

Dunajski-Tod equation and reductions of the generalized dispersionless 2DTL hierarchy

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Abstract

We transfer the scheme for constructing differential reductions recently developed for the Manakov-Santini hierarchy to the case of the two-component generalization of dispersionless 2DTL hierarchy. We demonstrate that the equation arising as a result of the simplest reduction is equivalent (up to a Legendre type transformation) to the Dunajski-Tod equation, locally describing general ASD vacuum metric with conformal symmetry. We consider higher reductions and corresponding reduced hierarchies also.

1 Introduction

In 1999 Dunajski and Tod [1] introduced the equation locally describing general ASD vacuum metric with conformal symmetry, which reads

$$(\eta F_{\bar{w}} + F_{u\bar{w}})(\eta F_w - F_{uw}) - (\eta^2 F - F_{uu})F_{w\bar{w}} = 4e^{2\rho u}. \quad (1)$$

They also demonstrated that this equation is integrable, representing it as the integrability condition for a linear system of equations. More recently [2] Dunajski considered a case of ASD Ricci-flat metric with a conformal Killing vector whose self-dual derivative is null, and discovered that the interpolating system describing this case is a simple differential reduction of the Manakov-Santini system [3, 4]. This crucial observation initiated a study of reductions of the Manakov-Santini hierarchy [5].

The Manakov-Santini system [3, 4] is a two-component generalization of the dKP (Khohlov-Zabolotskaya) equation to the case of general vector fields

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in the Lax pair, instead of Hamiltonian vector fields for the dKP case. If we don't impose Hamiltonian (area-preserving) reduction from the beginning, there is a freedom to consider more complicated reductions representing kind of twisted area-preservation conditions [5]. And it appeared that the reduction of the Manakov-Santini system corresponding to Dunajski interpolating system is the lowest order reduction of this class [5].

The Manakov-Santini hierarchy belongs to the class of multidimensional hierarchies studied in [6, 7]. The construction of the work [5] can also be transferred to the multidimensional case [8].

Similar to the Manakov-Santini system, it is possible to introduce a two-component generalization of the dispersionless 2DTL equation [9] and consider its differential reductions. The general construction of interpolating reductions for the two-component generalization of the dispersionless 2DTL equation is developed in the present work. The equation corresponding to the simplest reduction was presented in [9] in the form

$$m_{tt} = (m_t)^{\frac{1}{\alpha}}(m_{ty}m_x - m_{xy}m_t), \quad (2)$$

in the limit $\alpha \rightarrow 0$ it can be reduced to the d2DTL equation. Taking into account that the equation (1) for $\eta \rightarrow 0$ reduces to the dispersionless 2DTL equation too [2], it is a natural hypothesis that equations (1) and (2) should be connected, representing different forms of interpolating equation for the d2DTL case. And it is indeed so! In the present paper we demonstrate that these equations are equivalent up to a Legendre type transformation.

First we introduce the two-component generalization of the dispersionless 2DTL equation and give an elementary description of interpolating reduction in terms of the Lax pair. We derive equation (2) and demonstrate its equivalence to Dunajski-Tod equation. We also discuss a connection of interpolating equation (2) for some special rational values of parameter α with the generalized dispersionless Harry Dym equation constructed by Blaszk [10, 11]. Finally, we develop a general construction of differential (interpolating) reductions of the generalized dispersionless 2DTL hierarchy and present the hierarchy connected with equation (2).

2 Interpolating reduction: elementary description

First we will give a description of the simplest interpolating reduction of the two-component generalization of the dispersionless 2DTL equation in terms of the Lax pair. A general construction of reductions in terms of the hierarchy is presented below.

2.1 Two-component generalization of the dispersionless 2DTL equation

The simplest two-component generalization of the dispersionless 2DTL equation was introduced in [9] as a first equation of the hierarchy (see Section 3) generalizing dispersionless 2DTL hierarchy to the case of non-Hamiltonian vector fields, it reads

$$\begin{aligned}(e^{-\phi})_{tt} &= m_t \phi_{xy} - m_x \phi_{ty}, \\ m_{tt} e^{-\phi} &= m_{ty} m_x - m_{xy} m_t,\end{aligned}\tag{3}$$

and the corresponding Lax pair is

$$\begin{aligned}\partial_x \Psi &= \left(\left(\lambda + \frac{m_x}{m_t} \right) \partial_t - \left(\phi_t \frac{m_x}{m_t} - \phi_x \right) \lambda \partial_\lambda \right) \Psi, \\ \partial_y \Psi &= \left(-\frac{1}{\lambda} \frac{e^{-\phi}}{m_t} \partial_t - \frac{1}{\lambda} \frac{(e^{-\phi})_t}{m_t} \lambda \partial_\lambda \right) \Psi.\end{aligned}\tag{4}$$

