The Two-Component Camassa-Holm Equations CH(2,1) and CH(2,2): First-Order Integrating Factors and Conservation Laws

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Abstract:
Recently, Holm and Ivanov, proposed and studied a class of multi-component generalisations of the Camassa-Holm equations [D D Holm and R I Ivanov, Multi-component generalizations of the CH equation: geometrical aspects, peakons and numerical examples, *J. Phys A: Math. Theor* **43**, 492001 (20pp), 2010]. We consider two of those systems, denoted by Holm and Ivanov by CH(2,1) and CH(2,2), and report a class of integrating factors and its corresponding conservation laws for these two systems. In particular, we obtain the complete sent of first-order integrating factors for the systems in Cauchy-Kovalevskaya form and evaluate the corresponding sets of conservation laws for CH(2,1) and CH(2,2).

1 Introduction

It is well known that certain conservation laws of shallow water wave equations, such as the Camassa-Holm equation [4] and the the Degasperis-Procesi equation [8], are useful to prove blow-up, cf. the papers [5], [16] and [15]. Furthermore, conservation laws play a central role in the prove of the global existence (in time) for solutions evolving from certain initial data, cf. the paper [6], and for proving the stability of peakons for both model equations, cf. the papers [7], [12] and [13]. In the context of the Camassa-Holm equation they are instrumental in the set-up of a theory of global weak solutions for nonlinear nonlocal conservation laws, cf. the considerations in the papers [2], [3] and [10]

In the current paper we derive all first-order integrating factors and its corresponding conservation laws for some recently proposed multi-component generalizations of the Camassa-Holm equation [11]. We concentrate on two explicit systems, namely CH(2,1) and CH(2,2), proposed by Holm and Ivanov in [11] (see 1.1a) – (1.1b) and (1.6a) – (1.6b) below).

We recently reported in [9] the complete set of first-order integrating factors and conservation laws for a classs of Camassa-Holm type equations, which includes the Camassa-Holm equation [4] and the the Degasperis-Procesi equation [8]. Our approach is based on the direct method described by Anco and Bluman in their paper [1], which can be applied to derive conservation laws of evolution equations that are in Cauchy-Kovalevskaya form. We now apply this method for the derivation of integrating factors for CH(2,1) and CH(2,2).

Consider the two-component Camassa-Holm equations introduced and denoted by Holm and Ivanov [11] as CH(2,1), which has the following form:

$$\sigma_1 q_t + 2qu_x + uq_x + \sigma\rho\rho_x = 0 \tag{1.1a}$$

$$\rho_t + \rho u_x + u\rho_x = 0, \tag{1.1b}$$

where

$$q = \sigma_1 u - u_{xx} + s \tag{1.2}$$

and s, σ and σ_1 are arbitrary constants. The physically interesting cases are $\sigma = \pm 1$ and $\sigma_1 = 1$ or $\sigma_1 = 0$. By defining the new dependent variables

$$u := U_1,\ u_x := U_2,\ u_{xx} := U_3,\ \rho := U_4 \tag{1.3}$$

and the change of independenbt variables,

$$X := t,\ T := x, \tag{1.4}$$

we can write system (1.1a) – (1.1b) in the following Cauchy-Kovalevskaya form:

$$E_1 := U_{1,T} - U_2 = 0 \tag{1.5a}$$

$$E_2 := U_{2,T} - U_3 = 0 \tag{1.5b}$$

$$\begin{aligned} E_3 := {} & U_{3,T} - \sigma U_1^{-1} U_{1,X} + U_1^{-1} U_{3,X} - 3\sigma_1 U_2 + 2U_1^{-1} U_2 U_3 + \sigma U_1^{-2} U_4 U_{4,X} \\ & + \sigma U_1^{-2} U_2 U_4^2 - s U_1^{-1} U_2 = 0 \end{aligned} \tag{1.5c}$$

$$E_4 := U_{4,T} + U_1^{-1} U_{4,X} + U_1^{-1} U_2 U_4 = 0. \tag{1.5d}$$

The second 2-component Camassa-Holm equation that we study in the current paper, denoted by CH(2,2), has the form [11]

