forms of synchronization or desynchronous behaviors in complex networks. However, identifying the occurrence of generalized synchronization does not give much information about the behavior of the network, since the function that relates the trajectory among the nodes that are generalized synchronous is usually unknown. The CAS phenomenon allows one to calculate, at least in an approximate sense, the equations of motion that describes the pattern to which the nodes are locked to. More specifically, we can derive the set of equations governing, in an approximate sense, the time evolution of the nodes, not covered by the theorem in Ref. [24].

Finally, if there is a node whose modified dynamics describes a stable periodic behavior and its CAS pattern is also a stable periodic stable behavior, then the CAS phenomenon appears when the network presents generalized synchronization.

B. CAS and other synchronous and weak-synchronous phenomena

Consider a network of N nodes described by

$$\dot{\mathbf{x}}_i = \mathbf{F}_i(\mathbf{x}_i) + \sigma \sum_{j=1}^{N} \mathbf{A}_{ij} \mathbf{E}[\mathcal{H}(\mathbf{x}_j - \mathbf{x}_i)] + \zeta_i(t), \quad (11)$$

where $\mathbf{x}_i \in \Re^d$ is a d-dimensional vector describing the state variables of the node i, $\mathbf{F}_i$ is a d-dimensional vector function representing the dynamical system of the node i, $\mathbf{A}_{ij}$ is the adjacent connection matrix, $\mathbf{E}$ is the coupling function as defined in [7], $\mathcal{H}$ is an arbitrary differentiable transformation, and $\zeta_i(t)$ is an arbitrary random fluctuation. Assume in the following that $\zeta_i(t) = 0$.

Assume that the nodes in the network (11) have equal dynamical descriptions, i.e., $\mathbf{F}_i = \mathbf{F}$, that the network is fully connected, so every node has a degree $k_i = N - 1$, and that $\mathcal{H}(\mathbf{x}_j - \mathbf{x}_i) = (\mathbf{x}_j - \mathbf{x}_i)$. We can rewrite it in terms of the average field $\overline{\mathbf{x}}(t) = \frac{1}{N} \sum_{i=1}^{N} \mathbf{x}_i(t)$:

$$\dot{\mathbf{x}}_i = \mathbf{F}_i(\mathbf{x}_i) - p_i \mathbf{E}(\mathbf{x}_i - \overline{\mathbf{x}}), \quad (12)$$

where $p_i = \sigma k_i$. Therefore every node becomes "decoupled" from the network in the sense that their interaction is all mediated by the average field. Collective behavior is dictated by the behavior of the average field and the individual dynamics of the node. The linear stability of the network (12) was used in Ref. [12] as an approximation to justify how desynchronous behavior about the average field can appear in complex networks. Notice that this assumption can only be rigorously fulfilled if

the network is fully connected and, therefore, it is natural to understand why the desynchronous phenomena reported in Ref. [12] happens for nodes that are highly connected. One can interpret the desynchronous behavior observed in Ref. [12] as an almost synchronization between a node and the mean field $\overline{\mathbf{x}}$.

The differences between complete synchronization and synchronization in the collective sense can be explained through the following example. An interesting solution of Eq. (12) can be obtained when $\overline{\mathbf{x}} = \mathbf{x}_i(t)$, $\mathbf{x}_i(t)$ varying in time. In this case, the average field is along the synchronization manifold. The network being completely synchronous, all nodes having equal trajectories, and $\mathbf{F}_i(\mathbf{x}_i(t)) = \mathbf{x}_i(t)$. For such a special network, collective behavior and complete synchronization are the same. On the other hand, collective behavior typically appears when the coupling term $\sigma E(\mathbf{x}_i - \overline{\mathbf{x}})$ is different from zero for most of the time and $\mathbf{F}_i(\mathbf{x}_i) \neq \mathbf{x}_i$, but there is a majority of nodes with similar behavior. In this sense, the desynchronous behaviors reported in Ref. [12] can be considered as a collective phenomena that happens to parameters close to the ones that yields complete synchronization.

To understanding when the CAS phenomenon occurs, consider the solution of Eq. (12) in the thermodynamics limit $N \to \infty$ when $\overline{\mathbf{x}}$ is a constant in time, $\overline{\mathbf{x}} = C$. For such a situation, the evolution of a node can be described by the same following d-dimensional system of ODEs

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) - p\mathbf{E}(\mathbf{x} - \mathbf{C}), \qquad (13)$$

where $p = \sigma(N - 1)$. If complete synchronization takes place, then $\mathbf{F}_i(\mathbf{C}) = 0$, meaning that there can only exist complete synchronization if all the nodes lock into the same stable steady state equilibrium point, likely to happen if $\mathbf{F}_i$ is the same for all the nodes.

Another possible network configuration that leads to $\overline{\mathbf{x}} = \mathbf{C}$ happens when each node is only weakly coupled ("independent") with the others such that the Central Limit Theorem could be applied. If the network has only a finite number of nodes and $\overline{\mathbf{x}}(t)$ is not exactly constant in time, but $\overline{\mathbf{x}}(t) \cong \mathbf{C}$, the nodes still behave in the same predictable way if the dynamics described by $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) - p\mathbf{E}(\mathbf{x}) + p\mathbf{E}(\mathbf{C})$ is a sufficiently stable periodic orbit. This is how the CAS phenomenon appears in fully connected networks. All nodes become locked to the stable periodic orbit described by $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) - p\mathbf{E}(\mathbf{x}) + p\mathbf{E}(\mathbf{C})$.