For $m = t$ the system (3) reduces to the dispersionless 2DTL equation

$$(e^{-\phi})_{tt} = \phi_{xy},\tag{5}$$

Respectively, the reduction $\phi = 0$ gives an equation [12] (see also [13], [14])

$$m_{tt} = m_{ty} m_x - m_{xy} m_t.\tag{6}$$

2.2 Interpolating reduction

A standard way to define a reduction starting from the Lax pair is to suggest that Lax equations possess a solution f with some special analytic properties in λ invariant under dynamics (i.e., belonging to some invariant manifold of linear equations). Suggesting that that Lax equations (4) possess a solution polinomial in λ or λ^{-1} , we arrive to Gelfand-Dikii reductions of the two-component d2DTL (3), which for d2DTL equation (5) reduce to standard Gelfand-Dikii reductions. A simplest case of polynomial solution $f = \lambda$ leads to equation (6), and d2DTL equation just degenerates in this case.

Interpolating reductions of the Manakov-Santini hierarchy introduced in [5] generalize Gelfand-Dikii reductions, and, containing a parameter, 'interpolate' between the Manakov-Santini hierarchy Gelfand-Dikii reduction of the order n and the dKP hierarchy. To understand the origin of this type of reductions in terms of the Lax pair for the system (3), it is important to note that there exists a whole one-parametric family of Lax pairs (or

overdetermined systems) corresponding to the system (3), of which the Lax pair (4) is only a special representative. It is rather general fact for the Lax pairs in terms of non-Hamiltonian or not divergence-free vector fields.

Indeed, starting from the Lax pair (4), written symbolically in the form,

$$\begin{aligned}\partial_x \Psi &= \hat{U} \Psi, \\ \partial_y \Psi &= \hat{V} \Psi,\end{aligned}\tag{7}$$

where $\hat{U} = u_1 \partial_t + u_2 \lambda \partial_\lambda$, $\hat{V} = v_1 \partial_t + v_2 \lambda \partial_\lambda$ are vector fields defined by (4), we introduce a one-parametric family of Lax pairs

$$\begin{aligned}\partial_x \Phi &= \hat{U} \Phi + \beta \operatorname{div} \hat{U} \Phi, \\ \partial_y \Phi &= \hat{V} \Phi + \beta \operatorname{div} \hat{V} \Phi,\end{aligned}\tag{8}$$

where β is a parameter, $\operatorname{div} \hat{U} = \partial_t u_1 + \lambda \partial_\lambda u_2$, $\operatorname{div} \hat{V} = \partial_t v_1 + \lambda \partial_\lambda v_2$. These Lax pairs are no more in the form of pure vector fields, they also have a non-differential term. However, it is easy to check that compatibility conditions for them remain the same and provide system (3). It is convenient to rewrite the system (8) in terms of $\ln \Phi$, then it takes the form of nonhomogeneous linear system

$$\begin{aligned}\partial_x \ln \Phi &= \hat{U} \ln \Phi + \beta \operatorname{div} \hat{U}, \\ \partial_y \ln \Phi &= \hat{V} \ln \Phi + \beta \operatorname{div} \hat{V},\end{aligned}\tag{9}$$

Knowing the general solution of the linear system (7), it is not difficult to construct the general solution for the systems (8), (9). Indeed, let Ψ_1, Ψ_2 be two solutions of system (7) with nonzero Jacobian (Poisson bracket) $J = \{\Psi_1, \Psi_2\} \neq 0$, $\{f, g\} = \lambda(f_\lambda g_t - f_t g_\lambda)$. Then the general solution of the system (7) is of the form $F(\Psi_1, \Psi_2)$. The Poisson bracket of two solutions satisfies the system

$$\begin{aligned}\partial_x \ln J &= \hat{U} \ln J + \operatorname{div} \hat{U}, \\ \partial_y \ln J &= \hat{V} \ln J + \operatorname{div} \hat{V},\end{aligned}\tag{10}$$

