LARGE DEVIATIONS FOR A MEAN FIELD MODEL OF SYSTEMIC RISK

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Abstract. We consider a system of diffusion processes that interact through their empirical mean and have a stabilizing force acting on each of them, corresponding to a bistable potential. There are three parameters that characterize the system: the strength of the intrinsic stabilization, the strength of the external random perturbations, and the degree of cooperation or interaction between them. The latter is the rate of mean reversion of each component to the empirical mean of the system. We interpret this model in the context of systemic risk and analyze in detail the effect of cooperation between the components, that is, the rate of mean reversion. We show that in a certain regime of parameters increasing cooperation tends to increase the stability of the individual agents but it also increases the overall or systemic risk. We use the theory of large deviations of diffusions interacting through their mean field.

Key words. mean field, large deviations, systemic risk, dynamic phase transitions.

AMS subject classifications. 60F10, 60K35, 91B30, 82C26

1. Introduction.

Systemic risk is the risk that in an interconnected system of agents that can fail individually, a large number of them fails simultaneously or nearly so, leading to the overall failure of the system. It is a property of the interconnected system as a whole, and not only of the individual components, in the sense that assessment of the risk of individual failure alone cannot provide an assessment of the systemic risk. The interconnectivity of the agents, its form and evolution, play an essential role in systemic risk assessment [6].

In this paper we consider a simple model of interacting agents for which systemic risk can be assessed analytically in some interesting cases. Each agent can be in one of two states, a normal and a failed one, and it can undergo transitions between them. We assume that the dynamic evolution of each agent has the following features. First, there is an intrinsic stabilization mechanism that tends to keep the agents near the normal state. Second, there are external destabilizing forces that tend to push away from the normal state and are modeled by stochastic processes. Third, there is cooperation among the agents that acts as a stabilizer. In such a system we expect that there is a decrease in the risk of destabilization or "failure" for each agent because of the cooperation. What is less obvious is the effect of cooperation on the overall or system's risk, which can be defined in a precise way for the model considered here. We show in this paper that for the models under consideration and in a certain regime of parameters, the systemic risk increases with increasing cooperation. The aim of this paper is to analyze this tradeoff between individual risk and systemic risk for a class interacting systems subject to failure.

Perhaps a simple mathematical model of interacting agents having the features we want is a system of stochastic differential equations with mean-field interaction. Let $x_j(t)$ be the state of risk of agent or component j, taking real values. For $j = 1, \dots, N$, the $x_j(t)$'s are modeled as continuous-time stochastic processes satisfying the system

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of Itô stochastic differential equations:

$$dx_j(t) = -hU(x_j(t))dt + \theta(\bar{x}(t) - x_j(t))dt + \sigma dw_j(t), \tag{1.1}$$

with given initial conditions. Here $-hU(y) = -hV'(y)$ is the restoring force, V is a potential which we assume has two stable states, and $\{w_j(t)\}_{j=1}^N$ are independent, standard Brownian motions. The parameter h controls the level of intrinsic stabilization and σ the strength of the destabilizing random forces. The interaction or cooperation is the mean reversion term with rate of mean reversion θ and with $\bar{x}(t) := \frac{1}{N}\sum_{i=1}^N x_i(t)$ denoting the empirical mean of of the processes, that is, the empirical mean of the individual risks. For $\theta > 0$ the individual risk processes tend to mean-revert to their empirical mean, which is a simple but non-trivial form of cooperation. We take the empirical mean $\bar{x}(t)$ to be a measure of the systemic risk. The bi-stable-state structure of $V(y)$ determines the normal and failed states of the agents. If, for example, $U(y) = y^3 - y$, so that $V(y) = \frac{1}{4}y^4 - \frac{1}{2}y^2$, then the two stable states are ± 1 and we let -1 be the normal state and $+1$ to be the failed state. The potential $V(y)$ ensures that each agent or component stays around -1 (normal) or $+1$ (failed). The evolution of the system is characterized by the initial conditions, the three parameters (h, θ, σ) and by the system size N.

We have chosen a mean-field interaction because it is a simple form of cooperative behavior. More elaborate models are considered in Section 3, where some heterogeneity is introduced between the components of the system. For mean-field models a natural measure of systemic risk is the transition probability of the empirical mean $\bar{x}(t)$ from the normal state to the failed state. More precisely, the mathematical problem we address here is this: For N large we calculate approximately such transition probabilities and analyze how they depend on h, σ and θ, the three parameters of the system. We are interested in a regime of these parameters for which there are two collective, that is, large N, equilibria centered around the normal and failed states. These two equilibria can be identified through the mean-field limit of the system, that is, the weak limit in probability of the empirical density of the agents risk x_j.

The mathematical analysis of bistable mean field models like (1.1) was initiated by Dawson [9], including the mean field limit, the existence of multiple equilibria, and a fluctuation theory. Non-equilibrium statistical mechanics and phase transitions have been studied extensively in the sciences [18]. The large deviation theory that we use here was developed by Dawson and Gärtier [10]. In particular, they introduced and analyzed the rate function for large deviations associated with (1.1) when N is large. Their theory may be considered as an infinite dimensional extension of the Freidlin-Wentzell theory of large deviations for stochastic differential equations with small noise [15, 13]. The main result in this paper is the analysis of this rate function for small h. That is, for a shallow two-well potential, where transitions from one well (quasi-equilibrium) to the other are exponentially small in N, the "constant" in the exponent is small when h is small. Other mean field models have been studied in [31, 17, 25, 2, 26, 28, 14], and large deviations results for various models can be found in [11, 1, 27, 12, 21, 8, 7]. In [7] a general large deviations theory is developed for a model with both drift and volatility interactions, as well as with degenerate noise, using weak convergence and optimal control methods.

The main contribution of the paper as far as systemic risk theory is concerned is the demonstration that, within the range of the bistable mean field model (1.1), while cooperation between agents decreases the individual risk of each agent, the systemic or overall risk is increased. This is discussed in detail in Section 6.3, in terms of the three parameters (h, θ, σ), with h small. The fact that reducing individual risk by cooperation or diversification can lead to increased systemic risk has been anticipated in macroeconomics and elsewhere and it has been extensively discussed, modeled, and analyzed in [29, 4, 19, 16, 24, 30, 5, 3, 20, 22]. However, the dynamic phase transitions formulation and the large deviations theory exploited in this paper have not been used in the economics literature, to our knowledge.

The paper is organized as follows. In Section 2, we briefly review the classical mean-field limit in [9], and we discuss the intrinsic stability of equllibria [9] when h is small. Section 3 generalizes (1.1) by replacing the rate of mean reversion θ by an agent-dependent θ_j. The mean-field limit and the explicit conditions are also studied. In Section 4, we carry out numerical simulations of both the homogeneous and the heterogeneous model in various parameter ranges. Section 5 uses the large deviation principle in [10] to formulate the dynamic phase transition of interest here, that is, the system transition from the normal state to the failed state. In Section 6, we specialize the large deviations theory when h is small so as to obtain a result from which the systemic risk as a function the basic parameters (h, θ, σ) can be assessed and interpreted. Section 7 case where there is system diversity in mean reversion diversity and it is shown that under some natural conditions the heterogeneous model is systemically more unstable than the homogeneous one. The technical details of the proofs are in the appendices.

2. The Mean-Field Limit. We briefly review the mean field limit in [9] and carry out a small h analysis of results since they will be used in calculating large deviation probabilities. We want to analyze the systemic behavior of the interacting diffusion processes (1.1), through their empirical mean $\bar{x}(t)$, but this is not possible in a direct way since (1.1) is nonlinear. We consider instead the empirical density of $x_j(t)$, which is a measure valued process that has a limit as $N \to \infty$. Let $M_1(\mathbb{R})$ be the space of probability measures endowed with the weak (Prohorov) topology and let $C([0,T], M_1(\mathbb{R}))$ be the space of continuous $M_1(\mathbb{R})$-valued processes on $[0,T]$ endowed with the corresponding weak topology. Define the empirical probability measure process $X_N(t,dy) := \frac{1}{N}\sum_{j=1}^{N} \delta_{x_j(t)}(dy)$ and note that $X_N \in C([0,T], M_1(\mathbb{R}))$. The mean field limit theorem for X_N, proved in [9], is as follows:

THEOREM 2.1. *(Dawson, 1983) X_N converges in law as $N \to \infty$ to a deterministic process with density $u(t,y)dy \in C([0,T], M_1(\mathbb{R}))$ satisfying the Fokker-Planck equation:*

$$\frac{\partial}{\partial t}u = h\frac{\partial}{\partial y}[U(y)u] - \theta\frac{\partial}{\partial y}\left\{\left[\int yu(t,y)dy - y\right]u\right\} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial y^2}u. \qquad (2.1)$$

In [9], $U(y) = y^3 - y$ but obviously the same proof can be applied to a wide class of $U(y)$. By Theorem 2.1, we can analyze u and view X_N as a perturbation of u for N large. We may consider $\bar{x}(t)$ in the same way because $\bar{x}(t) = \int yX_N(t,dy)$. However, the limit problem is infinitely dimensional, as is expected.

Explicit solutions of (2.1) are not available in general, but we can find equilibrium solutions. Assuming that the first order moment is ξ, then an equilibrium solution u_ξ^e satisfies

$$h\frac{d}{dy}[(y^3 - y)u_\xi^e] - \theta\frac{d}{dy}[(\xi - y)u_\xi^e] + \frac{1}{2}\sigma^2\frac{d^2}{dy^2}u_\xi^e = 0,$$

and has the form

$$u^e_\xi(y) = \frac{1}{Z_\xi\sqrt{2\pi\frac{\sigma^2}{2\theta}}}\exp\left\{-\frac{(y-\xi)^2}{2\frac{\sigma^2}{2\theta}} - h\frac{2}{\sigma^2}V(y)\right\}, \tag{2.2}$$

with Z_ξ the normalization constant:

$$Z_\xi = \int \frac{1}{\sqrt{2\pi\frac{\sigma^2}{2\theta}}}\exp\left\{-\frac{(y-\xi)^2}{2\frac{\sigma^2}{2\theta}} - h\frac{2}{\sigma^2}V(y)\right\} dy.$$

Now ξ must satisfy the compatibility or consistency condition:

$$\xi = m(\xi) := \int y u^e_\xi(y)dy. \tag{2.3}$$

Finding equilibrium solutions has thus been reduced to finding solutions of this equation.

For $U(y) = y^3 - y$, $\xi = 0$ is a solution for (2.3). With the same $U(y)$, it can be shown (see also [9]) that there are two additional non-zero solutions $\pm\xi_b$ if and only if $\frac{d}{d\xi}m(0) > 1$, and for given h and θ, there exists a critical $\sigma_c(h,\theta) > 0$ such that $\frac{d}{d\xi}m(0) > 1$ if and only if $\sigma < \sigma_c(h,\theta)$.

An explanation for this bifurcation at equilibrium is that when $\sigma \geq \sigma_c$, randomness dominates the interaction among the components, i.e., $\theta(\bar{x}(t) - x_j(t))dt$ is negligible. In this case, the system behaves like N independent diffusions and hence, by the symmetry of $V(y)$, at any given time roughly half of them stay around -1 and half around $+1$ so the average is 0. When, however, $\sigma < \sigma_c$, then the interactive force is significantly larger (now $\sigma dw_j(t)$ is less important). Therefore all agents stay around the same place (either $-\xi_b$ or $+\xi_b$) and the zero average equilibrium is unstable. Since we want to model systemic risk phenomena, we assume that $\sigma < \sigma_c$ throughout this paper, and we regard $-\xi_b$ as the healthy state of the system and $+\xi_b$ as the failed state. The calculation of transitions probabilities between these two states is our objective.

For small h we can approximate the solution of (2.3) to order $O(h)$ as follows.

PROPOSITION 2.2. *For small h, the critical value σ_c can be expanded as*

$$\sigma_c = \sqrt{\frac{2\theta}{3}} + O(h). \tag{2.4}$$

In addition, the non-zero solutions $\pm\xi_b$ are

$$\pm\,\xi_b = \pm\sqrt{1 - 3\frac{\sigma^2}{2\theta}}\left(1 + h\frac{6}{\sigma^2}\left(\frac{\sigma^2}{2\theta}\right)^2\frac{1-2(\sigma^2/2\theta)}{1-3(\sigma^2/2\theta)}\right) + O(h^2). \tag{2.5}$$

Proof. See Appendix A. ☐

From Proposition 2.2, we see the relation between the existence of the bi-stable states and the ratio $\sigma^2/2\theta$: For a given θ, and for small h, (2.3) has non-zero solutions if and only if $3\sigma^2/2\theta < 1$. Moreover, these non-zero solutions $\pm\xi_b$ are generally not ± 1 since the magnitude $|\xi_b|$ is less than 1. Note that the coefficient of order h in the expansion (2.5) depends significantly on θ and σ. Thus, when $3\sigma^2/2\theta$ tends to 1, ξ_b in (2.5) will not go to $+\infty$ while, in fact, ξ_b goes to 0. From the $O(1)$ term in (2.5), we also see that ξ_b is roughly decreasing as $\sigma^2/2\theta$ is increasing.

3. Diversity of Sensitivities. We can generalize (1.1) by allowing for agent dependent coefficients. We consider a particular case in which each agent can have a different rate of mean reversion to the empirical mean, that is, for $j = 1, \dots, N$,

$$dx_j = -h\frac{\partial}{\partial x_j}V(x_j)dt + \sigma dw_j + \theta_j(\bar{x} - x_j)dt. \tag{3.1}$$

We consider the case where $\{\theta_j\}_{j=1}^N$ takes K values $\{\Theta_j\}_{j=1}^K$. We define $\mathcal{I}_l = \{j : \theta_j = \Theta_l\}$, $\rho_l = |\mathcal{I}_l|/N$ and $X_N^l = \frac{1}{\rho_l N}\sum_{j\in\mathcal{I}_l}\delta_{x_j}$. Assuming that $\lim_{N\to\infty}\rho_l$ exists for all l, the limit of $(X_N^1, \dots, X_N^K)$ as $N \to \infty$ are the solutions $(u_1, \dots, u_K)$ of the set of K coupled Fokker-Planck equations.