Now, we break the symmetry of the network, allowing the nodes to be connected arbitrarily to their neighbors. We still consider diffusive linear couplings, $\mathcal{H}(\mathbf{x}_j - \mathbf{x}_i) = (\mathbf{x}_j - \mathbf{x}_i)$. The equations of such a network can be written as

$$\dot{\mathbf{x}}_i = \mathbf{F}_i(\mathbf{x}_i) - p_i\mathbf{E}(\mathbf{x}_i) + p_i\mathbf{E}(\overline{\mathbf{x}}_i(t)), \qquad (14)$$

where k_i is the degree of node i with $k_l \leq k_m$, if $l < m$, and $\overline{\mathbf{x}}_i(t)$ is the local mean field defined as

$$\overline{\mathbf{x}}_i(t) = \frac{1}{k_i} \sum_{j=1}^{N} A_{ij}\mathbf{x}_j(t). \qquad (15)$$

Our main assumption is that the local mean field of a variable that is bounded, either $\overline{\mathbf{x}}_i(t)$ or $\overline{\dot{\mathbf{x}}}_i(t)$, exhibits small oscillations about an expected constant value $\mathbf{C}$. In other words, one can define a time average $\mathbf{C}$ by either

$$\mathbf{C}_i = \frac{1}{t} \int_0^t \overline{\mathbf{x}}_i(t)dt, \qquad (16)$$

or

$$\mathbf{C}_i = \frac{1}{t} \int_0^t \overline{\dot{\mathbf{x}}}_i(t)dt. \qquad (17)$$

Notice that $\mathbf{x}_i \in \Re^d$ (or $\dot{\mathbf{x}}_i \in \Re^d$), and so does $\mathbf{C} \in \Re^d$. The CAS phenomenon appears for a node that has at least one component of the local mean field ($\overline{\mathbf{x}}_i$ or $\overline{\dot{\mathbf{x}}}_i$) that is approximately constant. The appearance of this almost constant value is a consequence of the Central Limit Theorem. For networks whose nodes are described by only bounded variables, when calculating the local mean field we only take into consideration the component receiving the couplings from other nodes. For networks of Kuramoto oscillators that have one variable (the phase θ) that is not bounded, a constant local mean field appears in the component that describes the instantaneous frequency ($\dot{\theta}_i$).

In Ref. [24], it was shown that for chaotic networks described by a system of equations similar to Eq. (14), generalized synchronization can appear if the modified dynamics described by $\dot{\mathbf{x}}_i = \mathbf{F}_i(\mathbf{x}_i) - \sigma k_i\mathbf{E}(\mathbf{x}_i)$ of a certain number of nodes are either stable equilibrium points ($\dot{\mathbf{x}}_i=0$) or they describe stable periodic solutions (limit cycle). Generalized synchronization appears between the nodes that have modified dynamics describing stable periodic states and the nodes that have modified dynamics describing chaotic states.

To understand the phenomenon of *collective almost synchronization* (CAS), introduced in this work, consider that $\mathcal{H}(\mathbf{x}_j - \mathbf{x}_i) = (\mathbf{x}_j - \mathbf{x}_i)$. It is a phenomena that appears necessarily when $\overline{\mathbf{x}}_i \cong \mathbf{C}_i$ or $\overline{\dot{\mathbf{x}}}_i \cong \mathbf{C}_i$. The equations for the network can then be described by

$$\dot{\mathbf{x}}_i = F_i(\mathbf{x}_i) - p_i\mathbf{E}(\mathbf{x}_i) + p_i\mathbf{E}(\mathbf{C}_i) + \delta_i, \qquad (18)$$

where the residual term is $\delta_i = p_i(\overline{\mathbf{x}}_i - \mathbf{C}_i)$. This term is small most of the time but large for some intervals of time; $\delta_i(t) > 0$ for all time, but $\delta_i(t) < \epsilon$ for most of the times. Another requirement for the CAS phenomenon to appear is that the CAS pattern $\boldsymbol{\Xi}_i(t)$ of a node i that is described by Eq. (18) ignoring the residual term

$$\dot{\boldsymbol{\Xi}}_i = \mathbf{F}_i(\boldsymbol{\Xi}_i) - p_i\mathbf{E}(\boldsymbol{\Xi}_i) + p_i\mathbf{E}(\mathbf{C}_i). \qquad (19)$$

must be a stable periodic orbit. We define that a node presents collective almost synchronization (CAS) if

$$|\mathbf{x}_i(t) - \boldsymbol{\Xi}_i(t - \tau_i)| < \epsilon_i, \qquad (20)$$

for most of the time,

Notice from Eq. (19) that for $p_i > 0$, the CAS pattern will not be described by $\mathbf{F}(\mathbf{x}_i)$ and therefore does not belong to the synchronization manifold. On the other hand, Ξ_i is induced by the local mean field as typically happens in synchronous phenomenon due to collective behavior. This property of the CAS phenomenon shares similarities with the way complete synchronization appears in networks of nodes coupled under time-delay functions [9]. In such networks, nodes become completely synchronous to a solution of the network that is different from the solution of an isolated node of the network. Additionally, the trajectory of the nodes present a time-lag to this solution.