having a general solution $\{\Psi_1, \Psi_2\} + f(\Psi_1, \Psi_2)$, where the first term is a special solution of the nonhomogeneous equations, and the second term is a general solution of the homogeneous equations. Comparing this system to the system (9), we conclude that the general solution of the system (9) is $\beta\{\Psi_1, \Psi_2\} + f(\Psi_1, \Psi_2)$, and the general solution of the system (8) is

$$\Phi = \exp(\beta\{\Psi_1, \Psi_2\})F(\Psi_1, \Psi_2).$$

Suggesting the existence of solution f with some special analytic properties in λ for the Lax pairs (8) or (9), we will obtain one-parametric interpolating reduction, which for $\beta = 0$ implies the existence of solution f for standard Lax equations (4) (Gelfand-Dikii type reduction), and in the limit $\beta \rightarrow \infty$ corresponds to Hamiltonian (divergence-free) vector fields.

We define a simplest interpolating reduction for equation (3) by the condition that Lax equations (8) possess a solution $f = \lambda$ (equivalently, equations (9) possess a solution $\ln \lambda$ and equations (10) – solution $-\alpha \ln \lambda$, $\alpha = -\beta^{-1}$.) An explicit form of equations (9) corresponding to (4) is

$$\begin{aligned}\partial_x \Phi &= \left(\left(\lambda + \frac{m_x}{m_t} \right) \partial_t - \left(\phi_t \frac{m_x}{m_t} - \phi_x \right) \lambda \partial_\lambda \right) \Phi + \beta \partial_t \frac{m_x}{m_t}, \\ \partial_y \Phi &= \left(-\frac{1}{\lambda} \frac{e^{-\phi}}{m_t} \partial_t - \frac{1}{\lambda} \frac{(e^{-\phi})_t}{m_t} \lambda \partial_\lambda \right) \Phi - \beta \frac{e^{-\phi}}{\lambda} \partial_t \frac{1}{m_t},\end{aligned}\quad (11)$$

and it is easy to check that substitution of solution $\ln \lambda$ to both equations gives the same reduction condition

$$e^{\alpha\phi} = m_t, \quad \alpha = -\beta^{-1}, \quad (12)$$

which is an interpolating reduction of the system (3). This reduction makes it possible to rewrite the system (3) as one equation for m (2) or in the form of deformed d2DTL equation,

$$\begin{aligned}(e^{-\phi})_{tt} &= m_t \phi_{xy} - m_x \phi_{ty}, \\ m_t &= e^{\alpha\phi}.\end{aligned}$$

The limit $\alpha \rightarrow 0$ corresponds to the dispersionless 2DTL equation (5), and the limit $\alpha \rightarrow \infty$ gives equation (6).

2.3 Legendre transform and Dunajski-Tod interpolating equation

Equation (2) can be represented in exterior differential form

$$\beta^{-1} dm_t^\beta \wedge dx \wedge dy = dm_y \wedge dm \wedge dy, \quad (13)$$

where $\beta = 1 - \alpha^{-1}$.

Let us consider a Legendre type transform (where τ is a new independent variable and M is a new dependent variable)

$$m_t = e^\tau, \quad M = m - te^\tau.$$

Differential of M is of the form

$$dM = M_x dx + M_y dy - te^\tau d\tau.$$

Transformed equation (13) reads

$$\beta^{-1} de^{\beta\tau} \wedge dx \wedge dy = dM_y \wedge dM \wedge dy - dM_y \wedge dM_\tau \wedge dy,$$

and transformed equation (2) looks like

$$e^{\beta\tau} = (M_{y\tau}M_x - M_{yx}M_\tau) - (M_{y\tau}M_{x\tau} - M_{yx}M_{\tau\tau}). \quad (14)$$

Scaling the time $\tau \rightarrow 2\tau$, we get

$$4e^{2\beta\tau} = 2(M_{y\tau}M_x - M_{yx}M_\tau) - (M_{y\tau}M_{x\tau} - M_{yx}M_{\tau\tau})$$

In terms of the function $F = e^{-\tau}M$

$$(F_y + F_{y\tau})(F_x - F_{x\tau}) - (F - F_{\tau\tau})F_{xy} = 4e^{-2\alpha^{-1}\tau}.$$

Considering the scaling $x \rightarrow \eta^{-1}x$, $y \rightarrow \eta^{-1}y$, $\tau \rightarrow \eta\tau$, we obtain Dunajski-Tod equation

$$(\eta F_y + F_{y\tau})(\eta F_x - F_{x\tau}) - (\eta^2 F - F_{\tau\tau})F_{xy} = 4e^{2\rho\tau}, \quad (15)$$

where $\rho = -\alpha^{-1}\eta$.