Theorem 3.1. *The measure valued process $(X_N^1, \dots, X_N^K)$ converges weakly as $N \to \infty$ to the solution $(u_1, \dots, u_K)$ of the system of the Fokker-Planck equations:*

$$\frac{\partial}{\partial t}u_1 = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial y^2}u_1 - \Theta_1\frac{\partial}{\partial y}\left\{\left[\int y\sum_{l=1}^K \rho_l u_l(t,y)dy - y\right]u_1\right\} + h\frac{\partial}{\partial y}[U(y)u_1] \tag{3.2}$$

$$\vdots$$

$$\frac{\partial}{\partial t}u_K = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial y^2}u_K - \Theta_K\frac{\partial}{\partial y}\left\{\left[\int y\sum_{l=1}^K \rho_l u_l(t,y)dy - y\right]u_K\right\} + h\frac{\partial}{\partial y}[U(y)u_K].$$

Proof. See Appendix B.1. □

The equilibrium solutions $\{u_{l,\xi}^e\}_{l=1}^K$ have the form

$$u_{l,\xi}^e(y) = \frac{1}{Z_{l,\xi}\sqrt{2\pi\frac{\sigma^2}{2\Theta_l}}}\exp\left\{-\frac{(y-\xi)^2}{2\frac{\sigma^2}{2\Theta_l}} - h\frac{2}{\sigma^2}V(y)\right\} \tag{3.3}$$

$$Z_{l,\xi} = \int\frac{1}{\sqrt{2\pi\frac{\sigma^2}{2\Theta_l}}}\exp\left\{-\frac{(y-\xi)^2}{2\frac{\sigma^2}{2\Theta_l}} - h\frac{2}{\sigma^2}V(y)\right\}dy.$$

and ξ must satisfy the compatibility condition

$$\xi = m(\xi) := \sum_{l=1}^K \rho_l\int y u_{l,\xi}^e(y)dy. \tag{3.4}$$

For $U(y) = y^3 - y$, $\xi = 0$ is the trivial solution of (3.4), and a simple extension of Theorem 3.3.1 in [9], shows that there are two sets of non-trivial solutions $\{u_{l,\xi_b}^e\}_{l=1}^K$ and $\{u_{l,-\xi_b}^e\}_{l=1}^K$ if and only if $\frac{d}{d\xi}m(0) > 1$. The numerical simulations presented in the next section show that diversity in the rate of mean reversion can have significant impact on the stability of the mean-field model.

As in the homogeneous case, we can get an approximate condition for equilibrium bifurcation for small h.

Proposition 3.2. *The compatibility condition (3.4) has non-zero solutions if and only if $\sigma < \sigma_c^{div}$. For small h, σ_c^{div} has the expansion*

$$\sigma_c^{div} = \sqrt{\sum_{l=1}^K\frac{\rho_l}{\Theta_l}/\sum_{l=1}^K\frac{3\rho_l}{2\Theta_l^2}} + O(h).$$

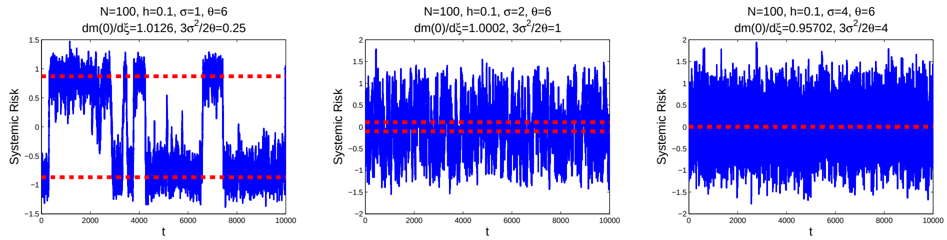

FIG. 4.1. *Simulations for different σ. The system has two (statistically) stable equilibria when σ is below the critical value or otherwise has single stable state* 0*. For small h, $3\sigma^2/2\theta < 1$ is the approximate criterion.*

Proof. See Appendix B.2. □

We note that diversity does affect the threshold condition and makes the analysis more difficult. The non-zero solutions $\pm\xi_b$ can be computed approximately when h is small:

$$\pm\,\xi_b = \pm\sqrt{\sum_{l=1}^{K}\frac{\rho_l}{\Theta_l}\left(1-3\frac{\sigma^2}{2\Theta_l}\right)/\sum_{l=1}^{K}\frac{\rho_l}{\Theta_l}} + O(h). \tag{3.5}$$

Higher order terms in the expansion of (3.5) can also be obtained. However, the expressions are complicated and so we will omit them in this paper. It can also be shown that $\sigma_c^{\mathrm{div}} \le \sigma_c^{\mathrm{homo}}$, where σ_c^{homo} is the critical value in the homogeneous case.

PROPOSITION 3.3. *With $\theta = \sum_{l=1}^{K}\rho_l\Theta_l$, we have $\sigma_c^{homo} \ge \sigma_c^{div}$ for small h.*

Proof. See Appendix B.3. □

This result shows that when there is diversity the parameter region of existence of equilibria $\pm\xi_b$ is smaller than in the homogeneous case . From this observation we can anticipate that these equilibria are less stable in the presence of diversity, and this is confirmed next by numerical simulations and analytically.

4. Numerical Simulations.

Before going into a detailed analysis of the models, we carry out numerical simulations of (1.1) and (3.1) so as to get a quick impression of their behavior. We discretize with a uniform time grid, and let X_j^n denote the simulated X_j at time $n\Delta t$.

4.1. Homogeneous Model.

We simulate (1.1) using the Euler scheme

$$X_j^{n+1} = X_j^n - hU(X_j^n)\Delta t + \sigma\Delta W_j^{n+1} + \theta(\frac{1}{N}\sum_{k=1}^{N}X_k^n - X_j^n)\Delta t. \tag{4.1}$$

We take $U(y) = y^3 - y$, $= 1$, $X_j^0 = -1$, $\Delta t = 0.02$, and let $\{\Delta W_j^n\}_{j,n}$ be independent Gaussian random variables with mean zero and variance Δt. In the figures presented, the dashed lines show the numerical solutions of the compatibility equation (2.3), $\xi = m(\xi)$. As noted earlier, if $\frac{d}{d\xi}m(0) \le 1$, then $0 = m(0)$ is the unique solution and 0 is a stable state. Therefore we should observe that the systemic risk fluctuates around 0. If $\frac{d}{d\xi}m(0) > 1$, there are two additional non-zero solutions $\pm\xi_b = m(\pm\xi_b)$ and $\pm\xi_b$ are stable while 0 is unstable. We also know that when h is small, the condition $\frac{d}{d\xi}m(0) > 1$ can be simplified to be $3\sigma^2/2\theta < 1$.

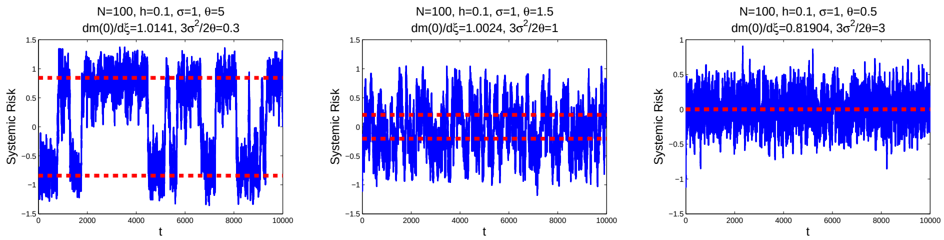

FIG. 4.2. *Simulations for different θ. The system has two stable equilibria if θ is above the critical value or otherwise has single stable state* 0. *For small h, $3\sigma^2/2\theta < 1$ is the approximate criterion.*

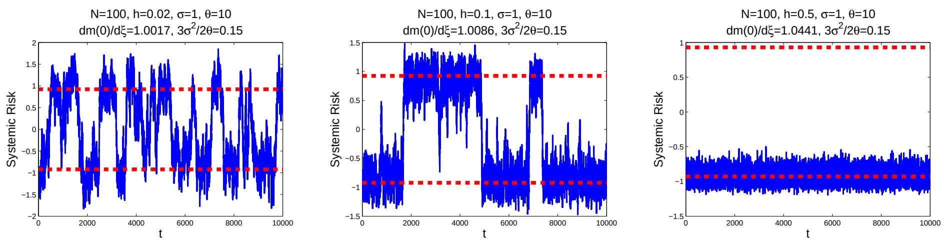

FIG. 4.3. *The effect of changing h. Increasing it stabilizes the system.*

Figure 4.1 and Figure 4.2 illustrate the behavior of the empirical mean as the system canoes from having two equilibria to having a single one, which is controlled by the value of $\frac{d}{d\xi}m(0)$. This is an instance of a bifurcation of equilibria. In Figure 4.2 we change θ but fix the other parameters, and consider the three cases $\frac{d}{d\xi}m(0) < 1$, $\frac{d}{d\xi}m(0) \approx 1$ and $\frac{d}{d\xi}m(0) > 1$. In Figure 4.1 we change σ. We can see that even though the controlled parameters are not the same the bifurcation behavior is similar. We also verify numerically that for h small, $\frac{d}{d\xi}m(0) > 1$ is well approximated by the condition $3\sigma^2/2\theta < 1$.

Figure 4.3 shows the effect of increasing h on the system stability. By stability we mean resistance to the transition of the empirical mean of the system from one state to the other (because the model is symmetric). The parameter h is proportional to the height of the potential barrier of each agent. Thus we increase the overall system stability if we increase the component's stability. This observation is analogous to comments in [29, 23, 24]. It is clear that h influences system stability substantially.

Figure 4.4 illustrates the effect of system size on its stability. Clearly a larger system is more stable. These stability phenomena will be quantifies with the large deviations analysis of Section 5.

4.2. Heterogeneous Model. For the heterogeneous model, θ is replaced by θ_j, and the discretization is

$$X_j^{n+1} = X_j^n - hU(X_j^n)\Delta t + \sigma \Delta W_j^{n+1} + \theta_j\left(\frac{1}{N}\sum_{k=1}^{N} X_k^n - X_j^n\right)\Delta t, \tag{4.2}$$

with the same parameter settings. The different values of θ_j are controlled by the parameters Θ_l and ρ_l. In the simulation, we take $\{\Theta_l\}_{l=1}^K = \{\Theta_L, \Theta_M, \Theta_H\}$ for a system a low, medium and high rates of mean reversion to the empirical mean, that

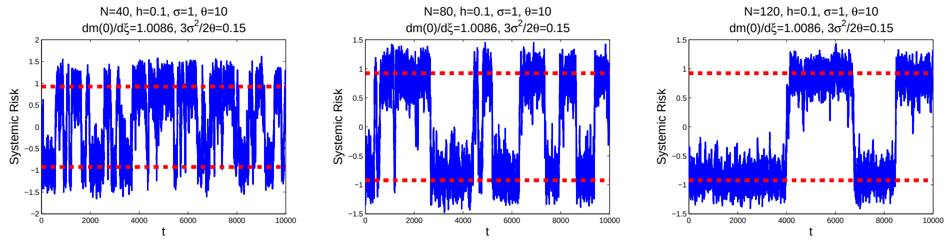

FIG. 4.4. *Influence of the system size N. A larger system tends to have a more stable behavior.*

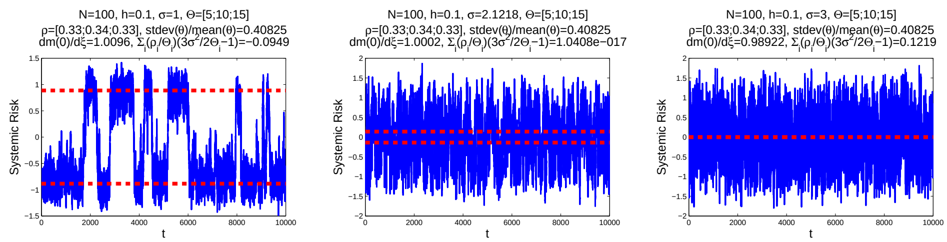

FIG. 4.5. *Effect of changes in σ. The system has two stable equllibria when σ is below the critical value and has single one otherwise. For small h, $\sum_{l=1}^{K}(\rho_l/\Theta_l)(3\sigma^2/2\Theta_l - 1) < 1$ is the approximate criterion.*

is, the systemic risk. We also take $\{\rho_l\}_{l=1}^{K} = \{\rho_L, \rho_M, \rho_H\}$ for the corresponding fractions. We use the normalized standard deviation of the distribution of θ_j values in order to quantify diversity. We find that the heterogeneous model behaves like the homogeneous one when h, σ and N change. But, diversity on the rates of mean reversion has significant impact on system stability.

As in the homogeneous case, we consider cases with σ below, close to and above the critical value. The results are similar to the homogeneous case as expected. For σ below the critical value we have two equlibria and for σ above the critical value one equilibrium. The condition $\frac{d}{d\xi}m(0) > 0$ is still necessary and sufficient for the existence two equlibria. The condition $\sum_{l=1}^{K}(\rho_l/\Theta_l)(3\sigma^2/2\Theta_l - 1) < 1$ is also a good approximation to the exact one when h is small.

The parameter h and the system size N are closely associated with system stability. We note that when h or N are increased, the system becomes visibly more stable. Another observation is that with h, σ and N fixed, and with the mean of θ_j of (4.2) equal to θ of (4.1), the heterogeneous system is consistently more unstable than the corresponding homogeneous model. Clearly diversity tends to destabilize the system.

We also change the diversity of θ_j by changing Θ_l and ρ_l. To compare with the homogeneous case, we change the standard deviation of θ_j while the mean of θ_j is fixed. In this most interesting part of the simulations we see that when we increase the standard deviation of diversity values, the number of transitions is notably larger than that in the homogeneous case.