$$q_{1,t} + u_0 q_{1,x} + 2q_1 u_{0,x} + u_1 q_{2,x} + 2q_2 u_{1,x} = 0 \tag{1.6a}$$

$$q_{2,t} + u_0 q_{2,x} + 2q_2 u_{0,x} = 0, \tag{1.6b}$$

where

$$q_1 = u_1 - u_{1,xx} + s_1 \tag{1.7a}$$

$$q_2 = u_0 - u_{0,xx} + 3u_1^2 - u_{1x}^2 - 2u_1 u_{1,xx} + 4s_1 u_1 + s_2. \tag{1.7b}$$

Here $s_1,\ s_2$ are arbitrary constants. By defining the new dependent variables

$$u_0 := U_1,\ u_{0,x} := U_2,\ u_{0,xx} := U_3,\ u_1 := U_4,\ u_{1,x} := U_5,\ u_{1,xx} := U_6 \tag{1.8}$$

and the change of independenbt variables (1.4), we can present (1.6a) – (1.6b) in the following Cauchy-Kovalevskaya form:

$$E_1 := U_{1,T} - U_2 = 0 \tag{1.9a}$$

$$E_2 := U_{2,T} - U_3 = 0 \tag{1.9b}$$

$$\begin{aligned}
E_3 :=\ & U_{3,T} + 12U_1^{-1}U_4^3U_5 - 4U_1^{-1}U_4U_{4,X} + 2U_1^{-1}U_5U_{5,X} - 4s_1U_1^{-1}U_{4,X} \\
& +4U_5U_6 - 4s_1U_5 + 2U_1^{-1}U_2U_3 - 6U_1^{-1}U_2U_4^2 + 2U_1^{-1}U_2U_5^2 - 2s_2U_1^{-1}U_2 \\
& -4s_1U_1^{-1}U_2U_4 - 12U_1^{-2}U_2U_4^4 + 2U_1^{-1}U_6U_{4,X} - 8U_1^{-1}U_4^2U_5U_6 \\
& +16s_1U_1^{-1}U_4^2U_5 + 4U_1^{-2}U_4^2U_6U_{4,X} - 8s_1U_1^{-2}U_4^2U_{4,X} + 4U_1^{-2}U_4^2U_2U_3 \\
& +4U_1^{-2}U_4^2U_2U_5^2 + 8U_1^{-2}U_2U_4^3U_6 - 16s_1U_1^{-2}U_2U_4^3 - 4s_2U_1^{-2}U_2U_4^2 \\
& +4U_1^{-2}U_4^2U_5U_{5,X} - 4U_1^{-1}U_3U_4U_5 + 4s_2U_1^{-1}U_4U_5 - 12U_1^{-2}U_4U_{4,X} \\
& +2U_1^{-2}U_4^2U_{3,X} + 4U_1^{-2}U_4^3U_{6,X} - 4U_1^{-1}U_4U_5^3 - 2U_1^{-2}U_4^2U_{1,X} \\
& -U_1^{-1}U_{1,X} + U_1^{-1}U_{3,X} - 3U_2 = 0
\end{aligned} \tag{1.9c}$$

$$E_4 := U_{4,T} - U_5 = 0 \tag{1.9d}$$

$$E_5 := U_{5,T} - U_6 = 0 \tag{1.9e}$$

$$\begin{aligned}
E_6 :=\ & U_{6,T} + 4U_1^{-1}U_4U_5U_6 - 8s_1U_1^{-1}U_4U_5 + 2U_1^{-1}U_5^3 - 3U_5 - U_1^{-1}U_{4,X} \\
& +U_1^{-1}U_{6,X} - 2U_1^{-2}U_4^2U_{6,X} + 6U_1^{-2}U_2U_4^3 - U_1^{-1}U_4U_{3,X} + U_1^{-2}U_4U_{1,X} \\
& +6U_1^{-2}U_4^2U_{4,X} - 2U_1^{-2}U_4U_6U_{4,X} + 4s_1U_1^{-2}U_4U_{4,X} - 2U_1^{-2}U_2U_3U_4 \\
& -2U_1^{-2}U_2U_4U_5^2 - 4U_1^{-2}U_2U_4^2U_6 + 8s_1U_1^{-2}U_2U_4^2 + 2s_2U_1^{-2}U_2U_4 \\
& -2U_1^{-2}U_4U_5U_{5,X} + 2U_1^{-1}U_3U_5 - 2s_2U_1^{-1}U_5 + 2U_1^{-1}U_2U_6 \\
& -2s_1U_1^{-1}U_2 - 6U_1^{-1}U_4^2U_5 = 0.
\end{aligned} \tag{1.9f}$$