2.4 Generalized dispersionless Harry Dym equation and d2DTL interpolating equation

Generalized dispersionless Harry Dym equation constructed by Blaszk [10, 11] can be written in the form of conservation law,

$$\partial_t u^{2-r} = \frac{(3-r)}{(r-1)(r-2)} (u^{2-r} \partial_x^{-1} \partial_y u^{r-1})_y, \quad (16)$$

where parameter r is integer, $r \in \mathbb{Z}$. This equation suggests the existence of potential v , such that

$$\begin{aligned} \partial_y v &= u^{2-r}, \\ \partial_t v &= \frac{(3-r)}{(r-1)(r-2)} u^{2-r} \partial_x^{-1} \partial_y u^{r-1}, \end{aligned}$$

and for the potential we get an equation

$$\frac{(3-r)}{(r-1)(r-2)}v_{yy} = v_y^{\frac{r-3}{2-r}}(v_{xt}v_y - v_tv_{xy}),$$

which after the change of variables $y \rightarrow t$, $x \rightarrow y$, $t \rightarrow x$ is equivalent to equation (2) with $\alpha = \frac{2-r}{r-3}$ (the constant can be eliminated by the rescaling of variables). Thus the generalized dispersionless Harry Dym equation is connected with equation (2) with some rational values of parameter α . The special values $\alpha = \infty$ corresponding to equation (6) and $\alpha = 0$ corresponding to d2DTL are related respectively with special values $r = 3$ and $r = 2$ for the generalized dispersionless Harry Dym equation (16).

Several examples of integrable equations with a structure similar to equations (16), (2) were provided in the work [15].

3 Interpolating reductions: general construction

3.1 Generalized dispersionless 2DTL hierarchy

First we briefly describe the generalized dispersionless 2DTL hierarchy, following the work [9] (on d2DTL hierarchy see [16], [17], [18]).

We consider formal series

$$\begin{aligned} \Lambda^{\text{out}} &= \ln \lambda + \sum_{k=1}^{\infty} l_k^+ \lambda^{-k}, & \Lambda^{\text{in}} &= \ln \lambda + \phi + \sum_{k=1}^{\infty} l_k^- \lambda^k, \\ M^{\text{out}} &= M_0^{\text{out}} + \sum_{k=1}^{\infty} m_k^+ e^{-k\Lambda^{\text{out}}}, & M^{\text{in}} &= M_0^{\text{in}} + m_0 + \sum_{k=1}^{\infty} m_k^- e^{k\Lambda^{\text{in}}}, \\ M_0 &= t + \sum_{k=1}^{\infty} x_k e^{k\Lambda} - \sum_{k=1}^{\infty} y_k e^{-k\Lambda}, \end{aligned} \tag{17}$$

where λ is a spectral variable, t, x_k, y_k are considered independent variables, and other coefficients of the series ($\phi, m_0, l_k^\pm, m_k^\pm$) – dependent variables. Usually we suggest that ‘out’ and ‘in’ components of the series define the functions outside and inside the unit circle in the complex plane of the variable λ respectively (in more detail in [9]). Generalized dispersionless 2DTL hierarchy is defined by the generating relation

$$(\{\Lambda, M\}^{-1} d\Lambda \wedge dM)^{\text{out}} = (\{\Lambda, M\}^{-1} d\Lambda \wedge dM)^{\text{in}}, \tag{18}$$

where $\{f, g\} = \lambda(f_\lambda g_t - f_t g_\lambda)$, which may be considered as a continuity condition on the unit circle for the differential two-form (or just in terms of

formal series); $\{\Lambda, M\}^{\text{out}} = 1 + O(\lambda^{-1})$, $\{\Lambda, M\}^{\text{in}} = 1 + \partial_t m_0 + O(\lambda)$, and we suggest that $\{\Lambda, M\} \neq 0$. The differential d is given by

$$df = \partial_\lambda f d\lambda + \partial_t f dt + \sum_{k=1}^{\infty} \partial_k^+ f dx_k + \sum_{k=1}^{\infty} \partial_k^- f dy_k, \quad (19)$$

where $\partial_k^+ f = \frac{\partial f}{\partial x_k}$, $\partial_k^- f = \frac{\partial f}{\partial y_k}$. As a result of condition (18), the coefficients of the differential two-form in the generating relation (18) are *meromorphic*.