5. Large Deviations. In the previous two sections we saw both analytically and numerically that for large N, the empirical mean $\bar{x}(t)$ in (1.1) stays around the first order moment of the deterministic limit, $\int_{-\infty}^{\infty} yu(t,y)dy$. If the condition of existence of two equilibria is satisfied, then $\bar{x}(t)$ will remain close to either $-\xi_b$ or $+\xi_b$ for relatively long time intervals, depending mostly on the parameter h. If all agents

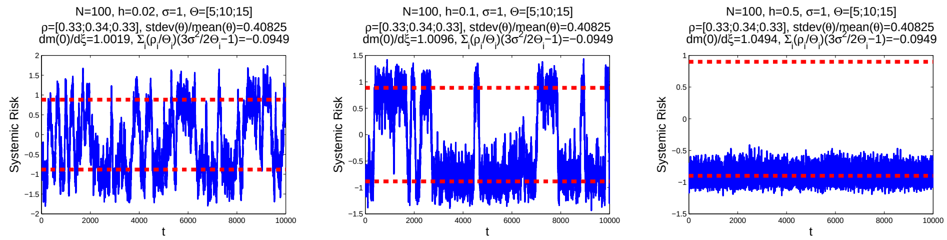

FIG. 4.6. *Effect of changing h. Increasing it stabilizes the system.*

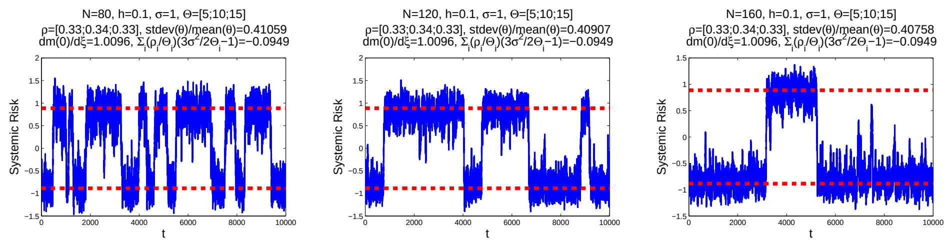

FIG. 4.7. *Effect of changing the system size N. Larger system have a more stable behavior.*

are in the normal state initially, that is, $x_j(0) = -1$ for all j, then $\bar{x}(t)$ is close to $-\xi_b$ for some $t > 0$. However, as long as $N < \infty$, as we have seen in the simulations, the randomness $\{w_j(t)\}_{j=1}^N$ will cause transitions with non-zero probability. A systemic transition is the event that $\bar{x}(t)$ jumps from $\pm\xi_b$ to $\mp\xi_b$ within a finite time horizon. Thus, systemic transition means a that large number of agents transition in a finite time. In this paper, we are interested in computing the probability of such a systemic transition. Mathematically, given a finite time horizon $[0, T]$ and the conditions for existence of two equilibria, we want to compute the probability

$$\mathbf{P}(\bar{x}(0) = -\xi_b, \bar{x}(T) = \xi_b). \tag{5.1}$$

5.1. Large Deviations of Mean-fields. According to [10], we can calculate this probability asymptotically for large N using the large deviations. To state the large deviations theory that we will use, we will review briefly some notation and terminology from [10].

- $M_1(\mathbb{R})$ is the space of probability measures on $\mathbb{R}$ with the Prohorov metric ρ, associated with weak convergence.
- $C([0, T], M_1(\mathbb{R}))$ is the space of continuous functions from $[0, T]$ to $M_1(\mathbb{R})$ with the metric $\sup_{0 \le t \le T} \rho(\phi_1(t), \phi_2(t))$.
- $M_R(\mathbb{R}) = \{\mu \in M_1(\mathbb{R}), \int \varphi(y)\mu(dy) \le R\}$, where $\varphi \in C^2(\mathbb{R})$ is a nonnegative function with $\lim_{|x|\to\infty} \varphi(x) = \infty$. From [10], if $U(y) = y^3 - y$, we can choose $\varphi(y) = 1 + y^2 + \gamma y^4$, $0 \le \gamma \le h/2$.
- $M_\infty(\mathbb{R}) = \cup_{R>0} M_R(\mathbb{R}) = \{\mu \in M_1(\mathbb{R}), \int \varphi(y)\mu(dy) < \infty\}$ endowed with the inductive topology: $\mu_n \to \mu$ in $M_\infty(\mathbb{R})$ if and only if $\mu_n \to \mu$ in $M_1(\mathbb{R})$ and $\sup_n \int \varphi(y)\mu(dy) < \infty$.
- $C([0, T], M_\infty(\mathbb{R}))$ is the space of continuous functions from $[0, T]$ to $M_\infty(\mathbb{R})$ endowed with the topology: $\phi_n(\cdot) \to \phi(\cdot)$ in $C([0, T], M_\infty(\mathbb{R}))$ if and only if $\phi_n(\cdot) \to \phi(\cdot)$ in $C([0, T], M_1(\mathbb{R}))$ and $\sup_{0\le t\le T} \sup_n \int \varphi(y)\phi_n(t, dy) < \infty$.
- Given $\nu \in M_\infty(\mathbb{R})$, we let $\mathcal{E}^\nu = \{\phi \in C([0, T], M_\infty(\mathbb{R})) : \phi(0) = \nu\}$, endowed with the relative topology.

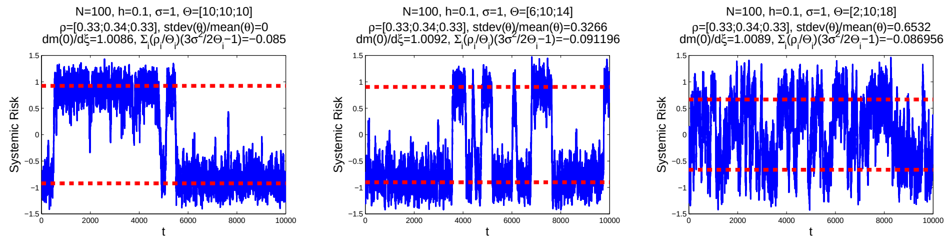

FIG. 4.8. *The effect of changes in* Θ_l*. The median of the diversity values is fixed but the low and high sensitivities are changed to adjust the level of diversity of* θ_j *while* ρ_l *and the mean of* θ_j *are the same. Increasing diversity tends to destabilize the system.*

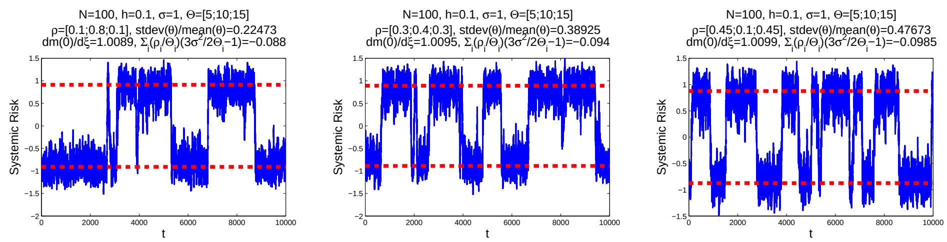

FIG. 4.9. *The effect of changes in* ρ_l*, with* Θ_l *and the mean of* θ_j *fixed. Increasing diversity tends to destabilize the system.*

To simplify the notation, we rewrite (2.1) as $u_t = \mathcal{L}_u^* u + h\mathcal{M}^* u$, where

$$\mathcal{L}_\psi^* \phi = \frac{1}{2}\sigma^2 \phi_{yy} + \theta \frac{\partial}{\partial y}\left\{\left[y - \int y\psi(t,y)dy\right]\phi\right\}, \quad \mathcal{M}^*\phi = \frac{\partial}{\partial y}\left[U(y)\phi\right].$$

THEOREM 5.1. *(Dawson and Gärtner, 1987) Given a finite horizon* $[0,T]$*,* $\nu \in M_\infty(\mathbb{R})$ *and* $A \subseteq \mathcal{E}^\nu$*, if* $X_N(0) = \frac{1}{N}\sum_{j=1}^N \delta_{x_j(0)} \to \nu$ *in* $M_\infty(\mathbb{R})$ *as* $N \to \infty$*, then the law of* $X_N(t) = \frac{1}{N}\sum_{j=1}^N \delta_{x_j(t)}$ *satisfies the large deviation principle with the good rate function* I_h*:*

$$\begin{aligned} -\inf_{\phi \in \mathring{A}} I_h(\phi) &\le \liminf_{N\to\infty} \frac{1}{N}\log \mathbf{P}(X_N \in A) \\ &\le \limsup_{N\to\infty} \frac{1}{N}\log \mathbf{P}(X_N \in A) \le -\inf_{\phi\in \bar{A}} I_h(\phi), \end{aligned}$$

where $\mathring{A}$ *and* $\bar{A}$ *are the interior and closure of* A *in* $\mathcal{E}^\nu$*, respectively, and*

$$I_h(\phi) = \frac{1}{2\sigma^2}\int_0^T \sup_{f:\langle \phi, f_y^2\rangle \neq 0} J_h(\phi, f)dt, \tag{5.2}$$

$$J_h(\phi,f) = \langle \phi_t - \mathcal{L}_\phi^*\phi - h\mathcal{M}^*\phi, f\rangle^2 / \langle \phi, f_y^2\rangle, \quad \langle \phi, f\rangle = \int_{-\infty}^\infty f(y)\phi(dy),$$

if $\phi(t)$ *is absolutely continuous in* $t \in [0,T]$ *and* $I_h(\phi) = \infty$ *otherwise.*

Here the definition of absolute continuity for the path of measures $(\phi(t))_{t\in[0,T]}$ is in the sense of Definition 4.1 in [10], that is to say: for each compact set $K \subset \mathbb{R}$

there exists a neighborhood U_K of the null function in the set of test functions with compact support in K and an absolutely continuous function H_K from $[0,T]$ to $\mathbb{R}$ such that $|\langle\phi(t), f\rangle - \langle\phi(s), f\rangle| \leq |H_K(t) - H_K(s)|$ for all $s, t \in [0,T]$ and $f \in U_K$.

In order to use Theorem 5.1, we let $\nu = u^e_{-\xi_b}$ in (2.2) and define the rare event A of systemic transition by

$$A = \left\{\phi \in \mathcal{E}^\nu : \phi(T) = u^e_{\xi_b}\right\}. \tag{5.3}$$

However, since $\mathring{A}$ is an empty set, Theorem 5.1 give a trivial lower bound for the probability in question. Therefore we consider instead the closed rare event A_δ:

$$A_\delta = \left\{\phi \in \mathcal{E}^\nu : \rho(\phi(T), u^e_{\xi_b}) \leq \delta\right\}.$$

Then Theorem 5.1 implies that

$$\begin{aligned} -\inf_{\phi\in\mathring{A}_\delta} I_h(\phi) &\leq \liminf_{N\to\infty} \frac{1}{N}\log \mathbf{P}(X_N \in A_\delta) \\ &\leq \limsup_{N\to\infty} \frac{1}{N}\log \mathbf{P}(X_N \in A_\delta) \leq -\inf_{\phi\in A_\delta} I_h(\phi). \end{aligned}$$

In addition, we show that $\inf_{\phi\in A_\delta} I_h(\phi)$ can be bounded from below by $\inf_{\phi\in A} I_h(\phi)$ as $\delta \to 0$.

Lemma 5.2. *By definition* $\inf_{\phi\in A_\delta} I_h(\phi)$ *is decreasing with* $\delta > 0$ *and bounded from above by* $\inf_{\phi\in A} I_h(\phi)$. *In addition,*

$$\lim_{\delta\to 0} \inf_{\phi\in A_\delta} I_h(\phi) \geq \inf_{\phi\in A} I_h(\phi).$$

Proof. See Appendix C. ◻

Combining Lemma 5.2 and the fact that $\inf_{\phi\in\mathring{A}_\delta} I_h(\phi) \leq \inf_{\phi\in A} I_h(\phi)$, for any $\epsilon > 0$, we have sufficiently small $\delta > 0$ such that

$$\begin{aligned} -\inf_{\phi\in A} I_h(\phi) &\leq \liminf_{N\to\infty} \frac{1}{N}\log \mathbf{P}(X_N \in A_\delta) \\ &\leq \limsup_{N\to\infty} \frac{1}{N}\log \mathbf{P}(X_N \in A_\delta) \leq -\inf_{\phi\in A} I_h(\phi) + \epsilon. \end{aligned}$$

Therefore for large N and sufficiently small δ,

$$\mathbf{P}(X_N \in A_\delta) \approx \exp\left(-N \inf_{\phi\in A} I_h(\phi)\right). \tag{5.4}$$

This tells us that a larger system has a more stable empirical mean trajectory, which is consistent with we have seen in the numerical simulation. Now the main step is finding $\inf_{\phi\in A} I_h(\phi)$, which is a min-max problem variational problem

$$\inf_{\phi\in A} I_h(\phi) = \inf_{\phi\in A} \frac{1}{2\sigma^2}\int_0^T \sup_{f:\langle\phi, f_y^2\rangle\neq 0} \langle\phi_t - \mathcal{L}^*_\phi\phi - h\mathcal{M}^*\phi, f\rangle^2/\langle\phi, f_y^2\rangle dt, \tag{5.5}$$

where f is any real Schwartz test function.

5.2. An Alternative Expression for the Rate Function. The representation of the rate function (5.2) is somewhat complicated, but we can simplify the representation if ϕ has the density function with nice properties. If ϕ is a density function such that $\phi(t,y)$ is smooth, rapid decreasing in $y \in \mathbb{R}$ for each $t \in [0,T]$ and is absolutely continuous in $t \in [0,T]$ for each $y \in \mathbb{R}$, then let $g(t,y)$ satisfy

$$\phi_t - \mathcal{L}_\phi^* \phi - h\mathcal{M}^* \phi = (\phi g)_y. \tag{5.6}$$

Note that because of the properties of ϕ, the left hand side of (5.6) is well-defined in $y \in \mathbb{R}$ and almost everywhere in $t \in [0,T]$. In addition, because ϕ is positive valued, g exists and is unique except on a measure zero set in $[0,T]$.

Note that for the pair (ϕ, g) satisfying (5.6)

$$\sup_{f:\langle \phi(t), f_y^2 \rangle \neq 0} J_h(\phi(t), f) = \sup_{f:\langle \phi(t), f_y^2 \rangle \neq 0} \langle \phi(t), f_y g \rangle^2 / \langle \phi(t), f_y^2 \rangle = \langle \phi(t), g^2 \rangle,$$

and therefore we have the following proposition.

PROPOSITION 5.3. *If ϕ is a density function such that $\phi(t)$ is a Schwartz function for each $t \in [0,T]$ and is absolutely continuous in $t \in [0,T]$ for each $y \in \mathbb{R}$, and $g(t,y)$ solves (5.6), the rate function $I_h(\phi)$ in (5.2) can be written in the form*

$$I_h(\phi) = \frac{1}{2\sigma^2} \int_0^T \langle \phi, g^2 \rangle dt. \tag{5.7}$$

We interpret (5.6) and (5.7) as follows. The function g is regarded as the driving force making ϕ deviate from the solution of the Fokker-Planck equation (2.1), and $I_h(\phi)$ is the $L^2(\phi)$ norm of g, which means how difficult to have this deviation ϕ.