To understand the reason why the CAS phenomenon appears when $\Xi_i(t)$ is a sufficiently stable periodic orbit, we study the variational equation of the CAS pattern (19)

$$\dot{\xi}_i = [D\mathbf{F}_i(\xi_i) - p_i\mathbf{E}]\xi_i. \tag{21}$$

obtained by linearizing Eq. (19) around Ξ_i by making $\xi_i = \mathbf{x}_i - \Xi_i$. This equation produces no positive Lyapunov exponents. As a consequence, neglecting the existence of the time-lag between $\mathbf{x}_i(t)$ and $\Xi(t)_i$, the trajectory of the node i oscillates about Ξ_i, and $\mathbf{x}_i - \Xi_i \leq \epsilon_i$, for most of the time, satisfying Eq. (20), where ϵ_i depends on δ_i. If there are two nodes i and j, which feel similar local mean fields, $\Xi_i \cong \Xi_j$, then $\mathbf{x}_i \cong \mathbf{x}_j$, for most of the time.

To understand why the nodes that present CAS have also between them a time-lag type of synchronization, integrate Eq. (18), using Eq. (19), to obtain

$$\mathbf{x}_i(t) = \int_0^t [\dot{\Xi}_i(t) + \delta_i(t)]dt. \tag{22}$$

This integral is not trivial in the general case. But we have a simple phenomenological explanation for its solution. When the CAS pattern is sufficiently stable, the asymptotic time limit state of the variable $\mathbf{x}_i(t)$ is the CAS pattern $\Xi_i(t)$. But due to the residual term $\delta_i(t)$, the trajectory of $\mathbf{x}_i(t)$ arrives in the neighborhood of $\Xi(t)$ at time t with a time-lag. As a result, nodes that are collectively almost synchronous obey Eq. (20). In addition, two nodes that present CAS have also a time-lag between their trajectories for the same reason. There is an extra contribution to the time-lag between the trajectories of two nodes if their initial conditions differ.

Phase synchronization [14] is a phenomena where the phase difference, denoted by $\Delta\phi_{ij}$ between the phases of two signals (or nodes in a network), $\phi_i(t)$ and $\phi_j(t)$, remains bounded for all time

$$\Delta\phi_{ij} = \left|\phi_i(i) - \frac{p}{q}\phi_j(t)\right| \leq S, \tag{23}$$

where $S = 2\pi$, and p and q are two rational numbers [14]. For coupled chaotic oscillators one can also find irrational phase synchronization [20], where Eq. (23) can be satisfied *for all time* with p and q irrational. S is a reasonably small constant, that can be larger than 2π in order to encompass oscillatory systems that either have a time varying time-scale or whose time-lag varies in time. This bound can be simply calculated by making S to represent the growth of the phase in the faster time scale after one period of the slower time scale.

The link between the CAS phenomenon and phase synchronization can be explained by thinking that it is a synchronous phenomenon among the nodes that is mediated by their CAS patterns. The phase of the periodic orbit of the CAS pattern of the node i grows as $\tilde{\phi}_i(t) = \omega_i t + \xi_i(t) + \phi_i^0$ and of the node j grows as $\tilde{\phi}_j(t) = \omega_j t + \xi_j(t) + \phi_j^0$. The quantities ϕ_i^0 and ϕ_j^0 are displacements of the phase caused by the existence of time-lag, and $\xi_i(t)$ and $\xi_j(t)$ are small fluctuations. For $t \to \infty$ these can be neglected and we have that

$$\frac{\tilde{\phi}_i(t)}{\tilde{\phi}_j(t)} = \frac{\omega_i}{\omega_j} = \frac{p}{q}, \tag{24}$$

where $\omega_i = \lim_{t\to\infty} \frac{\tilde{\phi}_i(t)}{t}$ gives the average frequency of oscillation of the CAS pattern of node i, and p and q are two real numbers.

The phase of the nodes can be written as a function of the phase of the periodic orbits of the CAS pattern. So, $\phi(t)_i = \tilde{\phi}(t)_i + \delta\phi_i(t)$ and $\phi(t)_j = \tilde{\phi}(t)_j + \delta\phi_j(t)$, $\delta_i(t)$ represents a variation of the phase of the node i with respect to the phase of the CAS pattern, and depends on the way the phase is defined [28]. The phase difference $\Delta\phi_{ij}(t)$, as written in Eq. (23), becomes equal to $|t(q\omega_i - p\omega_j) + q\delta_i(t) - p\delta\phi_j(t)|$. But, from Eq. (24), $q\omega_i - p\omega_j = 0$, and therefore, $\Delta\phi_{ij}(t) \leq \max(q\delta\phi_i(t) - p\delta\phi_j(t))$. But since the node orbit is locked to the CAS pattern, $\Delta\phi_{ij}(t)$ is always a small quantity.

In practice, for networks composed by a finite number of nodes, we do not expect that the quantities $\delta\phi_i(t)$ and $\delta\phi_j(t)$ to remain small for all the time. The reason is that the CAS pattern can only be approximately calculated and in general we do not know the precise real value of the local mean field. However, our simulations show that these quantities remain small for time intervals that comprise many periods of oscillations of the node trajectories. For networks having an expected value of the mean field $\mathbf{C}_i$ that is independent on the coupling strength σ, the ratio p/q does not change as one changes the value of σ, and then phase synchronization is stable under a parameter variation. For the network of Kuramoto oscillators, Eq. (23) can be verified for all time with a value of p/q that remains invariant as one changes σ.