The above first-order Cauchy-Kovalevskaya systems can now be investigated for integrating factors to derive conservation laws for the systems; which then leads to conservation laws of the systems CH(1,1) and CH(2,2) in the original variables.

2 General description

In this section we breifly describe the direct method [1] of integrating factors (or multipliers) for the general first-order Cauchy-Kovalevskaya system of six equations:

$$E_j := U_{j,T} - F_j(U_1, \ldots, U_6, U_{1,X}, \ldots, U_{6,X}) = 0, \quad j = 1, 2, \ldots, 6. \tag{2.1}$$

Every conserved density, Φ^T, and conserved flux, Φ^X, of system (2.1) must satisfy

$$D_T \Phi^T + D_X \Phi^X \Big|_{\vec{E}=\vec{0}} = 0, \tag{2.2}$$

where, in general, both Φ^T and Φ^X are functions of X, T, U_j as well as X-derivatives of U_j. Moreover, every Φ^T requires six integrating factors, $\{\Lambda_1,\ \Lambda_2, \ldots, \Lambda_6\}$, which are directly related to the conserved density by the relation [1]

$$\Lambda_k = \hat{E}[U_k]\Phi^T, \qquad k = 1, 2, \ldots, 6. \tag{2.3}$$

Here $\hat{E}$ is the Euler Operator,

$$\hat{E}[U_k] := \frac{\partial}{\partial U_k} - D_T \circ \frac{\partial}{\partial U_{k,T}} + \sum_{j=1}^{q} (-1)^j D_X^j \circ \frac{\partial}{\partial U_{k,jX}}, \tag{2.4}$$

where we use the notation

$$U_{k,jX} := \frac{\partial^j U_k}{\partial X^j}.$$

The conditions on the integrating factors, $\{\Lambda_j\}$, of system (2.1) are

$$\hat{E}[U_k]\,(\Lambda_1\, E_1 + \Lambda_2\, E_2 + \cdots + \Lambda_6\, E_6) = 0, \qquad k = 1, 2, \ldots, 6. \tag{2.5}$$

However, since all integrating factors of system (2.1) are adjoint symmetries of the system (2.1), we can calculate $\{\Lambda_j\}$ by the condition

$$\begin{pmatrix} L^*_{E_1}[U_1] & L^*_{E_2}[U_1] & \cdots & L^*_{E_6}[U_1] \\ L^*_{E_1}[U_2] & L^*_{E_2}[U_2] & \cdots & L^*_{E_6}[U_2] \\ \vdots & \vdots & \vdots & \vdots \\ L^*_{E_1}[U_6] & L^*_{E_2}[U_6] & \cdots & L^*_{E_6}[U_6] \end{pmatrix} \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_6 \end{pmatrix} \Bigg|_{\vec{E}=\vec{0}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.6}$$

and then require the self-adjointness condition on $\{\Lambda_j\}$ (as integrating factors are variational quatities), namely

$$\begin{pmatrix} L_{\Lambda_1}[U_1] & L_{\Lambda_1}[U_2] & \cdots & L_{\Lambda_1}[U_6] \\ L_{\Lambda_2}[U_1] & L_{\Lambda_2}[U_2] & \cdots & L_{\Lambda_2}[U_6] \\ \vdots & \vdots & \vdots & \vdots \\ L_{\Lambda_6}[U_1] & L_{\Lambda_6}[U_2] & \cdots & L_{\Lambda_6}[U_6] \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ \vdots \\ E_6 \end{pmatrix}$$