6. Small h Analysis. The goal of this section is to analyze the min-max problem (5.5) which controls the asymptotic systemic transition probability. This problem is nonlinear and infinitely dimensional and is difficult to analyze. To get some useful information about it we will assume that h is small and analyze it in this regime. We will first solve (5.5) when h is exactly 0, and then we will get rigorous upper and lower bounds for (5.5) when h is nonzero but small. We will then compare the large deviations result with the local fluctuation theory of a single agent so as to explain why interconnectedness destabilizes the system. Finally by assuming that (5.5) is sufficiently differentiable in h, we expand formally to obtain a more explicit result for $\inf_{\phi \in A} I_h(\phi)$ and the associated optimal ϕ.

6.1. The $h = 0$ and the Small h Analysis. We note that when $h = 0$, $u^e_{\pm\xi_b} = u^e_{\pm\xi_0}$, where

$$u^e_{\pm\xi_0}(y) = \frac{1}{\sqrt{2\pi\frac{\sigma^2}{2\theta}}} \exp\left\{ -\frac{(y-(\pm\xi_0))^2}{2\frac{\sigma^2}{2\theta}} \right\}, \qquad \xi_0 = \sqrt{1 - 3\frac{\sigma^2}{2\theta}}. \tag{6.1}$$

In this case, (5.5) is solvable and the optimal path is a Gaussian, starting from $u^e_{-\xi_0}$ and ending in $u^e_{+\xi_0}$.

THEOREM 6.1. *Let $h = 0$ and define*

$$p^e(t,y) = \frac{1}{\sqrt{2\pi\frac{\sigma^2}{2\theta}}} \exp\left\{ -\frac{(y-a^e(t))^2}{2\frac{\sigma^2}{2\theta}} \right\}, \quad a^e(t) = \frac{2\xi_0}{T} t - \xi_0. \tag{6.2}$$

Then $p^e \in A$ is the unique minimizer for (5.5) and

$$\inf_{\phi\in A} I_0(\phi) = I_0(p^e) = \frac{2\xi_0^2}{\sigma^2 T}.$$

Proof. See Appendix D.1. □

We show next that (5.5) is continuous at $h = 0$.

THEOREM 6.2. *There exists $\gamma(h)$ such that $\gamma(h) \to 0$ as $h \to 0$ and*

$$\Big| \inf_{\phi\in A} I_h(\phi) - \frac{2\xi_b^2}{\sigma^2 T}\Big| \le \gamma(h). \tag{6.3}$$

We recall here that

$$\xi_b = \xi_0 + h\xi_1 + O(h^2), \qquad \xi_1 = \sqrt{1-3\frac{\sigma^2}{2\theta}}\frac{6}{\sigma^2}\left(\frac{\sigma^2}{2\theta}\right)^2 \frac{1-2(\sigma^2/2\theta)}{1-3(\sigma^2/2\theta)}. \tag{6.4}$$

Proof. See Appendix D.2 and D.3. □

As it is stated we could replace ξ_b by ξ_0 in Theorem 6.2, since $\xi_b = \xi_0 + o(1)$ as $h \to 0$. We will see in the next section (in Proposition 6.8) that $\gamma(h) = O(h^2)$ (in fact we will show this rigorously for the upper bound and formally for the lower bound). Since $\xi_b = \xi_0 + h\xi_1 + O(h^2)$ and therefore the term $2\xi_b^2/(\sigma^2 T)$ contains the leading-order term and the first-order correction in the h-expansion of $\inf_{\phi\in A} I_h(\phi)$.

6.2. Comparison with the Fluctuation Theory of a Single Agent. To get a better understanding of the large deviations results we need to carry out a standard fluctuation theory for a single agent. We assume that $x_j(0) = -1$ for all j and that the $x_j(t)$'s are in the vicinity of -1 so that we can linearize (1.1):

$$x_j(t) = -1 + \delta x_j(t), \quad \bar{x}(t) = -1 + \delta\bar{x}(t), \quad \delta\bar{x}(t) = \frac{1}{N}\sum_{j=1}^{N}\delta x_j(t).$$

For $V(y) = \frac{1}{4}y^4 - \frac{1}{2}y^2$, $\delta x_j(t)$ and $\delta\bar{x}(t)$ satisfy the linear stochastic differential equations

$$d\delta x_j = -(\theta+2h)\delta x_j dt + \theta\delta\bar{x}dt + \sigma dw_j, \quad d\delta\bar{x} = -2h\delta\bar{x}dt + \frac{\sigma}{N}\sum_{j=1}^{N} dw_j,$$

with $\delta x_j(0) = \delta\bar{x}(0) = 0$. The processes $\delta x_j(t)$ and $\delta\bar{x}(t)$ are Gaussian and the mean and variance functions are easily calculated. We are especially interested in their behavior for large N.

LEMMA 6.3. *For all $t \ge 0$, $\mathbf{E}\delta x_j(t) = \mathbf{E}\delta\bar{x}(t) = 0$ and $\mathbf{Var}\delta\bar{x}(t) = \frac{\sigma^2}{N}(1-e^{-4ht})$. In addition, $\mathbf{Var}\delta x_j(t) \to \frac{\sigma^2}{2(\theta+2h)}(1-e^{-2(\theta+2h)t})$ as $N \to \infty$, uniformly in $t \ge 0$.*

From Lemma 6.3, we see that σ^2/N and $\sigma^2/2(\theta+2h)$ should be sufficiently small so that linearization is consistent with the results it produces.

6.3. Increased Probability of Large Deviations for Increased θ and Its Systemic Risk Interpretation. We have now the analytical results with which we may conclude that individual risk diversification may increase the systemic risk. Assume that σ^2/N and $\sigma^2/2(\theta+2h)$ are sufficiently small and N is large. From

Lemma 6.3, the risk $x_j(t)$ of the agent j is approximately a Gaussian process with the stationary distribution $\mathcal{N}(-1, \sigma^2/2(\theta+2h))$. If the external risk, σ is high, then in order to keep the risk $x_j(t)$ at an acceptable level, the agent may increase the intrinsic stability, h, or share the risk with other agents, that is, increase θ. Increasing h is in general more costly (cuts into profits) than increasing θ, and at the individual agent level there is no difference in risk assessment between increasing h and increasing θ. Therefore the agents are likely to increase θ and reduce individual risk by diversifying it. Note that $\sigma^2/2(\theta+2h) \lesssim \sigma^2/2\theta$ when σ^2 and θ are significantly larger than h. Thus, individual agents can maintain low locally assessed risk by diversification, even in a very uncertain environment.

What is not perceived by the individual agents, however, is that risk diversification may increases the systemic risk while it reduces their individual risk. Because σ^2 and θ are significantly larger than h, the small h analysis can be applied and from (5.4) and Theorem 6.2, the systemic risk (the probability of the system failure) is

$$\mathbf{P} \approx \exp\left(-N\frac{2\xi_b^2}{\sigma^2 T}\right), \quad \xi_b = \sqrt{1-3\frac{\sigma^2}{2\theta}}\left(1 + h\frac{6}{\sigma^2}\left(\frac{\sigma^2}{2\theta}\right)^2 \frac{1-2(\sigma^2/2\theta)}{1-3(\sigma^2/2\theta)}\right) + O(h^2).$$

We see that there are additional systemic-level σ^2 terms in the exponent and ξ_b, which can not be observed by the agents, increasing the systemic risk, even if the individual risk $\sigma^2/2\theta$ is fixed. In other words, the individual agents may believe that they are able to withstand larger external fluctuations as long as their risk can be diversified, but a higher σ tends to destabilize the system.

6.4. A Formal Expansion for Small h. Now we assume that (5.5) has a second order expansion in h we can formally compute it. Because the optimal path is p^e for $h = 0$, it is reasonable to assume that the optimal path ϕ is $p^e + O(h)$ for small h. Our goal is to obtain a reduced Freidlin-Wentzell theory. That is, obtain a reduced rate function corresponding to finite dimensional system after ignoring higher order terms. For this purpose, the reduced rate function should have the information up to $O(h^2)$ terms, and we also need to expand ϕ to $O(h^2)$.

For convenience, we assume that the optimal $\phi = p + hq^1 + h^2 q^2$, where

$$p(t,y) = \frac{1}{\sqrt{2\pi\frac{\sigma^2}{2\theta}}} \exp\left\{-\frac{(y-a(t))^2}{2\frac{\sigma^2}{2\theta}}\right\}, \quad a(t) = \langle \phi, y \rangle.$$

In other words, the first moment of ϕ is determined by $a(t)$, and from the zero h case we know that $a(t) = a^e(t) + O(h)$. From the form of p and (5.6), a natural parameterization for q^1 and q^2 is the Hermite expansion

$$q^1(t,y) = \sum_{n=2}^{\infty} b_n(t) \frac{\partial^n}{\partial y^n} p(t,y), \quad q^2(t,y) = \sum_{n=2}^{\infty} c_n(t) \frac{\partial^n}{\partial y^n} p(t,y).$$

Note that by the properties of p and $a(t)$, $\langle q^1, y^n \rangle = \langle q^2, y^n \rangle = 0$ for $n = 0, 1$, so we can start the Hermite expansion from $n = 2$.

An important remark about the expansion is that the Hermite functions are a basis of the L^2 space and thus $p + hq^1 + h^2 q^2$ is generally a signed measure. However, if q^1 and q^2 can be expressed as the linear combinations of finite Hermite functions, then we can easily see that for any $\epsilon > 0$, there exists a sufficiently small h such that the negative part of $p + hq^1 + h^2 q^2$ is less than ϵ.

6.4.1. Optimization over g. The first step in finding the optimal $\phi = p + hq^1 + h^2q^2$ is determining the optimal g by using (5.6) for ϕ. Once we obtain g, we can compute $I_h(\phi)$ by using (5.7). It is also natural to assume that $g = g^0 + hg^1 + h^2g^2$ along with the Hermite expansion:

$$g^0 = p^{-1}\sum_{n=0}^{\infty}\alpha_n(t)\frac{\partial^n}{\partial y^n}p, \quad g^1 = p^{-1}\sum_{n=0}^{\infty}\beta_n(t)\frac{\partial^n}{\partial y^n}p, \quad g^2 = p^{-1}\sum_{n=0}^{\infty}\gamma_n(t)\frac{\partial^n}{\partial y^n}p.$$

In addition, since $\langle q^1, y\rangle = \langle q^2, y\rangle = 0$, we can see that $\phi = p + hq^1 + h^2q^2$ satisfies

$$\mathcal{L}_\phi^*\phi = \mathcal{L}_p^*p + h\mathcal{L}_p^*q^1 + h^2\mathcal{L}_p^*q^2, \quad \mathcal{M}^*\phi = \mathcal{M}^*p + h\mathcal{M}^*q^1 + h^2\mathcal{M}^*q^2.$$

The force $U(y) = y^3 - y$ can also be expanded in Hermite polynomials:

$$U(y) = p^{-1}\sum_{n=0}^{3}\delta_n(t)\frac{\partial^n}{\partial y^n}p.$$

Now everything is expanded in the orthogonal basis and we can find the optimal g^0 and g^1 by putting everything into (5.6) and comparing coefficients.

LEMMA 6.4. *With the expansions mentioned above, the optimal g^0 is $-\frac{d}{dt}a$, and the optimal β_n for g^1 are*

$$\beta_n = \begin{cases} -\delta_0 = -\langle p, U(y)\rangle, & n = 0, \\ \frac{d}{dt}b_{n+1} + \theta(n+1)b_{n+1} - \delta_n, & 1 \le n \le 3, \\ \frac{d}{dt}b_{n+1} + \theta(n+1)b_{n+1}, & n \ge 4. \end{cases} \tag{6.5}$$

Proof. See Appendix D.4. □

It remains to determine g^2. From (5.7) we see that the only contribution of g^2 to I_h up to $O(h^2)$ is $\langle p, 2g^0g^2\rangle = -2\gamma_0\frac{d}{dt}a$. Thus it suffices to determine γ_0, which can also be obtained from (5.6).

LEMMA 6.5. *With the expansions mentioned above, the optimal γ_0 is*

$$\gamma_0 = -\langle q^1, U(y) + g^1\rangle.$$

Proof. See Appendix D.5. □

6.4.2. Optimization over ϕ. We are now ready to find the optimal ϕ. For given $\phi = p + hq^1 + h^2q^2$ and the corresponding optimal $g = g^0 + hg^1 + h^2g^2$, (5.7) gives

$$\begin{aligned} I_h(\phi) &= \int_0^T \langle p + hq^1 + h^2q^2, (g^0 + hg^1 + h^2g^2)^2\rangle dt \\ &= \int_0^T \langle p, (g^0)^2\rangle dt + h\int_0^T \langle p, 2g^0g^1\rangle dt \\ &\quad + h^2\int_0^T \left(\langle p, (g^1)^2 + 2g^0g^2\rangle + \langle q^1, 2g^0g^1\rangle\right) dt + O(h^3). \end{aligned}$$

From Lemma 6.5, $\langle p, 2g^0g^2\rangle = -2g^0\langle q^1, U(y) + g^1\rangle$, and therefore

$$\langle p, 2g^0g^2\rangle + \langle q^1, 2g^0g^1\rangle = -2g^0\langle q^1, U(y)\rangle = -2g^0\sum_{n=2}^{3}H_n\delta_n b_n,$$

where $H_n(t) := \langle p^{-1}, (\partial^n p/\partial y^n)^2\rangle$. We note that

$$\langle p, 2g^0g^1\rangle = -2g^0\delta_0, \quad \langle p, (g^1)^2\rangle = \delta_0^2 + \sum_{n=1}^{\infty} H_n\beta_n^2, \quad \langle p, (g^0)^2\rangle = (g^0)^2.$$

Then $I_h(\phi)$ can be written as

$$\begin{aligned} I_h(\phi) = &\int_0^T (g^0 - h\delta_0)^2 dt + h^2 \int_0^T (H_1\beta_1^2 - 2H_2 g^0\delta_2 b_2)dt \\ &+ h^2 \int_0^T (H_2\beta_2^2 - 2H_3 g^0\delta_3 b_3)dt + h^2 \sum_{n=3}^{\infty} \int_0^T H_n\beta_n^2 dt + O(h^3). \end{aligned} \tag{6.6}$$

We see that a and b_n are coupled at the $O(h^2)$ level of (6.6). However, from the results of the zero h case, $a = a^e + O(h)$ and $p = p^e + O(h)$ so we can decouple a and b_n and express the expanded $I_h(\phi)$ up to $O(h^2)$ as the sum of independent terms.