Assume for now that the nodes have equal dynamics, so $\mathbf{F}_i = \mathbf{F}$. If a node i with degree k_i has a periodic CAS pattern that is sufficiently stable under Eq. (21), all the nodes with degrees close to k_i also have similar CAS patterns that are sufficiently stable under Eq. (21). Node i is locked to Ξ_i and node j is locked to Ξ_j. But since Ξ_i is approximately equal to Ξ_j, thus, $\mathbf{x}_i \cong \mathbf{x}_j$, for most of the time. So, if the pattern solution is sufficiently stable, the external noise $\zeta_i(t)$ can be different from zero,

and still have similar trajectories for that interval of time. The same argument remains valid if $\mathbf{F}_i \neq \mathbf{F}_j$, as long as the CAS pattern is sufficiently stable.

In Ref. [26], synchronization was defined in terms of the node $\mathbf{x}_N$ that has the largest number of connections, when $\mathbf{x}_i(t) \cong \mathbf{x}_N$ (which is equivalent to stating that $|\mathbf{x}_i(t) - \mathbf{x}_N| < \epsilon$), where $\mathbf{x}_N$ is assumed to be very close to the synchronization manifold $\mathbf{s}$ defined by $\dot{\mathbf{s}} = \mathbf{F}(\mathbf{s})$. This type of synchronous behavior was shown to exist in scaling free networks whose nodes have equal dynamics and that are linearly connected. This was called hub synchronization.

The link between the CAS phenomenon with the hub synchronization phenomenon [26], and generalized synchronization can be explained as in the following. It is not required for nodes that present the CAS phenomenon for their error dynamics $\mathbf{x}_j - \mathbf{x}_i$ to be small. But for the following comparison, assume that $\vartheta_{ij} = \mathbf{x}_j - \mathbf{x}_i$ is small so that we can linearise Eq. (14) about another node j. Assume also that $\mathbf{F}_i = \mathbf{F}$. The variational equations of the error dynamics between two nodes i and j that have equal degrees are described by

$$\dot{\vartheta}_{ij} = [D\mathbf{F}(\mathbf{x}_i) - p_i E]\vartheta_{ij} + \eta_i. \qquad (25)$$

In Ref. [26], hub synchronization exists if Eq. (25), neglecting the coupling term η_i, has no positive Lyapunov exponents. That is another way of stating that hub synchronization between i and j occurs when the variational equations of the modified dynamics $[\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) - p_i E(\mathbf{x}_i)]$ presents no positive Lyapunov exponent. In other words, in order to have hub synchronization it is necessary that the modified dynamics of both nodes be describable by stable periodic oscillations. Hub synchronization is the result of a weak form of generalized synchronization, defined in terms of the linear stability of the error dynamics between two highly connected nodes. Unlike generalized synchronization, hub synchronization offers a way to predict, in an approximate sense, the trajectory of the synchronous nodes.

In contrast, the CAS phenomenon appears when the CAS pattern, which is different from the solution of the modified dynamics, becomes periodic. Another difference between the CAS and the hub synchronization phenomenon is that whereas $\overline{\mathbf{x}}_i \cong \mathbf{C}$ in the CAS phenomenon, $\overline{\mathbf{x}}_i \cong \mathbf{x}_i$ in the hub synchronization, in order for η_i to be very small, and $\mathbf{x}_i$ to be close to the synchronization manifold. So, whereas hub synchronization can be interpreted as being a type of practical synchronization [15], CAS is a type of almost synchronization.

In the work of Refs. [29, 30], it was numerically reported a new desynchronous phenomenon in complex networks. The network has no positive Lyapunov exponents but it presents a desynchronous non-trivial collective behavior. A possible situation for the phenomenon to appear is when δ_i and $\mathbf{C}_i$ in Eq. (18) are either zero or sufficiently small such that the stability of the network is completely determined by Eq. (21), and this equation produces no positive Lyapunov exponent. Assume now that p_i in Eq. (19) is appropriately adjusted such that the CAS pattern for every node i is a stable periodic orbit. The variational Eqs. (21) for all nodes have no positive Lyapunov exponents. If additionally, $\overline{\mathbf{x}}_i(t) \cong \mathbf{C}$, then the network in Eq. (14) possesses no positive Lyapunov exponent. Therefore, networks that present the CAS phenomenon for all nodes might present the desynchronous phenomenon reported in Refs. [29, 30]. The CAS phenomenon becomes different from the phenomenon of Refs.[29, 30] if for at least one node, Eq. (19) produces a chaotic orbit.