$$= \begin{pmatrix} L^*_{\Lambda_1}[U_1] & L^*_{\Lambda_2}[U_1] & \cdots & L^*_{\Lambda_6}[U_1] \\ L^*_{\Lambda_1}[U_2] & L^*_{\Lambda_2}[U_2] & \cdots & L^*_{\Lambda_6}[U_2] \\ \vdots & \vdots & \vdots & \vdots \\ L^*_{\Lambda_1}[U_6] & L^*_{\Lambda_2}[U_6] & \cdots & L^*_{\Lambda_6}[U_6] \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ \vdots \\ E_6 \end{pmatrix}. \tag{2.7}$$

(2.8)

Here L is the linear operator and L^* its adjoint:

$$L_P[U_j] := \frac{\partial P}{\partial U_j} + \sum_{i=1}^{p} \frac{\partial P}{\partial U_{j,iT}}\ D_T^i + \sum_{k=1}^{q} \frac{\partial P}{\partial U_{j,kX}}\ D_X^k \tag{2.9a}$$

$$L^*_P[U_j] := \frac{\partial P}{\partial U_j} + \sum_{i=1}^{p} (-1)^i\ D_T^i \circ \frac{\partial P}{\partial U_{j,iT}} + \sum_{k=1}^{q} (-1)^k\ D_X^k \circ \frac{\partial P}{\partial U_{j,kX}}. \tag{2.9b}$$

Note that the self-adjointness condition, (2.7), is independent of the form of the evolution system (2.1) and only depends on the functional arguments of $\{\Lambda_j\}$ as well as the number of equations in the system.

3 Integrating factors for system (1.5a) – (1.5d) and conservation laws for (1.1a) – (1.1b):

Solving conditions (2.6) and (2.7) for system (1.5a) – (1.5d), the complete set of first-order integrating factors $\{\Lambda_1, \dots, \Lambda_4\}$, of the form

$$\Lambda_j = \Lambda_j(X, T, U_1, \dots, U_4, U_{1,X}, \dots, U_{4,X}), \quad j = 1, 2, \dots, 4$$

give two cases, depending on the relations between σ and σ_1:

Case 1: $\sigma = \sigma_1$. The first-order integrating factors for system (1.5a) – (1.5d) are then given by

$$\Lambda_1 = \lambda_1 \left(U_4^{-1} U_3 - 2\sigma U_4^{-1} U_1 - \frac{1}{2} U_4^{-1} s \right) + \lambda_2 \left(U_{2,X} - sU_1 + 2U_1 U_3 - 3\sigma U_1^2 - \sigma U_4^2 \right)$$

$$+\lambda_3 \left(U_3 - 3\sigma U_1 - s \right) + \lambda_4 U_4 \tag{3.1a}$$

$$\Lambda_2 = -\lambda_2 U_{1,X} + \lambda_3 U_2 \tag{3.1b}$$

$$\Lambda_3 = \lambda_1 U_4^{-1} U_1 + \lambda_2 U_1^2 + \lambda_3 U_1 \tag{3.1c}$$

$$\Lambda_4 = \lambda_1 \left(\sigma U_4^{-2} U_1^2 - U_4^{-2} U_1 U_3 + \frac{1}{2} s U_4^{-2} U_1 - \sigma \right) - 2\lambda_2 \sigma U_1 U_4$$

$$-\lambda_3 \sigma U_4 + \lambda_4 U_1, \tag{3.1d}$$

where λ_j are arbitrary constants. This leads to the following three sets of conserved density, Φ^t, and conserved flux, Φ^x, for the original system (1.1a) – (1.1b) for this case (separated by means of the arbitrary λ's):

$$\Phi_1^t = \rho^{-1} u_{xx} - \sigma \rho^{-1} u - \frac{1}{2} \rho^{-1} s \tag{3.2a}$$

$$\Phi_1^x = \rho^{-1} u u_{xx} - \sigma \rho^{-1} u^2 - \sigma \rho - \frac{1}{2} s \rho^{-1} u \tag{3.2b}$$

$$\Phi_2^t = u u_{xx} + \frac{1}{2} u_x^2 - \frac{1}{2} \sigma u^2 - \frac{1}{2} \sigma \rho^2 \tag{3.3a}$$

$$\Phi_2^x = u^2 u_{xx} - u_x u_t - \frac{1}{2} s u^2 - \sigma \rho^2 u - \sigma u^3 \tag{3.3b}$$

$$\Phi_3^t = u_{xx} - \sigma u \tag{3.4a}$$

$$\Phi_3^x = u u_{xx} + \frac{1}{2} u_x^2 - su - \frac{1}{2} \sigma \rho^2 - \frac{3}{2} \sigma u^2. \tag{3.4b}$$