Generating equation (18) implies Lax-Sato equations of the hierarchy. In explicit form, a complete set of Lax-Sato equations reads

$$\left(\partial_n^+ - \left(\frac{e^{n\Lambda} \lambda \partial_\lambda \Lambda}{\{\Lambda, M\}} \right)_+^{\text{out}} \partial_t + \left(\frac{e^{n\Lambda} \partial_t \Lambda}{\{\Lambda, M\}} \right)_+^{\text{out}} \lambda \partial_\lambda \right) \begin{pmatrix} \Lambda \\ M \end{pmatrix} = 0, \quad (20)$$

$$\left(\partial_n^- + \left(\frac{e^{-n\Lambda} \lambda \partial_\lambda \Lambda}{\{\Lambda, M\}} \right)_-^{\text{in}} \partial_t - \left(\frac{e^{-n\Lambda} \partial_t \Lambda}{\{\Lambda, M\}} \right)_-^{\text{in}} \lambda \partial_\lambda \right) \begin{pmatrix} \Lambda \\ M \end{pmatrix} = 0, \quad (21)$$

where $(\dots)_-$, $(\dots)_+$ are standard projections respectively to negative and nonnegative powers of λ . The Lax-Sato equations for the times $x = x_1$, $y = y_1$, $\partial_1^+ = \partial_x$, $\partial_1^- = \partial_y$,

$$\partial_x \Psi = ((\lambda + (m_1^+)_t - l_1^+) \partial_t - \lambda (l_1^+)_t \partial_\lambda) \Psi, \quad (22)$$

$$\partial_y \Psi = \left(-\frac{1}{\lambda} \frac{e^{-\phi}}{m_t} \partial_t - \frac{(e^{-\phi})_t}{m_t} \partial_\lambda \right) \Psi, \quad (23)$$

where $\Psi = \begin{pmatrix} \Lambda \\ M \end{pmatrix}$, $m = m_0 + t$, correspond to the Lax pair (4), where the coefficients in the first Lax-Sato equation can be transformed to the form (4) by taking its expansion at $\lambda = 0$, and the system (3) arises as a compatibility condition.

Lax-Sato equations (20,21) define the evolution of the series $\Lambda^{\text{in}}, \Lambda^{\text{out}}, M^{\text{in}}, M^{\text{out}}$. The only term containing an interaction between Λ and M is $\{\Lambda, M\}$. The condition $\{\Lambda, M\} = 1$ splits out equations for Λ and reduces the hierarchy (20,21) to the d2DTL hierarchy, while the condition $\Lambda = \ln \lambda$ – to the hierarchy, considered by Martínez Alonso and Shabat [13, 14], see also Pavlov [12].

3.2 Interpolating reduction for the hierarchy

Following the scheme of the work [5], we start from nonhomogeneous Lax-Sato equations for the Jacobian (Poisson bracket) $J_0 = \{\Lambda, M\}$. We define

reductions by the condition that some specific solution of these equations is continuous on the unit circle ('in' and 'out' components are equal). This condition is preserved by the dynamics because the coefficients of vector fields in the Lax-Sato equations are meromorphic with respect to λ , and thus it defines a reduction of the hierarchy.

Rewriting Lax-Sato equations (20,21) symbolically as (compare (7))

$$(\partial_n^+ - \hat{U}_n) \begin{pmatrix} \Lambda \\ M \end{pmatrix} = 0, \quad (\partial_n^- - \hat{V}_n) \begin{pmatrix} \Lambda \\ M \end{pmatrix} = 0, \quad (24)$$

where corresponding vector fields are defined explicitly by Lax-Sato equations (20,21), we obtain nonhomogeneous linear equations for the Jacobian in the form (see also (10))

$$\begin{aligned} \partial_n^+ \ln J_0 &= \hat{U}_n \ln J_0 + \operatorname{div} \hat{U}_n, \\ \partial_n^- \ln J_0 &= \hat{V}_n \ln J_0 + \operatorname{div} \hat{V}_n. \end{aligned} \quad (25)$$

We define interpolating reduction for the hierarchy by the condition

$$(\ln J_0 - \alpha \Lambda)^{\text{out}} = (\ln J_0 - \alpha \Lambda)^{\text{in}} \quad (26)$$

Both sides of this relation satisfy equations (25), which preserve the continuity, and thus condition (26) indeed defines a reduction.