To understand the occurrence of CAS in networks formed by heterogeneous nodes connected by nonlinear functions such as networks of Kuramoto oscillators, we rewrite the Kuramoto's network model in terms of the local mean field, $\overline{\theta}_i = \frac{1}{k_i} \sum_{j=1}^N A_{ij}\theta_j$. Using the coordinate transformation

$$\frac{1}{k_i} \sum_{j=1}^N A_{ij} \exp^{\jmath(\theta_j - \theta_i)} = \tilde{r}_i \exp^{\jmath(\overline{\theta}_i - \theta_i)}, \qquad (26)$$

the dynamics of the node i is described by

$$\dot{\theta}_i = \omega_i + p_i \tilde{r}_i \sin(\overline{\theta}_i - \theta_i). \qquad (27)$$

The phase θ_i is not a bounded variable and therefore we expect that typically $\overline{\theta}_i$ has not a well defined average. But, $\dot{\overline{\theta}}_i(t)$ is bounded and has a well defined average value which is an approximately constant quantity (C_i) for nodes in networks with sufficiently large number of connections and with sufficiently small coupling strengths. When $\dot{\overline{\theta}}_i \cong C_i$, the node i has the propensity to exhibit the CAS phenomenon, and the CAS pattern is calculated by Eq. (27) considering that $\overline{\theta}_i = C_i t$. Notice that $\overline{\theta}_i = \dot{\overline{\theta}}_i t \cong C_i t$.

Phase synchronization between two nodes in the networks of Eq. (27) is stable under parameter variations (coupling strength in this case) if these nodes present the CAS phenomenon. There is irrational (rational) phase synchronization if $\frac{\dot{\overline{\theta}}_i}{\dot{\overline{\theta}}_j}$ is irrational (rational). If nodes are sufficiently "decoupled" we expect that $\frac{\dot{\overline{\theta}}_i}{\dot{\overline{\theta}}_j} \approx \omega_i/\omega_j$. Phase synchronization will be rational whenever nodes with different natural frequencies become locked to Arnold tongues's, induced by the coupling $p_i \tilde{r}_i \sin(\overline{\theta}_i - \theta_i)$.

There is a special solution of Eq. (27) that produces a bounded state in the variable θ_i when the network is complete synchronous to an equilibrium point. In such case, $\overline{\theta}_i$ becomes constant, and Eq. (27) has one stable equilibrium $\theta_i = \arcsin\left(\frac{\omega_i}{p_i}\right)$, obtained when $p_i > \omega_i$. But, the local mean field becomes constant due to complete synchronization and not due to the fact that the nodes are "decoupled". These conditions do not produce the CAS phenomenon.

We take the thermodynamics limit when the network has infinite nodes with infinite degrees. C_i calculated

using Eq. (17) does not change as one change the coupling σ, since $\overline{\dot{\theta}_i} = \lim_{k_i,N\to\infty} \frac{1}{k_i}[\sum_{j=1}^{N} A_{ij}(\omega_i + p_i\tilde{r}_i sin(\overline{\theta_i} - \theta_i))] = \lim_{k_i,N\to\infty} \frac{1}{k_i}[\sum_{j=1}^{N} A_{ij}\omega_j] + [\sum_{j=1}^{N} A_{ij}(\sigma\tilde{r}_j sin(\overline{\theta_j} - \theta_i))] = \frac{1}{k_i}[\sum_{j=1}^{N} A_{ij}\omega_j] + \sigma\sum_{j=1}^{N} A_{ij}(\tilde{r}_j sin(\overline{\theta_j} - \theta_i))$. But, if nodes are sufficiently decoupled $\sum_{j=1}^{N} A_{ij}(\tilde{r}_j sin(\overline{\theta_j} - \theta_i))$ approaches zero, and therefore, C_j only depends on the natural frequencies: $\overline{\dot{\theta}_i} = C_i = \frac{1}{k_i}[\sum_{j=1}^{N} A_{ij}\omega_j]$.

Assume that there are two nodes, i and j, and that for most of the time $\Xi_i \cong \Xi_j$. Then, for most of the time it is also true that $\Xi_i - \theta_i \cong \Xi_j - \theta_j$, which allow us to write that $sin(\Psi_j - \theta_j) - sin(\Psi_i - \theta_i) \cong \cos(\Psi_i - \theta_i)[(\Psi_j - \theta_j) - (\Psi_i - \theta_j)] \cong \cos(\Psi_i - \theta_i)[\theta_j - \theta_j]$. Since $\Psi_i \cong \theta_i$, then $\cos(\Psi_i - \theta_i) \cong 1$ and $sin(\Psi_j - \theta_j) - sin(\Psi_i - \theta_i) \cong [\theta_j - \theta_j]$. Defining the error dynamics between the two nodes to be $\xi_{ij} = \theta_j - \theta_i$, we arrive that

$$\dot{\xi}_{ij} \cong (\omega_j - \omega_i) - p_i\xi_{ij}. \tag{28}$$

Therefore, it implies that we expect to find two nodes having the same similar CAS behavior when both the local mean field is close and when the difference between their natural frequencies $(\omega_j - \omega_i)$ is small.

The CAS phenomenon can also appear in a system of driven particles [31] that is a simple but powerful model for the onset of pattern formation in population dynamics [2], economical systems [32] and social systems [3]. In the work of Ref. [31], it was assumed that individual particles were moving at a constant speed but with an orientation that depends on the local mean field of the orientation of the individual particles within a local neighborhood and under the effect of additional external noise. Writing an equivalent time-continuous description of the Vicsek particle model [31], the equations of motion for the direction of movement of a particle i, can be written as

$$\dot{\mathbf{x}}_i = -\mathbf{x}_i + \overline{\mathbf{x}}_i + \Delta\theta_i, \tag{29}$$

where $\overline{\mathbf{x}}_i$ represents the local mean field of the orientation of the particle i within a local neighborhood and $\Delta\theta_i$ represents a small noise term. When $\overline{\mathbf{x}}_i$ is approximately constant, the CAS pattern is described by a solution of $\dot{\mathbf{x}}_i = -\mathbf{x}_i + \overline{\mathbf{x}}_i$, which will be a stable equilibrium point as long as $\Delta\theta_i$ is sufficiently small. From the Central Limit Theorem, $\overline{\mathbf{x}}_i$ will be approximately constant as long as the neighborhood considered is sufficiently large or the density of particles is sufficiently large.