Case 2: $\sigma \neq \sigma_1$. The first-order integrating factors for system (1.5a) – (1.5d) are then given by

$$\Lambda_1 = \lambda_2 \left(U_{2,X} - sU_1 + 2U_1 U_3 - 3\sigma_1 U_1^2 - \sigma U_4^2 \right)$$

$$+\lambda_3 \left(U_3 - 3\sigma_1 U_1 - s \right) + \lambda_4 U_4 \tag{3.5a}$$

$$\Lambda_2 = -\lambda_2 U_{1,X} + \lambda_3 U_2 \tag{3.5b}$$

$$\Lambda_3 = \lambda_2 U_1^2 + \lambda_3 U_1 \tag{3.5c}$$

$$\Lambda_4 = -2\lambda_2 \sigma U_1 U_4 - \lambda_3 \sigma U_4 + \lambda_4 U_1, \tag{3.5d}$$

where Λ_j are arbitrary constants. This leads to the following two sets of conserved density, Φ^t, and conserved flux, Φ^x, for the original system (1.1a) – (1.1b) for this case:

$$\Phi_1^t = uu_{xx} + \frac{1}{2}u_x^2 - \frac{1}{2}\sigma u^2 - \frac{1}{2}\sigma\rho^2 \tag{3.6a}$$

$$\Phi_1^x = u^2 u_{xx} - u_x u_t - \frac{1}{2}su^2 - \sigma\rho^2 u - \sigma_1 u^3 \tag{3.6b}$$

$$\Phi_2^t = u_{xx} - \sigma u \tag{3.7a}$$

$$\Phi_2^x = uu_{xx} + \frac{1}{2}u_x^2 - su - \frac{1}{2}\sigma\rho^2 - \frac{3}{2}\sigma_1 u^2 \tag{3.7b}$$

Remark: The obvious conservation law for system (1.1a) – (1.1b), namely $\Phi^t = \rho$, $\Phi^x = \rho u$, has not been included in the above list.

4 Integrating factors for system (1.9a) – (1.9f) and conservation laws for (1.6a) – (1.6b):

Solving conditions (2.6) and (2.7) for system (1.9a) – (1.9f), the complete set of first-order integrating factors $\{\Lambda_1, \ldots, \Lambda_6\}$, of the form

$$\Lambda_j = \Lambda_j(X, T, U_1, \ldots, U_6, U_{1,X}, \ldots, U_{6,X}), \quad j = 1, 2, \ldots, 6$$

are the following:

$$\begin{aligned}\Lambda_1 = {} & \lambda_1\left(2U_6U_1 + 2U_3U_4 + 2U_4U_5^2 + 4U_4^2U_6 - 6U_4^3 - 2s_1U_1 - 8s_1U_4^2 - 6U_1U_4\right.\\ & \left.-2s_2U_4 + U_{5,X}\right) + \lambda_2\left(U_3 + 2U_5^2 - 4s_1U_4 - 3U_1 - 2s_2\right)\\ & +\lambda_3\left(U_6 - 3U_4 - 2s_1\right)\end{aligned} \tag{4.1a}$$

$$\Lambda_2 = -\lambda_1 U_{4,X} + \lambda_2 U_2 + \lambda_3 U_5 \tag{4.1b}$$

$$\Lambda_3 = 2\lambda_1 U_1 U_4 + \lambda_2\left(U_1 - 2U_4^2\right) + \lambda_3 U_4 \tag{4.1c}$$