This relation implies that

$$(\ln J_0 - \alpha \Lambda) = -\alpha \ln \lambda, \quad (27)$$

thus nonhomogeneous linear equations of the hierarchy (25) possess a solution $f = -\alpha \ln \lambda$. This property was used above to define interpolating reduction in terms of Lax pair for the system (2). It is possible to obtain reduction condition (12) directly from relation (27), taking its expansion at $\lambda = 0$, where at order zero we get

$$\alpha \phi = \ln(1 + \partial_t m_0) = \ln m_t. \quad (28)$$

Substituting the expression for the Poisson bracket implied by relation (27),

$$J_0 = \{\Lambda, M\} = \lambda^{-\alpha} \exp(\alpha \Lambda),$$

to the generating relation (18), we obtain the generating relation for the reduced hierarchy

$$(\exp(-\alpha \Lambda) d\Lambda \wedge dM)^{\text{out}} = (\exp(-\alpha \Lambda) d\Lambda \wedge dM)^{\text{in}}.$$

The two-form Ω defined by the generating relation,

$$\Omega = (d\Lambda \wedge d(\exp(-\alpha\Lambda)M))^{\text{out}} = (d\Lambda \wedge d(\exp(-\alpha\Lambda)M))^{\text{in}},$$

is (up to a factor $\lambda^{-\alpha}$) meromorphic in the complex plane having poles only at zero and infinity, and it is evidently closed. The condition of conservation of this form can be used to define a reduction in terms of nonlinear vector Riemann-Hilbert problem and develop a dressing scheme for the reduced hierarchy similar to [5] (see also [9]).

The Lax-Sato equations for the reduced hierarchy read

$$\begin{aligned} \left(\partial_n^+ - \left(\lambda^\alpha e^{(n-\alpha)\Lambda} \lambda \partial_\lambda \Lambda \right)_+^{\text{out}} \partial_t + \left(\lambda^\alpha e^{(n-\alpha)\Lambda} \partial_t \Lambda \right)_+^{\text{out}} \lambda \partial_\lambda \right) \begin{pmatrix} \Lambda \\ M \end{pmatrix} &= 0, \\ \left(\partial_n^- + \left(\lambda^\alpha e^{(-n-\alpha)\Lambda} \lambda \partial_\lambda \Lambda \right)_-^{\text{in}} \partial_t - \left(\lambda^\alpha e^{(-n-\alpha)\Lambda} \partial_t \Lambda \right)_-^{\text{in}} \lambda \partial_\lambda \right) \begin{pmatrix} \Lambda \\ M \end{pmatrix} &= 0. \end{aligned}$$

Similar to d2DTL hierarchy, Lax-Sato equations for Λ split out, having no interaction with M .

3.3 Higher reductions

Here we define a class of reductions interpolating between Gelfand-Dikii reductions of the order n for the generalized d2DTL hierarchy and d2DTL hierarchy proper. Gelfand-Dikii reductions suggest existence of rational solution for Lax-Sato equations (20), (21). In the case of d2DTL hierarchy Gelfand-Dikii reduction of the order n implies stationarity with respect to a higher flow $\partial_n = a\partial_n^+ + b\partial_n^-$, where a, b are some constants. However, for the case of non-Hamiltonian vector fields this is not true.

We define higher interpolating reductions by the condition

$$(\ln J_0 - aL^n - bL^{-n})^{\text{out}} = (\ln J_0 - aL^n - bL^{-n})^{\text{in}},$$

where $L = e^\Lambda$. Both sides of this condition are solutions of nonhomogeneous Lax-Sato equations (25), and due to continuity they are equal to a single rational function $f = -aL_+^n - bL_-^{-n}$. Thus the reduction condition (29) implies that nonhomogeneous Lax-Sato equations (25) possess a rational solution f . This property can be used to define a reduction in terms of the Lax pair for the system (3) and calculate a differential reduction in terms of the functions ϕ, m . In the limit $a, b \rightarrow 0$ the reduction leads to $J_0 = 1$ and thus to d2DTL hierarchy, the limit $a, b \rightarrow \infty$ implies that homogeneous Lax-Sato equations (20), (21) possess a rational solution f , and thus it is a Gelfand-Dikii reduction of the order n .