PROPOSITION 6.6. *To order $O(h^2)$, the rate function $I_h(\phi)$ can be written as the sum of independent terms:*

$$\begin{aligned} I_h(\phi) = &\int_0^T (g^0 - h\delta_0)^2 dt + h^2 \int_0^T (\tilde{H}_1\tilde{\beta}_1^2 + 2\frac{d}{dt}a^e\tilde{H}_2\tilde{\delta}_2 b_2)dt \\ &+ h^2 \int_0^T (\tilde{H}_2\tilde{\beta}_2^2 + 2\frac{d}{dt}a^e\tilde{H}_3\tilde{\delta}_3 b_3)dt + h^2 \sum_{n=3}^{\infty} \int_0^T \tilde{H}_n\tilde{\beta}_n^2 dt + O(h^3), \end{aligned} \tag{6.7}$$

where $\tilde{H}_n(t) = \langle (p^e)^{-1}, (\partial^n p^e/\partial y^n)^2\rangle$, $U(y) = (p^e)^{-1}\sum_{n=0}^{3}\tilde{\delta}_n(t)\frac{\partial^n}{\partial y^n}p^e$, *and*

$$\tilde{\beta}_n = \begin{cases} -\tilde{\delta}_0 = -\langle p^e, U(y)\rangle, & n = 0, \\ \frac{d}{dt}b_{n+1} + \theta(n+1)b_{n+1} - \tilde{\delta}_n, & 1 \le n \le 3, \\ \frac{d}{dt}b_{n+1} + \theta(n+1)b_{n+1}, & n \ge 4. \end{cases}$$

We can see from (6.7) that q^2 does not appear in terms up to $O(h^2)$, and therefore we can let $q^2 \equiv 0$. From the h expansion of $u^e_{\pm\xi_b}$ in (2.2), $b_n(0) = b_n(T) = 0$ for $n \ge 5$. Then we have that the optimal $b_n \equiv 0$ for $n \ge 5$ so that $\tilde{\beta}_n \equiv 0$ for $n \ge 4$. Consequently, in order to find the optimal ϕ for $I_h(\phi)$ in (6.7), we may solve separately the variational problems for a, b_1, b_2 and b_3.

6.4.3. Probability of Systemic Transitions for Small h. We consider the small probability of systemic transitions for large N and small h through the large deviation $\inf_{\phi\in A} I_h(\phi)$. Here we consider the solution up to $O(h)$ terms. That is, using (6.7), we solve the variational problem for $a(t)$:

$$\inf_{\substack{a(t):0\le t\le T \\ a(0)=-\xi_b \\ a(T)=\xi_b}} \int_0^T (g^0 - h\delta_0)^2 dt = \inf_{\substack{a(t):0\le t\le T \\ a(0)=-\xi_b \\ a(T)=\xi_b}} \int_0^T (\frac{d}{dt}a + h(a^3 + 3\frac{\sigma^2}{2\theta}a - a))^2 dt. \tag{6.8}$$

By simple calculus of variations methods we find the optimal a.

LEMMA 6.7. *The optimal $a(t)$ for (6.8) satisfies the second order ordinary differential equation*

$$\frac{d^2}{dt^2}a = h^2(a^3 + (3\frac{\sigma^2}{2\theta} - 1)a)(3a^2 + (3\frac{\sigma^2}{2\theta} - 1))$$

with $a(0) = -\xi_b$ and $a(T) = \xi_b$. Consequently, the optimal path is

$$a(t) = \frac{2\xi_b}{T} t - \xi_b + O(h^2). \tag{6.9}$$

By inserting (6.9) into (6.8) we obtain $\inf_{\phi \in A} I_h(\phi)$ up to $O(h)$.

PROPOSITION 6.8. *For small h, the large deviations problem, $\inf_{\phi \in A} I_h(\phi)$, up to $O(h)$, is*

$$\inf_{\phi \in A} I_h(\phi) = \frac{2\xi_0}{\sigma^2 T}(\xi_0 + 2h\xi_1) + O(h^2), \tag{6.10}$$

where $\xi_b = \xi_0 + h\xi_1 + O(h^2)$ from (2.5). Note that ξ_1 is positive because $2\theta > 3\sigma^2$.

Proof. See Appendix D.6. ▯

The asymptotic probability of systemic transition for large N and sufficiently small δ and h has the form

$$\mathbf{P}(X_N \in A_\delta) \approx \exp\left(-N \inf_{\phi \in A} I_h(\phi)\right) = \exp\left(-N \left\{ \frac{2\xi_0}{\sigma^2 T}(\xi_0 + 2h\xi_1) + O(h^2) \right\}\right).$$

7. Effect of Diversity of Sensitivities on the Transition Probability. We consider case considered in Section 3 in which $h = 0$. We aim at computing the transition probability in this situation. The K partial empirical averages

$$\bar{x}_k(t) := \frac{1}{|\mathcal{I}_k|} \sum_{j \in \mathcal{I}_k} x_j(t), \quad k = 1, \dots, K \tag{7.1}$$

then satisfy a closed system of stochastic differential equations

$$d\bar{x}_k = \frac{\sigma}{\sqrt{\rho_k N}} d\bar{w}_k(t) - \theta_k(\bar{x}_k - \bar{x})dt \tag{7.2}$$

where $\bar{w}_k$ are independent Brownian motions and the empirical mean $\bar{x}(t)$ can be expressed in terms of the partial averages as

$$\bar{x}(t) = \sum_{k=1}^{K} \rho_k \bar{x}_k(t)$$

PROPOSITION 7.1. *If $\bar{x}_k(0) = -\xi_b$ for all $k = 1, \dots, K$, then $\bar{x}(T)$ is a Gaussian random variable with mean $-\xi_b$ and variance $\sigma_T^2 := \mathrm{Var}(\bar{x}(T))$ given by*

$$\sigma_T^2 = \frac{\sigma^2}{N} \int_0^T \rho^T e^{Ms} R^{-1} (e^{Ms})^T \rho ds \tag{7.3}$$

where ρ is the K-dimensional vector $(\rho_k)_{k=1,\dots,K}$, M and R are the $K \times K$ matrices defined by

$$M_{ij} = -\theta_i(\delta_{ij} - \rho_j), \qquad R_{ij} = \rho_i \delta_{ij}, \qquad i,j = 1, \dots, K,$$

and ^{T} stands for the transpose.

Proof. See Appendix E.1. ▯

We can then deduce that the transition probability is

$$p_T \approx \exp\Big(-\frac{2\xi_b^2}{\sigma_T^2}\Big) \tag{7.4}$$

Our next goal is to study the impact of the diversity on the transition probability.

PROPOSITION 7.2. *Let us assume that the diversity is small:*

$$\theta_k = \bar{\theta}(1+\delta\alpha_k), \qquad \delta \ll 1$$

where $\sum_k \rho_k \alpha_k = 0$ *so that* $\bar{\theta}$ *is the mean value of the* θ_k*'s. The equilibrium position* ξ_b*, the variance* σ_T^2 *and the transition probability* p_T *can be expanded as powers of* δ *as*

$$\xi_b^2 = \Big(1-\frac{3\sigma^2}{2\bar{\theta}}\Big) - \delta^2\Big(\sum_k \rho_k\alpha_k^2\Big)\frac{3\sigma^2}{2\bar{\theta}} + O(\delta^3),$$

$$\sigma_T^2 = \frac{\sigma^2 T}{N}\Big[1+\delta^2\Big(\sum_k \rho_k\alpha_k^2\Big)\Big(\frac{1}{T}\int_0^T (1-e^{-\bar{\theta}s})^2 ds\Big) + O(\delta^3)\Big],$$

$$p_T \approx \exp\Big\{-\frac{2N}{\sigma^2 T}\Big[\Big(1-\frac{3\sigma^2}{2\bar{\theta}}\Big) - \delta^2\Big(\sum_k \rho_k\alpha_k^2\Big)\Big(\frac{3\sigma^2}{2\bar{\theta}} + \frac{1}{T}\int_0^T (1-e^{-\bar{\theta}s})^2 ds\Big)\Big]\Big\}.$$

Proof. See Appendix E.2. ◻

This proposition shows that the diversity reduces the gap between the two equilibrium states and enhances the fluctuations of the empirical mean. Both effects contribute to the increase of the systemic transition probability.

8. Summary and Conclusions. The aim of this paper is to introduce and analyze a mathematical model for the evolution of risk in a system of interacting agents where cooperation between them can reduce their individual risk of failure but increase the systemic or overall risk. The model we use is a system of bistable diffusion processes that interact through their empirical mean, a mean field model. We take the rate of mean reversion to the empirical mean θ as a measure of cooperation, the depth of the bistable potential h as a measure of intrinsic stability of each agent, and the strength of the external random perturbations σ as the level of uncertainty in which the agents function. Using the theory of large deviations we calculate the probability that the system will transition from one of the two bistable states to the other during a time interval of length T, when the number of agents N is large and when h is small. In this regime of parameters we find that systemic risk increases with cooperation. The formula from which we draw this conclusion is given is Section 6.3. We also show that when the rate of mean reversion to the empirical mean varies among the different agents, that is, when there is diversity in the cooperative behavior then the probability of transitions increases, which means that the systemic risk increases.

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Appendix A. Proof of Proposition 2.2.

For small h, we view u_ξ^e as a perturbed Gaussian density function. Let $p_\xi(y)$ be the Gaussian density function with mean ξ and variance $\sigma^2/2\theta$, Y be the Gaussian random variable with the density p_ξ, and $\eta = 2/\sigma^2$. By using the expansion $\exp(-h\eta V) = 1 - h\eta V + h^2\eta^2 V^2/2 + O(h^3)$, we have

$$Z_\xi = 1 - h\eta \mathbf{E}V(Y) + \frac{1}{2}h^2\eta^2\mathbf{E}V^2(Y) + O(h^3)$$
$$Z_\xi^{-1} = 1 + h\eta\mathbf{E}V(Y) - \frac{1}{2}h^2\eta^2\mathbf{E}V^2(Y) + h^2\eta^2(\mathbf{E}V(Y))^2 + O(h^3).$$

Then we calculate $m(\xi)$ as follows:

$$\begin{aligned}
m(\xi) &= Z_\xi^{-1}\int y\left(1 - h\eta V + \frac{1}{2}h^2\eta^2V^2 + O(h^3)\right)p_\xi(y)dy \\
&= Z_\xi^{-1}\left(\xi - h\eta\mathbf{E}[YV(Y)] + \frac{1}{2}h^2\eta^2\mathbf{E}[YV^2(Y)] + O(h^3)\right) \\
&= \xi + h\eta\{\xi\mathbf{E}V(Y) - \mathbf{E}[YV(Y)]\} + h^2\eta^2\{-\frac{1}{2}\xi\mathbf{E}V^2(Y) + \xi(\mathbf{E}V(Y))^2 \\
&\quad - \mathbf{E}V(Y)\mathbf{E}[YV(Y)] + \frac{1}{2}\mathbf{E}[YV^2(Y)]\} + O(h^3) \\
&= \xi - h\eta\frac{\sigma^2}{2\theta}\mathbf{E}V_y(Y) + h^2\eta^2\frac{\sigma^2}{2\theta}\{\mathbf{E}[V(Y)V_y(Y)] - \mathbf{E}V(Y)\mathbf{E}V_y(Y)\} + O(h^3) \\
&= \xi - h\eta\frac{\sigma^2}{2\theta}\mathbf{E}V_y(Y) + h^2\eta^2\frac{\sigma^2}{2\theta}\mathbf{Cov}(V_y(Y), V(Y)) + O(h^3).
\end{aligned}$$

The compatibility condition $\xi_b = m(\xi_b)$ gives

$$\mathbf{E}V_y(Y) - h\eta\mathbf{Cov}(V_y(Y), V(Y)) + O(h^2) = 0. \tag{A.1}$$

Assuming that $\xi_b = \xi_0 + h\xi_1 + O(h^2)$, the $O(1)$ terms in (A.1) give

$$\xi_0^3 + 3\frac{\sigma^2}{2\theta}\xi_0 - \xi_0 = \xi_0(\xi_0^2 + 3\frac{\sigma^2}{2\theta} - 1) = 0.$$

Then $\xi_0 = 0, \pm\sqrt{1 - 3\sigma^2/2\theta}$ if $3\sigma^2 < 2\theta$, or otherwise $\xi_0 = 0$. In order to obtain the nontrivial result, we suppose that $3\sigma^2 < 2\theta$ and ξ_0 takes $\pm\sqrt{1 - 3\sigma^2/2\theta}$ in the later calculations. Note that $\mathbf{E}V_y(Y) = \xi^3 + (3\sigma^2/2\theta - 1)\xi = 2h\xi_0^2\xi_1 + O(h^2)$, and

$$\begin{aligned}
\mathbf{Cov}(V_y(Y), V(Y)) &= \mathbf{E}[V(Y)V_y(Y)] + O(h) = \mathbf{E}[(\frac{1}{4}Y^4 - \frac{1}{2}Y^2)(Y^3 - Y)] + O(h) \\
&= \mathbf{E}[\frac{1}{4}Y^7 - \frac{3}{4}Y^5 + \frac{1}{2}Y^3] + O(h).
\end{aligned}$$

Along with the identity $\xi_0^2 + 3\sigma^2/2\theta = 1$, we have

$$\mathbf{E}Y^3 = \xi_0 + O(h), \quad \mathbf{E}Y^5 = \left(1 + 4\frac{\sigma^2}{2\theta} - 6\left(\frac{\sigma^2}{2\theta}\right)^2\right)\xi_0 + O(h),$$
$$\mathbf{E}Y^7 = \left(1 + 12\frac{\sigma^2}{2\theta} + 6\left(\frac{\sigma^2}{2\theta}\right)^2 - 48\left(\frac{\sigma^2}{2\theta}\right)^3\right)\xi_0 + O(h).$$

Then $\mathbf{Cov}(V_y(Y), V(Y)) = 6(\sigma^2/2\theta)^2(1 - 2\sigma^2/2\theta)\xi_0 + O(h)$. The $O(h)$ terms in (A.1) imply $\xi_1 = 3\eta(\sigma^2/2\theta)^2(1 - 2\sigma^2/2\theta)/\xi_0$.