C. About the expected value of the local mean field: the Central Limit Theorem

The Theorem states that, given a set of t observations, each set of observation containing k measurements $(x_1, x_2, x_3, x_4, \ldots, x_k)$, the sum $S_N = \sum_{i=1}^{k} x_i(N)$ (for $N = 1, 2, \ldots, t$), with the variables $x_i(N)$ drawn from an independent random process that has a distribution with finite variance μ^2 and mean $\overline{x}$, converges to a Normal distribution for sufficiently large k. As a consequence, the expected value of these t observations is given by the mean $\overline{x}$ (additionally, $\overline{x} = \frac{1}{t}\sum_{N=1}^{t} S_N$), and the variance of the expected value is given by $\frac{\mu^2}{k}$. The larger the number k of variables being summed, the larger is the probability with which one has a sum close to the expected value. There are many situations when one can apply this theorem for variables with some sort of correlation [33], as it is the case for variables generated by deterministic chaotic systems with strong mixing properties, for which the decay of correlation is exponentially fast. In other words, a deterministic trajectory that is strongly chaotic behaves as an independent random variable in the long-term. For that reason, the Central Limit Theorem holds for the time average value $\overline{x}(t)$ produced by summing up chaotic trajectories from nodes belonging to a network that has nodes weakly connected. Consequently, the distribution of $\overline{x}_i(t) = \frac{1}{N}\sum_j A_{ij}x_j(t)$ for node i should converge to a Gaussian distribution centered at $C_i = \frac{1}{t}\int_0^t \overline{x}_i(t)dt$ as the degree of the node is sufficiently large. In addition, the variance μ_i^2 of the local mean field $\overline{x}(t)_i$ decreases proportional to k_i^{-1}, as we have numerically verified for networks of Hindmarsh-Rose neurons ($\mu_i^2 \propto k_i^{-1.0071}$) and networks of Kuramoto oscillators ($\mu_i^2 \propto k_i^{-1.055}$).

If the network has no positive Lyapunov exponents, we still expect to find an approximately constant local mean field at a node i, as long as the nodes are weakly connected and its degree is sufficiently large. To understand why, imagine that every node in the network stays close to a CAS pattern and one of its coordinates is described by $sin(\omega_i t)$. Without loss of generality we can make that every node has the same frequency $\omega_i = \omega$. The time-lag property in the node trajectories, when they exhibit the CAS pattern, results in that every node is close to $sin(\omega_i t)$ but they will have a random time-lag in relation to the CAS pattern (due to the decorrelated property between the node trajectories). So, the selected coordinate can be described by $sin(\omega t + \phi_i^0) + \delta_i(t)$, where ϕ_i^0 is a random initial phase and $\delta_i(t)$ is a small random term describing the distance between the node trajectory and the CAS pattern. Neglecting the term $\delta_i(t)$, the distribution of the sum $\sum_{i=1}^{k} sin(\omega t + \phi_i^0)$ converges to a normal distribution with a variance that depends on the variance of $sin(\phi_i^0)$.

From previous considerations, if the degree of some of the nodes tend to infinite, the variance of the local mean field for those nodes tends to zero and, in this limit, the residual term δ_i in Eq. (18) is zero and the local mean field of these nodes is a constant. As a consequence, the node is perfectly locked with the CAS pattern ($\epsilon = 0$ in Eq. (20)).

D. CAS in a network of coupled maps

As another example to illustrate how the CAS phenomenon appears in a complex network, we consider a network of maps whose node dynamics is described by $F_i(x_i) = 2x_i \bmod(1)$. The network composed, say, by $N = 1000$ maps, is represented by $x_i^{(n+1)} = F_i(x_i^{(n)}) + \sigma \sum_{j=1}^{N} A_{ij}(x_j^{(n)} - x_i^{(n)}) \bmod(1)$, where the upper index n represents the discrete iteration time, and A_{ij} is the adjacency matrix of a scaling-free network. The map has a constant probability density. When such a map is connected in a network, the density is no longer constant, but still symmetric and having an average value of 0.5. As a consequence, nodes that have a sufficient amount of connections ($k \geq 10$) feel a local mean field, say, within $[0.475, 0.525]$, (deviating of 5% about C_i=0.5) and $\mu_i^2 \propto k_i^{-1}$ (**criterion 1**), as shown in Fig. 2(a). Therefore, such nodes have propensity to present the CAS phenomenon. In (b) we show a bifurcation diagram of the CAS pattern, Ξ_i, obtained from Eq. (19) by using $C_i = C = 0.5$, as we vary p_i. Nodes in this network that have propensity to present the CAS phenomenon will present it if additionally $p_i \in [1,3]$; the CAS pattern is described by a period-2 stable orbit (**criterion 2**). This interval can be calculated by solving $|2 - p_i| \leq 1$. In (c) we show the probability density function of the trajectory of a node that present the CAS phenomenon. The density is centered at the position of the period-2 orbit of the CAS pattern and for most of the time Eq. (20) is satisfied. The filled circles are fittings assuming that the probability density is given by a Gaussian distribution. Therefore, there is a high probability that ϵ_i in Eq. (20) is small. In (d) we show a plot of the trajectories of two nodes that have the same degree which is equal to 80. We chose nodes which present no time-lag between their trajectories and the trajectory of the pattern. If there was a time-lag, the points in (d) would not be only aligned along the diagonal (identity) line, but they would also appear off-diagonal.