Reduction condition (29) implies the expression for the Poisson bracket,

$$J_0 = \{\Lambda, M\} = \exp(aL^n + bL^{-n} - aL_+^n - bL_-^n), \quad (29)$$

which is valid for both 'in' and 'out' components. Substituting this expression to the generating relation (18), we obtain the generating relation for the reduced hierarchy

$$\begin{aligned} & (\exp(-aL^n - bL^{-n} + aL_+^n + bL_-^n)d\Lambda \wedge dM)^{\text{out}} \\ &= (\exp(-aL^n - bL^{-n} + aL_+^n + bL_-^n)d\Lambda \wedge dM)^{\text{in}}. \end{aligned}$$

We will not write down explicitly the Lax-Sato equations for the reduced hierarchy, it is rather straightforward substituting (29) to (20), (21). A characteristic feature of these equations is that due to the reduction the equations for Λ split out, similar to d2DTL hierarchy.

Let us consider a case $n = 1$ in more detail. Nonhomogeneous Lax-Sato equations (25) in this case possess a rational solution

$$f = -aL_+ - bL_-^{-1} = -a(\lambda + l_1^+) - b\frac{e^{-\phi}}{\lambda}$$

(we use the series (17), $L = e^\Lambda$). There are two strategies to calculate the differential reduction for the system (3) in terms of the functions ϕ, m (which can be used for reductions of arbitrary order n). One is to substitute the rational function f to nonhomogeneous Lax pair (10). The arising conditions define all coefficients of the function and give a reduction condition. Another way is to take an expansion of relation (29) and use the Lax pair (4) (see also (23)) to express higher coefficients of the series for Λ, M through ϕ, m (in general in the form of recursion relations). In the case $n = 1$ the zero order of expansion of relation (29) at $\lambda = 0$ gives

$$\ln m_t = al_1^+ + be^{-\phi}l_1^-.$$

Using the pair (23), we obtain following relations for l_1^+, l_1^- :

$$\begin{aligned} \partial_t l_1^+ &= \phi_t \frac{m_x}{m_t} - \phi_x, & \partial_y l_1^+ &= -\frac{(e^{-\phi})_t}{m_t}, \\ \partial_t (e^{-\phi} l_1^-) &= -m_t \phi_y, & \left(\frac{m_x}{m_t} \partial_t - \partial_x \right) (e^{-\phi} l_1^-) &= (e^{-\phi})_t. \end{aligned}$$

The simplest form of the differential reduction for the system (3) in the case $n = 1$ reads

$$m_{tt} = a(\phi_t m_x - \phi_x m_t) - b(m_t)^2 \phi_y.$$

However, in this form the differential relation contains all the variables x , y , t . It is possible to rewrite it in equivalent form in (x, t) plane or (y, t) plane. The differential reduction in (y, t) plane reads

$$\partial_y \partial_t \ln m_t = -a \partial_t \left(\frac{(e^{-\phi})_t}{m_t} \right) - b \partial_y (m_t \phi_y).$$

Considering this reduction together with interpolating reduction (12) (see also (28))

$$e^{\alpha\phi} = m_t,$$

we obtain a (1+1)-dimensional system which represents a reduction of equation (2) and can be rewritten as (1+1)-dimensional equation for the function m

$$am_{tt} = (m_t)^{\frac{1}{\alpha}+1} (\alpha m_{ty} + b m_{yy} m_t). \quad (30)$$

It is possible to transform this equation to the system of hydrodynamic type.

The differential reduction in (x, t) plane reads

$$\partial_t \left(a \frac{m_x}{m_t} \left(\phi_t \frac{m_x}{m_t} - \phi_x \right) + \partial_t \frac{m_x}{m_t} + b (e^{-\phi})_t \right) - a \partial_x \left(\phi_t \frac{m_x}{m_t} - \phi_x \right) = 0.$$

Considered together with interpolating reduction (12), it forms a (1+1)-dimensional system representing a reduction of equation (2), which in terms of one function m reads

$$b m_{tt} m_t^{-\frac{1}{\alpha}-1} - \alpha \partial_t \frac{m_x}{m_t} + a \left(\frac{m_x}{m_t} \partial_t - \partial_x \right) \frac{m_x}{m_t} = 0. \quad (31)$$

A common solution of (1+1)-dimensional equations (30), (31) gives a solution of equation (2).

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