Appendix B. Proofs in Section 3.

B.1. Proof of Theorem 3.1. For a test function $f \in \mathcal{S}(\mathbb{R})$, we define $X_N^{f,l}(t) = \langle f(y), X_N^l(t,y)\rangle = \sum_{j\in\mathcal{I}_l} f(x_j(t))/|\mathcal{I}_l|$. By Itô's formula,

$$dX_N^{f,l} = \frac{1}{|\mathcal{I}_l|}\sum_{j\in\mathcal{I}_l}[-hU(x_j)dt + \sigma dw_j + \Theta_l(\bar{x} - x_j)dt]f_y(x_j) + \frac{1}{2}\sigma^2 f_{yy}(x_j)dt$$
$$= \langle -hUf_y + \Theta_l(\langle y, \sum_{l=1}^K \rho_l X_N^l\rangle - y)f_y + \frac{\sigma^2}{2}f_{yy}, X_N^l\rangle dt + \langle f_y, \frac{\sigma}{|\mathcal{I}_l|}\sum_{j\in\mathcal{I}_l}\delta_{x_j}dw_j\rangle.$$

Then by the integration by parts, we write

$$dX_N^l = \{(hUX_N^l)_y - [\Theta_l(\langle y, \sum_{l=1}^K \rho_l X_N^l\rangle - y)X_N^l]_y + \frac{\sigma^2}{2}(X_N^l)_{yy}\}dt - \frac{\sigma}{|\mathcal{I}_l|}\sum_{j\in\mathcal{I}_l}(\delta_{x_j})_y dw_j.$$

For simplicity, we prove the case that $K = 2$ and the general case is similar. We let $X_N^{1,\times n} \times X_N^{2,\times n}$ denote the product measure on $\mathbb{R}^{2n}$:

$$X_N^{1,\times n} \times X_N^{2,\times n}(y_1, \ldots, y_{2n}) = X_N^1(t,y_1)\cdots X_N^1(t,y_n)X_N^2(t,y_{n+1})\cdots X_N^2(t,y_{2n}).$$

For a test function $f \in \mathcal{S}(\mathbb{R}^{2n})$, we have

$$d\langle f, X_N^{1,\times n} \times X_N^{2,\times n}\rangle = d\langle f, X_N^{1,\times n} \times X_N^{2,\times n}\rangle^{(1)} + d\langle f, X_N^{1,\times n} \times X_N^{2,\times n}\rangle^{(2)},$$

where (1) and (2) denote the first and the second order terms of $d\langle f, X_N^{1,\times n} \times X_N^{2,\times n}\rangle$, respectively:

$$d\langle f, X_N^{1,\times n} \times X_N^{2,\times n}\rangle^{(1)} = \sum_{j=1}^n \langle f, dX_N^1(t,y_j) \times X_N^{1,\times(n-1),j} \times X_N^{2,\times n}\rangle$$
$$+ \sum_{j=n+1}^{2n}\langle f, dX_N^2(t,y_j) \times X_N^{1,\times n,j} \times X_N^{2,\times(n-1),j}\rangle$$

$$d\langle f, X_N^{1,\times n} \times X_N^{2,\times n}\rangle^{(2)} = \frac{1}{2}\sum_{\substack{j,k=1\\ j\neq k}}^n \langle f, dX_N^1(t,y_j) \times dX_N^1(t,y_k) \times X_N^{1,\times(n-2),j,k} \times X_N^{2,\times n}\rangle$$
$$+ \frac{1}{2}\sum_{\substack{j,k=n+1\\ j\neq k}}^{2n}\langle f, dX_N^2(t,y_j) \times dX_N^2(t,y_k) \times X_N^{1,\times n} \times X_N^{2,\times(n-2),j,k}\rangle$$
$$+ \frac{1}{2}\sum_{j=1}^n\sum_{k=n+1}^{2n}\langle f, dX_N^1(t,y_j) \times dX_N^2(t,y_k) \times X_N^{1,\times(n-1),j} \times X_N^{2,\times(n-1),k}\rangle.$$

Note that for $j \neq k$, $dX_N^l(t,y_j) \times dX_N^l(t,y_k) = \frac{\sigma^2}{|\mathcal{I}_1|^2}\sum_{i\in\mathcal{I}_1}(\delta_{x_i}(y_j))_j(\delta_{x_i}(y_k))_k dt = \frac{\sigma^2}{\rho_1^2 N}(\delta(y_k - y_j)X_N^1(t,y_j))_{jk}dt$, and $dX_N^1(t,y_j) \times dX_N^2(t,y_k) = 0$. If we analogously represent the generator $G_{(X_N^{1,\times n}, X_N^{2,\times n})}f$ of $\langle f, X_N^{1,\times n} \times X_N^{2,\times n}\rangle$ as

$$G_{(X_N^{1,\times n}, X_N^{2,\times n})}f = G^{(1)}_{(X_N^{1,\times n}, X_N^{2,\times n})}f + G^{(2)}_{(X_N^{1,\times n}, X_N^{2,\times n})}f,$$

then $G^{(2)}_{(X_N^{1,\times n},X_N^{2,\times n})}f \to 0$ as $N\to\infty$ and $G^{(1)}_{(X_N^{1,\times n},X_N^{2,\times n})}f = G_{(u_1^{\times n},u_2^{\times n})}f$, the generator of $\langle f, u_1^{\times n}\times u_2^{\times n}\rangle$, where (u_1,u_2) satisfying (3.2). Consequently, for all $f\in\mathcal{S}(\mathbb{R}^{2n})$, $G_{(X_N^{1,\times n},X_N^{2,\times n})}f \to G_{(u_1^{\times n},u_2^{\times n})}f$ as $N\to\infty$ and equivalently $(X_N^1,X_N^2)\to(u_1,u_2)$ weakly as $N\to\infty$.

B.2. Proof of Proposition 3.2. All we need to show is that for small h, $\frac{d}{d\xi}m(0) > 1$ if and only if $\sigma < \sigma_c^{\text{div}}$, where $m(\xi)$ is defined by (3.4). We obtain $\frac{d}{d\xi}m$ by calculate $\frac{d}{d\xi}\int yu^e_{l,\xi}(y)dy$. Note that $\frac{d}{d\xi}Z_{l,\xi} = (2\Theta_l/\sigma^2)(\int yu^e_{l,\xi}dy-\xi)Z_{l,\xi}$ and

$$\begin{aligned}\frac{d^2}{d\xi^2}Z_{l,\xi} &= \frac{2\Theta_l}{\sigma^2}Z_{l,\xi}\left(\frac{d}{d\xi}\int yu^e_{l,\xi}dy-1\right)+\frac{2\Theta_l}{\sigma^2}\left(\int yu^e_{l,\xi}dy-\xi\right)\frac{d}{d\xi}Z_{l,\xi} \qquad \text{(B.1)}\\ &= \frac{2\Theta_l}{\sigma^2}Z_{l,\xi}\left(\frac{d}{d\xi}\int yu^e_{l,\xi}dy-1\right)+\left(\frac{2\Theta_l}{\sigma^2}\right)^2 Z_{l,\xi}\left(\int yu^e_{l,\xi}dy-\xi\right)^2.\end{aligned}$$

On the other hand, we can also compute $\frac{d^2}{d\xi^2}Z_{l,\xi}$ by directly taking the twice derivatives of $Z_{l,\xi}$:

$$\frac{d^2}{d\xi^2}Z_{l,\xi} = -\frac{2\Theta_l}{\sigma^2}Z_{l,\xi}+\left(\frac{2\Theta_l}{\sigma^2}\right)^2 Z_{l,\xi}\int(y-\xi)^2u^e_{l,\xi}dy. \qquad \text{(B.2)}$$

By comparing (B.1) and (B.2),

$$\frac{d}{d\xi}\int yu^e_{l,\xi}dy = \frac{2\Theta_l}{\sigma^2}\left[\int y^2u^e_{l,\xi}dy - (\int yu^e_{l,\xi}dy)^2\right].$$

Note that $\int yu^e_{l,0}dy = 0$, so $\frac{d}{d\xi}m(0) = \sum_{l=1}^K \rho_l(2\Theta_l/\sigma^2)\int y^2u^e_{l,0}dy$. By using the same trick in the proof of Proposition 2.2, let $p_l(y)$ be the Gaussian density function with mean 0 and variance $\sigma^2/2\Theta_l$, Y_l be the Gaussian random variable with the density p_l, and $\eta = 2/\sigma^2$. Then for small h, $Z_{l,0}^{-1} = 1 + h\eta\mathbf{E}V(Y_l)+O(h^2)$, and

$$\begin{aligned}\int y^2u^e_{l,0}dy &= Z_{l,0}^{-1}\int y^2(1-h\eta V+O(h^2))p_l(y)dy\\ &= Z_{l,0}^{-1}(\mathbf{E}Y_l^2 - h\eta\mathbf{E}[Y_l^2V(Y_l)]+O(h^2))\\ &= \mathbf{E}Y_l^2 + h\eta(\mathbf{E}Y_l^2\mathbf{E}V(Y_l)-\mathbf{E}[Y_l^2V(Y_l)])+O(h^2).\end{aligned}$$

Therefore $\frac{d}{d\xi}m(0) > 1$ if and only if $\sum_{l=1}^K \rho_l(2\Theta_l/\sigma^2)(\mathbf{E}Y_l^2\mathbf{E}V(Y_l)-\mathbf{E}[Y_l^2V(Y_l)]) > 0$. Note that $\mathbf{E}Y_l^2 = \sigma^2/2\Theta_l$, $\mathbf{E}V(Y_l) = (3/4)(\mathbf{E}Y_l^2)^2-(1/2)\mathbf{E}Y_l^2$, and $\mathbf{E}[Y_l^2V(Y_l)] = (15/4)(\mathbf{E}Y_l^2)^3-(3/2)(\mathbf{E}Y_l^2)^2$. Then the sufficient and necessary condition becomes

$$\sum_{l=1}^K\frac{\rho_l}{\Theta_l}\left(1-3\frac{\sigma^2}{2\Theta_l}\right) > 0.$$

B.3. Proof of Proposition 3.3. It is equivalent to show that $\sum_{l=1}^K\rho_l/\Theta_l \le \sum_{l=1}^K\rho_l\Theta_l\sum_{l=1}^K\rho_l/\Theta_l^2$. First note that by the Cauchy-Schwarz inequality,

$$\left(\sum_{l=1}^K\frac{\rho_l}{\Theta_l}\right)^2 = \left(\sum_{l=1}^K\frac{\sqrt{\rho_l}}{\Theta_l}\times\sqrt{\rho_l}\right)^2 \le \sum_{l=1}^K\frac{\rho_l}{\Theta_l^2}\sum_{l=1}^K\rho_l = \sum_{l=1}^K\frac{\rho_l}{\Theta_l^2}.$$

Then it suffices to show that $1 \leq \sum_{l=1}^K \rho_l \Theta_l \sum_{l=1}^K \rho_l/\Theta_l$. Again by the Cauchy-Schwarz inequality,

$$\sum_{l=1}^K \rho_l \Theta_l \sum_{l=1}^K \frac{\rho_l}{\Theta_l} \geq \sum_{l=1}^K \sqrt{\rho_l \Theta_l}\sqrt{\frac{\rho_l}{\Theta_l}} = \sum_{l=1}^K \rho_l = 1.$$

Appendix C. Proof of Lemma 5.2.

It suffices to show the case that $\delta = 1/n$. For each n, let $\phi_n \in A_{1/n}$, such that $\inf_{\phi \in A_{1/n}} I_h(\phi) \leq I_h(\phi_n) < \inf_{\phi \in A_{1/n}} I_h(\phi) + 1/n$; $\{I_h(\phi_n)\}$ are bounded from above by $\inf_{\phi\in A} I_h(\phi)+1 < \infty$. Because I_h is a good rate function, and by Proposition B.13 of [17], compactness is equivalent to sequentially compactness in $C([0,T], M_\infty(\mathbb{R}))$, $\{\phi_n\}$ has a convergent subsequence $\{\phi_{n_k}\}$ whose limit ϕ^* is in A. As I_h is lower semicontinuous, then

$$\lim_n \inf_{\phi\in A_{1/n}} I_h(\phi) = \lim_k I_h(\phi_{n_k}) = \liminf_k I_h(\phi_{n_k}) \geq I_h(\phi^*) \geq \inf_{\phi \in A} I_h(\phi).$$

Appendix D. Proofs in Section 6.

D.1. Proof of Theorem 6.1. We prove it in three steps. The first step is to show that there exists a uniform lower bound of $I_0(\phi)$, for all $\phi \in A$.

LEMMA D.1. *If $h = 0$, then $\inf_{\phi\in A} I_0(\phi) \geq 2\xi_0^2/(\sigma^2 T)$.*

Proof. For any $\phi \in A$, $a(t)$ denotes $\int y\phi(t, dy)$. We observe that

$$J_h(\phi) = \sup_{f:\langle \phi, f_y^2\rangle \neq 0} \langle \phi_t - \mathcal{L}_\phi^* \phi, f\rangle^2 / \langle \phi, f_y^2\rangle \overset{f\equiv y}{\geq} \langle \phi_t - \mathcal{L}_\phi^*\phi, y\rangle^2,$$

because $\langle \phi, 1\rangle = 1$. Note that $\langle \phi_t, y\rangle = \frac{d}{dt}\langle \phi, y\rangle = \frac{d}{dt} a(t)$, and

$$\langle \mathcal{L}_\phi^* \phi, y\rangle = \langle \frac{1}{2}\sigma^2 \phi_{yy} + \theta \frac{\partial}{\partial y}\left[(y - a(t))\phi\right], y\rangle = -\theta\langle (y-a(t))\phi, 1\rangle = 0.$$

Then after taking the infimum over $\phi \in A$, we have

$$\inf_{\phi\in A} I_0(\phi) \geq \inf_{\phi \in A} \frac{1}{2\sigma^2}\int_0^T \left(\frac{d}{dt}a\right)^2 dt = \inf_{\substack{a(t):0\leq t\leq T\\ a(0)=-\xi_0\\ a(T)=\xi_0}} \frac{1}{2\sigma^2}\int_0^T \left(\frac{d}{dt}a\right)^2 dt = \frac{2\xi_0^2}{\sigma^2 T}.$$

The last equality is obtained by a simple calculus of variation with the optimal path $a(t) = 2\xi_0 t/T - \xi_0$. ▯

The second step is to show that $I_0(p^e) = 2\xi_0^2/(\sigma^2 T)$. Then $\inf_{\phi\in A} I_0(\phi) = 2\xi_0^2/(\sigma^2 T)$ and therefore p^e is a minimizer for (5.5).