E. CAS in the Kuramoto network

An illustration of this phenomenon in a network composed by nodes having heterogeneous dynamical descriptions and a nonlinear coupling function is presented in a random network of N=1000 Kuramoto oscillators. We rewrite the Kuramoto network model in terms of the local mean field, $\overline{\theta}_i = \frac{1}{k_i} \sum_{j=1}^{N} A_{ij}\theta_j$. Using the coordinate transformation $\frac{1}{k_i} \sum_{j=1}^{N} A_{ij} \exp^{\mathfrak{j}(\theta_j - \theta_i)} = \tilde{r}_i \exp^{\mathfrak{j}(\overline{\theta}_i - \theta_i)}$, the dynamics of node i is described by

$$\dot{\theta}_i = \omega_i + p_i \tilde{r}_i \sin(\overline{\theta}_i - \theta_i), \qquad (30)$$

where ω_i is the natural frequency of the node i, taken from a Gaussian distribution centered at zero and with standard deviation of 4. If $\tilde{r}_i$=1, all nodes coupled to node i are completely synchronous with it. If $\tilde{r}_i$=0, there
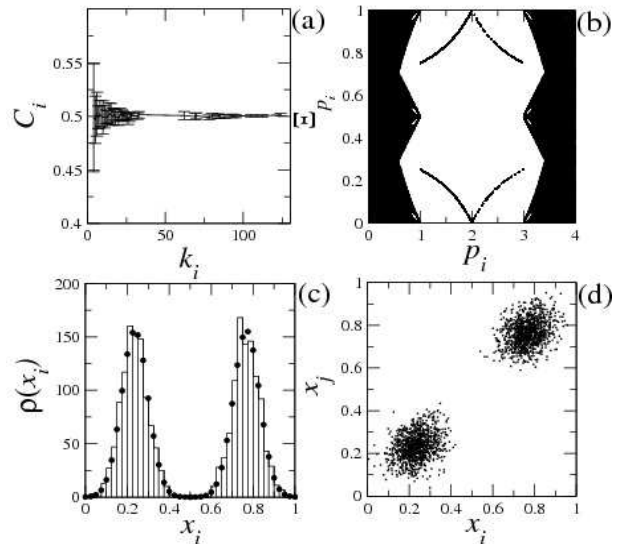

FIG. 2: (a) Expected value of the local mean field of the node i against the node degree k_i. The error bar indicates the variance (μ_i^2) of $\overline{x}_i$. (b) A bifurcation diagram of the CAS pattern [Eq. (19)] considering $C_i = 0.5$. (c) Probability density function of the trajectory of a node with degree k_i=80 (therefore, $p_i = \sigma k_i = 1.3$, $\sigma = 1.3/80$). (d) A return plot considering two nodes (i and j) with the same degree $k_i = k_j$ =80.

is no synchronization between the nodes that are coupled to the node i. Since the phase is an unbounded variable, the CAS phenomenon should be verified by the existence of an approximate constant local mean field in the frequency variable $\dot{\theta}_i$. If $\overline{\dot{\theta}}_i(t) \cong C_i$, which means that $\overline{\theta}_i = \dot{\overline{\theta}}_i t \cong C_i t$, then Eq. (30) describes a periodic orbit (the CAS pattern), regardless the values of ω_i, p_i, and $\tilde{r}_i$, since it is an autonomous two-dimensional system; chaos cannot exist. Therefore, **criterion 2** is always satisfied in a network of Kuramoto oscillators. We have numerically verified that **criterion 1** is satisfied for this network for $\sigma \leq \sigma^{CAS}(N = 1000)$, where $\sigma^{CAS}(N = 1000) \cong 0.075$. Complete synchronization is achieved in this network for $\sigma \geq \sigma^{CS} = 1.25$. So, the CAS phenomenon is observed for a coupling strength that is 15 times smaller than the one that produces complete synchronization.