$$\begin{aligned}\Lambda_4 = {} & \lambda_1\left(2U_1U_5^2 + 2U_1U_3 + 2U_4U_{5,X} - 2s_2U_1 - 3U_1^2 - 18U_1U_4^2 + U_{2,X}\right.\\ & \left.-16s_1U_1U_4 + 8U_1U_4U_6\right) + \lambda_2\left(24U_4^3 - 4U_3U_4 - 4U_4U_5^2 - 12U_4^2U_6 - 2U_{5,X}\right.\\ & \left.+24s_1U_4^2 - 4s_1U_1 + 4s_2U_4\right) + \lambda_3\left(U_3 + 4U_4U_6 - 3U_1 + 2U_5^2 - 12U_4^2\right.\\ & \left.-12s_1U_4 - 2s_2\right)\end{aligned} \tag{4.1d}$$

$$\begin{aligned}\Lambda_5 = {} & \lambda_1\left(4U_1U_4U_5 - U_{1,X} - 2U_4U_{4,X}\right) + \lambda_2\left(4U_1U_5 - 4U_4^2U_5 + 2U_{4,X}\right)\\ & +\lambda_3\left(U_2 + 4U_4U_5\right)\end{aligned} \tag{4.1e}$$

$$\Lambda_6 = \lambda_1\left(U_1^2 + 4U_1U_4^2\right) - 4\lambda_2U_4^3 + \lambda_3\left(U_1 + 2U_4^2\right). \tag{4.1f}$$

This leads to the following set of three conserved densities and conserved flux for the system (1.6a) – (1.6b):

$$\Phi_1^t = u_1 u_{0,xx} + u_1^2 u_{1,xx} - u_0 u_1 - 2s_1 u_1^2 - 2u_1^3 \tag{4.2a}$$

$$\Phi_1^x = \left(u_0 + u_1^2\right) u_{1,xt} + 2u_0 u_1 u_{0,xx} + 2u_0 u_1 u_{1,x}^2 + \left(4u_0 u_1^2 + u_0^2\right) u_{1,xx}$$
$$-\frac{1}{2} u_0^2 \left(6u_1 + 2s_1\right) - u_0 \left(6u_1^3 + 2s_2 u_1 + 8s_1 u_1^2\right) - u_{0,x} u_{1,t} \tag{4.2b}$$

$$\Phi_2^t = 2u_1 u_{1,xx} + u_{0,xx} - u_0 - 2u_1^2 + 2u_{1,x}^2 - 4s_1 u_1 \tag{4.3a}$$

$$\Phi_2^x = -2u_1 u_{1,xt} + (u_0 - 2u_1^2) u_{0,xx} - 4u_1^3 u_{1,xx} + \frac{1}{2} u_{0,x}^2 + 2(u_0 - u_1^2) u_{1,x}^2$$
$$-2(s_2 + 2s_1 u_1) u_0 - \frac{3}{2} u_0^2 + 2u_1^2 (s_2 + 4s_1 u_1 + 3u_1^2) \tag{4.3b}$$

$$\Phi_3^t = u_{1,xx} - u_1 \tag{4.4a}$$

$$\Phi_3^x = (u_0 + 2u_1^2) u_{1,xx} + u_1 u_{0,xx} + u_{0,x} u_{1,x} + 2u_1 u_{1,x}^2 - (2s_1 + 3u_1) u_0$$
$$-2u_1 (s_2 + 3s_1 u_1 + 2u_1^2). \tag{4.4b}$$

5 Concluding remarks

We have derived the complete set of first-order integrating factors for the systems CH(2,1) and CH(2,2) in Cauchy-Kovalevskaya form. The corresponding sets of conservation laws related to these integrating factors have been derived for both these systems. It would certainly be interesting to calculate higher-order integrating factors, although the computations involved for such calculations appear to be rather challenging. We aim to report some results in a future paper.

We expect that the same method than was applied here could also be used to find conservation laws for more general CH-systems proposed in [11] and [14]. However, for larger systems of equations, the computations involved in deriving the complete sets of integrating factors (even of first-order) can pose significant difficulties and computer algebra systems should be implemented to overcome these computational problems.

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