LEMMA D.2. *If $h = 0$, and*

$$p^e(t,y) = \frac{1}{\sqrt{2\pi\frac{\sigma^2}{2\theta}}} \exp\left\{-\frac{(y - a^e(t))^2}{2\frac{\sigma^2}{2\theta}}\right\}, \quad a^e(t) = \frac{2\xi_0}{T}t - \xi_0,$$

then $p^e \in A$ and $I_0(p^e) = 2\xi_0^2/(\sigma^2 T)$.

Proof. By reading (5.6) with $\phi = p^e$ and $h = 0$, we have $p^e_t = \mathcal{L}^*_{p^e} p^e + (p^e g)_y$. One can easily check that $\mathcal{L}^*_{p^e} p^e = 0$ and $p^e_t = -p^e_y \frac{d}{dt} a^e(t)$. Then we have $g = -\frac{d}{dt} a^e(t)$ and by (5.7),

$$I_0(p^e) = \frac{1}{2\sigma^2} \int_0^T \langle p^e, g^2 \rangle dt = \frac{1}{2\sigma^2} \int_0^T \left(\frac{d}{dt} a^e \right)^2 dt = \frac{2\xi_0^2}{\sigma^2 T}.$$

□

Finally we prove that for $h = 0$, the minimizer p^e is unique.

LEMMA D.3. *For $h = 0$, p^e is the unique minimizer for (5.5).*

Proof. From the previous lemmas, we find that if ϕ is a minimizer then $a(t) = \int y \phi(t, dy)$ must be $a^e(t)$, and $f = -\frac{d}{dt} a^e(t) y$ is a global maximizer of $J_0(\phi, \cdot)$. Then for any test function $\tilde{f}$, $\frac{d}{d\epsilon} J_0(\phi, -\frac{d}{dt} a^e(t) y + \epsilon \tilde{f}) = 0$ at $\epsilon = 0$. By a simple calculus of variations, ϕ satisfies the linear parabolic PDE:

$$\phi_t = \frac{1}{2} \sigma^2 \phi_{yy} + \theta \frac{\partial}{\partial y} \left[(y - a^e(t)) \phi \right] - \frac{d}{dt} a^e(t) \phi_y,$$

with the initial condition $\phi(0) = u^e_{-\xi_0}$, and that implies the uniqueness of the minimizer, which is p^e. □

D.2. Proof of Theorem 6.2 (Upper Bounds). Define the test function:

$$p^u(t, y) = \frac{1}{\sqrt{2\pi \frac{\sigma^2}{2\theta}}} \exp \left\{ - \frac{(y - a^u(t))^2}{2 \frac{\sigma^2}{2\theta}} \right\}, \quad a^u(t) = \frac{2\xi_b}{T} t - \xi_b.$$

We recall that from (2.3) and (2.5), ξ_b depends on h and $\xi_b \to \xi_0$ as $h \to 0$.

PROPOSITION D.4. *For any $\epsilon > 0$, then for all sufficiently small h,*

$$\inf_{\phi \in A} I_h(\phi) \leq \frac{1}{2\sigma^2} \int_0^T \langle p^u, (\frac{d}{dt} a^u - h(y^3 - y))^2 \rangle dt + \epsilon. \tag{D.1}$$

It is not difficult to see that the first term of the right hand side of (D.1) is equal to $2\xi_b^2/(\sigma^2 T)$ up to a term of order h as $h \to 0$.

Proof. We construct the test function $\phi^u \in A$ as follows:

$$\phi^u(t) = \begin{cases} (1 - \frac{t}{\delta T}) u^e_{-\xi_b} + \frac{t}{\delta T} p^u(t), & t \in [0, \delta T], \\ p^u(t), & t \in (\delta T, T - \delta T), \\ (1 - \frac{t - (T - \delta T)}{\delta T}) p^u(t) + \frac{t - (T - \delta T)}{\delta T} u^e_{\xi_b}, & t \in [T - \delta T, T], \end{cases}$$

where δT will be determined later. Note that $\inf_{\phi \in A} I_h(\phi) \leq I_h(\phi^u)$ so we just need to compute $I_h(\phi^u)$. Let g^u satisfy (5.6) for $\phi = \phi^u$. For $t \in (\delta T, T - \delta T)$, $\phi^u(t) = p^u(t)$, and it is easy to see that $p^u_t = -\frac{d}{dt} a^u p^u_y$ and $\mathcal{L}^*_{p^u} p^u = 0$. Therefore for $t \in (\delta T, T - \delta T)$, $g^u = -\frac{d}{dt} a^u - h(y^3 - y)$ by (5.6). From (5.7), we have

$$\begin{aligned} I_h(\phi^u) &= \frac{1}{2\sigma^2} \left(\int_0^{\delta T} + \int_{\delta T}^{T - \delta T} + \int_{T - \delta T}^T \right) \langle \phi^u, (g^u)^2 \rangle dt \\ &\leq \frac{1}{2\sigma^2} \int_0^T \langle p^u, (-\frac{d}{dt} a^u - h(y^3 - y))^2 \rangle dt + \frac{1}{2\sigma^2} \left(\int_0^{\delta T} + \int_{T - \delta T}^T \right) \langle \phi^u, (g^u)^2 \rangle dt. \end{aligned}$$

The rest is to show that for any $\epsilon > 0$, there exists a sufficiently small h such that the last term in the last equation is bounded by ϵ. It suffices to show that for any $\delta T > 0$, we can choose a sufficiently small h such that $\langle \phi^u, (g^u)^2\rangle$ is bounded by a δT-independent constant for $t \in [0, \delta T] \cup [T - \delta T, T]$. For $t \in [0, \delta T]$, because ϕ^u is simply the convex combination of $u^e_{-\xi_b}$ and p^u, the only term we need to worry is $(p^u(t) - u^e_{-\xi_b})/\delta T$ from computing $\phi^u_t(t)$. However, $p^u(t)$ is differentiable at $t = 0$ and $p^u(0) \to u^e_{-\xi_b}$ as $h \to 0$ so we can bound $(p^u(t) - u^e_{-\xi_b})/\delta T$ by a δT-independent constant with suitable h and we have the desired result. ◻

D.3. Proof of Theorem 6.2 (Lower Bounds). From (D.1), there exists some constant C such that $\inf_{\phi\in A} I_h(\phi) \le C$ for all $h \le h_0$. Then we can assume that $I_h(\phi) \le C$ for all $\phi \in A$ and all $h \le h_0$ without loss of generality. The following lemma shows that the first and second moments of all $\phi \in A$ are uniformly bounded.

LEMMA D.5. *Given $C > 0$, there exists $R > 0$ such that for any $\phi \in A$ with $I_h(\phi) \le C$ for some $h \ge 0$, then*

$$\sup_{t\in[0,T]} \langle \phi(t), y\rangle^2 \le \sup_{t\in[0,T]} \langle \phi(t), y^2\rangle \le R.$$

Proof. Recall that $M_R(\mathbb{R}) = \{\phi \in M_1(\mathbb{R}), \int \varphi(y)\phi(dy) \le R\}$ and $M_\infty(\mathbb{R}) = \cup_{R>0} M_R(\mathbb{R})$ with the inductive topology. Here we focus on the case that $\varphi = 1 + y^2$ in order to obtain the uniform result, and let $M^2_R(\mathbb{R})$ and $M^2_\infty(\mathbb{R})$ denote the spaces with the quadratic Lyapunov function φ.

The proof is an application of Theorem 5.1(c), Theorem 5.3 and Lemma 5.5 of [10]. By Theorem 5.1(c), if $\phi \in C([0,T], M^2_\infty(\mathbb{R}))$ with $\phi(0) = u^e_{-\xi_b}$ and $I_h(\phi) \le C$ for some $h \ge 0$, then ϕ is in an h-dependent compact set K. By Theorem 5.3 the compact set K is contained in $C([0,T], M^2_R(\mathbb{R}))$ for an h-dependent $R > 0$. Finally, by Lemma 5.5 and Theorem 5.1(c), it suffices to let $R \ge e^{\lambda T}(C + r)$, where r and λ satisfy

$$r \ge 2\int \varphi(y) u^e_{-\xi_b}(y)dy, \quad \lambda \ge \sup_{\mu\in M_1(\mathbb{R})} \langle \mu, \mathcal{L}_\mu\varphi + h\mathcal{M}\varphi + \frac{1}{2}\varphi_y^2\rangle / \langle \mu, \varphi\rangle,$$

with $\varphi(y) = 1 + y^2$. Obviously we can find the uniform r and λ for all $h \ge 0$ and also the uniform R. Then any ϕ of interest are in $C([0,T], M^2_R(\mathbb{R}))$ and thus have the uniform bounded first and second order moments. ◻

Now we derive that lower bound. The key idea is that because we have the universal upper bound for the first and second moments of all $\phi \in A$ and for all $h \le h_0$, Chebyshev's inequality implies the uniform convergence.

PROPOSITION D.6. *For any $\epsilon > 0$, then for all sufficiently small h,*

$$\inf_{\phi\in A} I_h(\phi) \ge \frac{1}{2\sigma^2}\int_0^T \langle p^u, (\frac{d}{dt}a^u - h(y^3 - y))^2\rangle dt - \epsilon. \tag{D.2}$$

Proof. Define $f^M = \iota * \hat{f}^M$, where $\hat{f}^M$ is a piecewise linear function and ι is the standard mollifier:

$$\hat{f}^M(y) = \begin{cases} y, & y \in (-M, M) \\ -y + 2M, & y \in [M, 2M] \\ -y - 2M, & y \in [-2M, -M] \\ 0, & \text{otherwise} \end{cases}, \quad \iota(y) = \begin{cases} Z\exp(\frac{1}{y^2-1}), & y^2 < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then f^M is a smooth function with the compact support $[-2M-1, 2M+1]$. In addition, $f^M(y) \equiv y$ on $(-M+1, M-1)$, $|f_x^M| \leq 1$, and $|f_{xx}^M|$ is uniformly bounded for all M and is nonzero only on $\cup_{i=-2}^{2}(iM-1, iM+1)$.

Because for all $\phi \in A$, $\langle \phi(t), (f_y^M)^2 \rangle \leq 1$, we can estimate the rate function:

$$\begin{aligned} I_h(\phi) &\geq \frac{1}{2\sigma^2} \int_0^T \langle \phi_t - \mathcal{L}_\phi^* \phi - h\mathcal{M}^* \phi, f^M \rangle^2 dt \\ &\geq \frac{1}{2\sigma^2 T} \left(\int_0^T \langle \phi_t - \mathcal{L}_\phi^* \phi - h\mathcal{M}^* \phi, f^M \rangle dt \right)^2 . \end{aligned}$$

Then we estimate the integrand term by term. By Lemma D.5, the following convergences are all uniform in $\phi \in A$ and $h \leq h_0$.

First we have

$$\int_0^T \langle \phi_t, f^M \rangle dt = \langle u^e_{\xi_b}, f^M \rangle - \langle u^e_{-\xi_b}, f^M \rangle.$$

$u^e_{\pm\xi_b}$ are exponentially decaying functions so $\langle u^e_{\pm\xi_b}, f^M \rangle$ converges to $\pm\xi_b$ rapidly as $M \to \infty$.

We note that $\langle \mathcal{L}_\phi^* \phi, f^M \rangle = \sigma^2 \langle \phi, f_{yy}^M \rangle / 2 - \theta \langle \phi, (y-a) f_y^M \rangle$. By reading the properties of f_{yy}^M and Chebyshev's inequality, we have $\langle \phi, f_{yy}^M \rangle \to 0$ as $M \to \infty$. We write $\langle \phi, (y-a) f_y^M \rangle$ as

$$\langle \phi, (y-a) f_y^M \rangle = a(1 - \langle \phi, f_y^M \rangle) + (\langle \phi, y f_y^M \rangle - a).$$

Since a is bounded and $\langle \phi, f_y^M \rangle \to 1$ as $M \to \infty$, $a(1 - \langle \phi, f_y^M \rangle) \to 0$ as $M \to \infty$. We see that

$$\begin{aligned} |\langle \phi, y f_y^M \rangle - a|^2 &\leq \left(2 \int_{(-M+1, M+1)^c} |y| \phi(dy) \right)^2 \\ &\leq 4 \int_{(-M+1, M+1)^c} y^2 \phi(dy) \int_{(-M+1, M+1)^c} \phi(dy). \end{aligned}$$

Again by Chebyshev's inequality, the right hand side vanishes as $M \to \infty$.

Finally we estimate $\langle \mathcal{M}^* \phi, f^M \rangle$. Since f^M is compactly supported,

$$|\langle \mathcal{M}^* \phi, f^M \rangle| = |\langle \phi, (y^3 - y) f_y^M \rangle| \leq (2M+1)^3 + (2M+1).$$

For a fixed M, we can choose a sufficiently small h such that $h|\langle \mathcal{M}^* \phi, f^M \rangle|$ is small.

Consequently, for any $\epsilon > 0$, we can first choose a sufficiently large M and then there exists a sufficiently small h such that

$$\inf_{\phi \in A} I_h(\phi) \geq \frac{2\xi_b^2}{\sigma^2 T} - \epsilon.$$

□

D.4. Proof of Lemma 6.4. We note that $p_t = -p_y \frac{d}{dt}a$ and therefore

$$\begin{aligned}\phi_t = -p_y \frac{d}{dt}a + h\sum_{n=2}^{\infty}\frac{d}{dt}b_n\frac{\partial^n}{\partial y^n}p - h\frac{d}{dt}a\sum_{n=2}^{\infty}b_n\frac{\partial^{n+1}}{\partial y^{n+1}}p \\ + h^2\sum_{n=2}^{\infty}\frac{d}{dt}c_n\frac{\partial^n}{\partial y^n}p - h^2\frac{d}{dt}a\sum_{n=2}^{\infty}c_n\frac{\partial^{n+1}}{\partial y^{n+1}}p.\end{aligned}$$

After collecting $O(1)$ terms in (5.6) and integrating over y, we have

$$-p\frac{d}{dt}a = \frac{1}{2}\sigma^2 p_y + \theta(y-a)p + pg^0 = pg^0.$$

Then $g^0 = -\frac{d}{dt}a$.