For the following results, we choose $\sigma = 0.001$. Since the natural frequencies have a distribution centered at zero, it is expected that, for nodes with higher degrees, the local mean field is close to zero (see Fig. 3(a)). In (b), we show the variance of the local mean field of the nodes with degree k_i. The fitting produces $\mu_i^2 \propto k_i^{-1.055}$ (**criterion 1**). In (c), we show the relationship between the value of $p_i \tilde{r}_i$ and the value of the degree k_i. In order to calculate the CAS pattern of a node with degree k_i, we need to use the value of $p_i \tilde{r}_i$ (which is obtained from this figure) and the measured C_i as an input in Eq. (30). We pick two arbitrary nodes, i and j, with
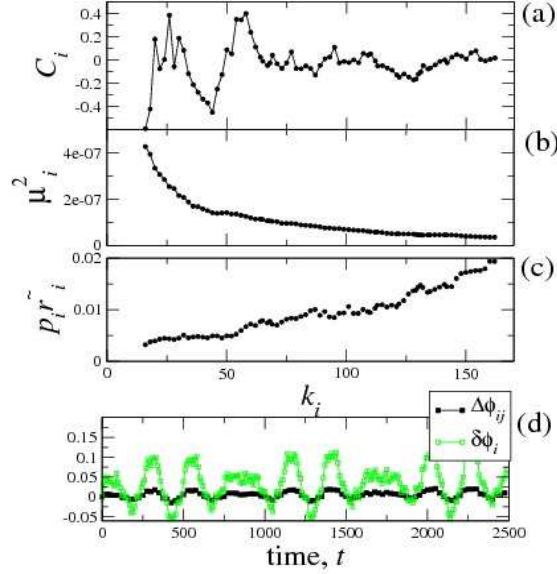
FIG. 3: Results for $\sigma = 0.001$. (a) Expected value of the local mean field $\overline{\overline{\theta}}_i$ of a node with degree k_i. (b) The variance μ_i^2 of the local mean field. (c) Relationship between the value of $p_i\tilde{r}_i$ and k_i. (d) Phase difference $\Delta\phi_{ij} = \theta_i - p/q\theta_j$ between two nodes, one with degree $k_i = 96$ and the other with degree $k_j = 56$; the phase difference $\delta\phi_i = \theta_i - \Xi_{\theta_i}$ between the phases of the trajectory of the node i with degree $k_i = 96$ and the phase of its CAS pattern.

degrees $k_i = 96$ and $k_j = 56$, respectively, with natural frequencies $\omega_i \approx -5.0547$ and $\omega_j \approx -5.2080$. In (d), we show that phase synchronization is verified between these two nodes, with $p/q = \omega_i/\omega_j$. We also show the phase difference $\delta\phi_j = \theta_j - \Xi_{\theta_j}$ between the phases of the trajectory of the node i with degree $k_j = 96$ and the phase of its CAS pattern, for a time interval corresponding to approximately $2500/P$ cycles, where the period of the cycles in node i is calculated by $P = \frac{2\pi}{5.0547}$. Phase synchronization between nodes i and j is a consequence of the fact that the phase difference between the nodes and their CAS patterns is bounded.

In the thermodynamic limit, when a fully connected network has an infinite number of nodes, C_i does not change as one changes the coupling σ, since it only depends on the mean field of the frequency variable $(\overline{\overline{\theta}})$. As a consequence, if there is the CAS phenomenon and phase synchronization between two nodes with a ratio of p/q for a given value of σ, changing σ does not change

the ratio p/q. Therefore phase synchronization is stable under alterations in σ. Phase synchronization will be rational and stable whenever nodes with different natural frequencies ω_i become locked to Arnold tongues [34, 35] induced by the coupling $p_i\tilde{r}_i sin(\overline{\overline{\theta}}_i - \theta_i)$.

There is a special solution of Eq. (30) that produces a bounded state in the variable θ_i when the network is complete synchronous to an equilibrium point. In such case, $\overline{\overline{\theta}}_i$ becomes constant, and Eq. (30) has one stable equilibrium $\theta_i = \arcsin\left(\frac{\omega_i}{p_i}\right)$, obtained when $p_i > \omega_i$. But, the local mean field becomes constant due to complete synchronisation and not due to the fact that the nodes are "decoupled". These conditions do not produce the CAS phenomenon.

F. Preserving the CAS pattern in different networks: a way to predict the onset of the CAS phenomenon in larger networks

Consider two networks, n_1 and n_2, whose nodes have equal dynamical descriptions, the network n_1 with N_1 nodes and the network n_2 with N_2 nodes $(N_2 > N_1)$, and two nodes, i in the network n_1 and j in the network n_2. Furthermore, assume that both nodes have stable periodic CAS patterns (**criteria 1** is satisfied), and assume that the nodes have sufficiently large degrees such that the local mean field of node i is approximately equal to node j. Then the CAS pattern of node i will be approximately the same as the one of node j if

$$\sigma^{CAS}(n_1)k_i(n_1) = \sigma^{CAS}(n_2)k_j(n_2). \qquad (31)$$

$\sigma^{CAS}(n_1)$ and $\sigma^{CAS}(n_2)$ represent the largest coupling strengths for which the variance of the local mean field of a node decays with the inverse of the degree of the node (**criterion 2** is satisfied) in the networks, respectively, and $k_i(n_1)$ and $k_j(n_2)$ are the degrees of the nodes i and j, respectively. In other words, the CAS phenomenon occur in the network if $\sigma \leq \sigma^{CAS}$.

Therefore, if $\sigma^{CAS}(N_1)$ is known, $\sigma^{CAS}(N_2)$ can be calculated from Eq. (31). In other words, if the CAS phenomenon is observed at node i for $\sigma \leq \sigma^{CAS}(N_1)$, the CAS phenomenon will also be observed at node j for $\sigma(n_2) \leq \sigma^{CAS}(n_2)$, where $\sigma^{CAS}(n_2)$ satisfies Eq. (31).

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