Now we collect $O(h)$ terms in (5.6) and integrating over y. We get

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{d}{dt}b_{n+1}\frac{\partial^n}{\partial y^n}p - \frac{d}{dt}a\sum_{n=2}^{\infty}b_n\frac{\partial^n}{\partial y^n}p = \frac{1}{2}\sigma^2\sum_{n=2}^{\infty}b_n\frac{\partial^{n+1}}{\partial y^{n+1}}p \\ + \theta(y-a)\sum_{n=2}^{\infty}b_n\frac{\partial^n}{\partial y^n}p + \sum_{n=0}^{3}\delta_n\frac{\partial^n}{\partial y^n}p + g^0\sum_{n=2}^{\infty}b_n\frac{\partial^n}{\partial y^n}p + \sum_{n=0}^{\infty}\beta_n\frac{\partial^n}{\partial y^n}p.\end{aligned}$$

Using the fact that

$$\frac{1}{2}\sigma^2\frac{\partial^{n+1}}{\partial y^{n+1}}p = -\theta(y-a)\frac{\partial^n}{\partial y^n}p - n\theta\frac{\partial^{n-1}}{\partial y^{n-1}}p,$$

we have

$$\sum_{n=1}^{\infty}\frac{d}{dt}b_{n+1}\frac{\partial^n}{\partial y^n}p = -\theta\sum_{n=1}^{\infty}(n+1)b_{n+1}\frac{\partial^n}{\partial y^n}p + \sum_{n=0}^{3}\delta_n\frac{\partial^n}{\partial y^n}p + \sum_{n=0}^{\infty}\beta_n\frac{\partial^n}{\partial y^n}p,$$

and the optimal β_n are obtained by comparing the coefficients.

D.5. Proof of Lemma 6.5. Let ψ^2 denote the anti-derivative of q^2. After collecting $O(h^2)$ terms in (5.6) and integrating over y. We have

$$\psi^2_t = \frac{1}{2}\sigma^2 q^2_y + \theta(y-a)q^2 + U(y)q^1 + q^2g^0 + q^1g^1 + pg^2. \tag{D.3}$$

Note that $pg^2 = \sum_{n=0}^{\infty}\gamma_n\frac{\partial^n}{\partial y^n}p$, so γ_0 is obtained by integrating (D.3) from $y=-\infty$ to $y=\infty$. Then we have $\gamma_0 = -\langle q^1, U(y)+g^1\rangle$.

D.6. Proof of Proposition 6.8. We write $a(t) = a_0(t) + ha_1(t) + O(h^2)$ with $a_0(t) = 2\xi_0 t/T - \xi_0$ and $a_1(t) = 2\xi_1 t/T - \xi_1$. Then we put $a(t)$ into (6.8) and we have

$$\inf_{\phi\in A} I_h(\phi) = \frac{1}{2\sigma^2}\int_0^T\left\{(\frac{d}{dt}a_0)^2 + 2h(\frac{d}{dt}a_0)(a_0^3 + (3\frac{\sigma^2}{2\theta}-1)a_0 + \frac{d}{dt}a_1)\right\}dt + O(h^2).$$

We note that $\frac{d}{dt}a_0$ is a constant, and $a_0(t)$ and $a_0^3(t)$ are odd functions with respect to $t = T/2$. Then

$$\begin{aligned}\inf_{\phi\in A} I_h(\phi) &= \frac{1}{2\sigma^2}\int_0^T\left\{\left(\frac{d}{dt}a_0\right)^2 + 2h\frac{d}{dt}a_0\frac{d}{dt}a_1\right\}dt + O(h^2) \\ &= \frac{1}{2\sigma^2}\int_0^T\left\{\left(\frac{2\xi_0}{T}\right)^2 + 2h\frac{2\xi_0}{T}\frac{2\xi_1}{T}\right\}dt + O(h^2) = \frac{2\xi_0}{\sigma^2 T}(\xi_0 + 2h\xi_1) + O(h^2).\end{aligned}$$

Appendix E. Proofs in Section 7.

E.1. Proof of Proposition 7.1. The system of SDEs (7.1) for the vector $\bar{X}(t) = (\bar{x}_k(t))_{k=1,\dots,K}$ has the form

$$d\bar{X}(t) = M\bar{X}(t) + \frac{\sigma}{\sqrt{N}} R^{-1/2} d\bar{W}(t)$$

where $\bar{W}(t) = (\bar{w}_k(t))_{k=1,\dots,K}$. This system can be solved:

$$\bar{X}(t) = e^{Mt}\bar{X}(0) + \frac{\sigma}{\sqrt{N}} \int_0^t e^{M(t-s)} R^{-1/2} d\bar{W}(s)$$

If $\bar{x}_k(0) = -\xi_b$, then, using the fact that the uniform vector is in the null space of M, we have $e^{Mt}\bar{X}(0) = \bar{X}(0)$. As a corollary we get the explicit representation of the empirical mean:

$$\bar{x}(t) = -\xi_b + \frac{\sigma}{\sqrt{N}} \int_0^t \rho^T e^{M(t-s)} R^{-1/2} d\bar{W}(s)$$

This shows the desired result.

E.2. Proof of Proposition 7.2. The expansion of ξ_b^2 follows from the explicit expression (3.5). The expansion of σ_T^2 follows from the expansion of (7.3) and uses the properties of the matrix M. We have $M = -\bar{\theta}\bar{M} - \delta\bar{\theta}N$, with

$$\begin{aligned} \bar{M} &= I - u\rho^T, \text{ with } \quad u_i = 1, \quad i = 1, \dots, K, \\ N_{ij} &= \alpha_i(\delta_{ij} - \rho_j), \quad i, j = 1, \dots, K, \end{aligned}$$

The matrix $\bar{M}$ satisfies $\bar{M}^n = \bar{M}$ for all $n \geq 1$ and therefore

$$e^{-\bar{\theta}\bar{M}t} = \sum_{n=0}^{\infty} \frac{(-\bar{\theta}t)^n}{n!} \bar{M}^n = I + \sum_{n=1}^{\infty} \frac{(-\bar{\theta}t)^n}{n!} \bar{M} = I + (e^{-\bar{\theta}t} - 1)\bar{M}.$$

We have

$$e^{Mt} = \sum_{n=0}^{\infty} \frac{(-\bar{\theta}t)^n}{n!} (\bar{M} + \delta N)^n$$

Using the fact that $\bar{M}^T\rho = 0$ (and again that $\bar{M}^n = \bar{M}$ for $n \geq 1$), we can expand

$$\begin{aligned} \rho^T e^{Mt} = \rho^T &+ \delta\rho^T \big\{ (-\bar{\theta}t)N + (e^{-\bar{\theta}t} - 1 + \bar{\theta}t)N\bar{M} \big\} \\ &+ \delta^2 \rho^T \Big\{ \frac{(\bar{\theta}t)^2}{2} N^2 + \big[e^{-\bar{\theta}t} - 1 + \bar{\theta}t - \frac{(\bar{\theta}t)^2}{2} \big] \big[N^2\bar{M} - 3(N\bar{M})^2 + N\bar{M}N \big] \\ &- \bar{\theta}t \big[e^{-\bar{\theta}t} - 1 + \bar{\theta}t \big] (N\bar{M})^2 \rho \Big\} + O(\delta^3). \end{aligned}$$

Using the fact that $\bar{M}^T N^T \rho = N^T \rho$ and $\bar{M}^T (N^T)^2 \rho = (N^T)^2 \rho$ this can be simplified into

$$\rho^T e^{Mt} = \rho^T + \delta\rho^T (e^{-\bar{\theta}t} - 1)N + \delta^2 \rho^T \big[(\bar{\theta}t)^2 - (1 + \bar{\theta}t)(e^{-\bar{\theta}t} - 1 + \bar{\theta}t) \big] N^2 + O(\delta^3)$$

Consequently

$$\begin{aligned}\rho^T e^{Mt} R^{-1} (e^{Mt})^T \rho &= \rho^T (I + (e^{-\bar{\theta}t} - 1)\bar{M}) R^{-1} (I + (e^{-\bar{\theta}t} - 1)\bar{M}^T)\rho \\ &+ 2\delta \rho^T (e^{-\bar{\theta}t} - 1) N R^{-1} (I + (e^{-\bar{\theta}t} - 1)\bar{M}^T)\rho \\ &+ 2\delta^2 \rho^T \left[(\bar{\theta}t)^2 - (1 + \bar{\theta}t)(e^{-\bar{\theta}t} - 1 + \bar{\theta}t)\right] N^2 R^{-1} (I + (e^{-\bar{\theta}t} - 1)\bar{M}^T)\rho \\ &+ \delta^2 \rho^T (e^{-\bar{\theta}t} - 1) N R^{-1} (e^{-\bar{\theta}t} - 1) N^T \rho + O(\delta^3).\end{aligned}$$

Using the fact that $\bar{M}^T \rho = 0$ and $N R^{-1} \rho = Nu = 0$, we obtain

$$\rho^T e^{Mt} R^{-1} (e^{Mt})^T \rho = \rho^T R^{-1} \rho + \delta^2 (1 - e^{-\bar{\theta}t})^2 \rho^T N R^{-1} N^T \rho + O(\delta^3)$$

We have $\rho^T R^{-1} \rho = 1$ and $\rho^T N R^{-1} N^T \rho = \sum_k \rho_k \alpha_k^2$ which gives the expansion of the variance σ_T^2.

Finally the expansion of the transition probability can be obtained by substituting the expansions of ξ_b^2 and σ_T^2 into (7.4).

REFERENCES

[1] G. Ben Arous and A. Guionnet, *Large deviations for Langevin spin glass dynamics*, Probab. Theory Related Fields, 102 (1995), pp. 455–509.

[2] G. Ben Arous and O. Zeitouni, *Increasing propagation of chaos for mean field models*, Ann. Inst. H. Poincaré Probab. Statist., 35 (1999), pp. 85–102.

[3] S. Battiston, D. D. Gatti, M. Gallegati, B. Greenwald, and J. E. Stiglitz, *Default Cascades: When Does Risk Diversification Increase Stability?*, (2011).

[4] S. Battiston, D. Delli Gatti, M. Gallegati, B. C. Greenwald, and J. E. Stiglitz, *Liaisons Dangereuses: Increasing Connectivity, Risk Sharing, and Systemic Risk*, Working Paper 15611, National Bureau of Economic Research, January 2009.

[5] N. Beale, D. G. Rand, H. Battey, K. Croxson, R. M. May, and M. A. Nowak, *Individual versus systemic risk and the Regulator's Dilemma*, Proceedings of the National Academy of Sciences, 108 (2011), p. 12647.

[6] D. Bisias, M. Flood, A. Lo, and S. Valavanis, *A Survey of Systemic Risk Analytics*, (2012).

[7] A. Budhiraja, P. Dupuis, and M. Fischer, *Large deviation properties of weakly interacting processes via weak convergence methods*, ArXiv e-prints, (2010).

[8] A. Budhiraja, P. Dupuis, and V. Maroulas, *Large deviations for infinite dimensional stochastic dynamical systems*, Ann. Probab., 36 (2008), pp. 1390–1420.

[9] D. A. Dawson, *Critical dynamics and fluctuations for a mean-field model of cooperative behavior*, J. Statist. Phys., 31 (1983), pp. 29–85.

[10] D. A. Dawson and J. Gärtner, *Large deviations from the McKean-Vlasov limit for weakly interacting diffusions*, Stochastics, 20 (1987), pp. 247–308.

[11] ———, *Multilevel large deviations and interacting diffusions*, Probab. Theory Related Fields, 98 (1994), pp. 423–487.

[12] D. A. Dawson and P. Del Moral, *Large deviations for interacting processes in the strong topology*, in Statistical modeling and analysis for complex data problems, vol. 1 of GERAD 25th Anniv. Ser., Springer, New York, 2005, pp. 179–208.

[13] A. Dembo and O. Zeitouni, *Large deviations techniques and applications*, vol. 38 of Stochastic Modelling and Applied Probability, Springer-Verlag, Berlin, 2010. Corrected reprint of the second (1998) edition.

[14] J.-P. Fouque and L.-H. Sun, *Systemic Risk Illustrated*, (2012). preprint.

[15] M. I. Freidlin and A. D. Wentzell, *Random perturbations of dynamical systems*, vol. 260 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, New York, second ed., 1998. Translated from the 1979 Russian original by Joseph Szücs.

[16] P. Gai and S. Kapadia, *Contagion in financial networks*, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science, 466 (2010), p. 2401.

[17] J. Gärtner, *On the McKean-Vlasov limit for interacting diffusions*, Math. Nachr., 137 (1988), pp. 197–248.

[18] H. Haken, *Synergetics: an introduction*, vol. 1 of Springer Series in Synergetics, Springer-Verlag, Berlin, third ed., 1983. Nonequilibrium phase transitions and self-organization in physics, chemistry, and biology.

[19] A. G. Haldane, *Rethinking the financial network*, Speech delivered at the Financial Student Association, Amsterdam, (2009).

[20] A. G. Haldane and R. M. May, *Systemic risk in banking ecosystems*, Nature, 469 (2011), pp. 351–355.

[21] S. Herrmann, P. Imkeller, and D. Peithmann, *Large deviations and a Kramers' type law for self-stabilizing diffusions*, Ann. Appl. Probab., 18 (2008), pp. 1379–1423.

[22] R. Ibragimov, D. Jaffee, and J. Walden, *Diversification disasters*, Journal of Financial Economics, 99 (2011), pp. 333 – 348.

[23] J. Lorenz, S. Battiston, and F. Schweitzer, *Systemic risk in a unifying framework for cascading processes on networks*, Eur. Phys. J. B, 71 (2009), pp. 441–460.

[24] R. M. May and N. Arinaminpathy, *Systemic risk: the dynamics of model banking systems*, Journal of the Royal Society Interface, 7 (2010), pp. 823–838.

[25] S. Méléard, *Asymptotic behaviour of some interacting particle systems; McKean-Vlasov and Boltzmann models*, in Probabilistic models for nonlinear partial differential equations (Montecatini Terme, 1995), vol. 1627 of Lecture Notes in Math., Springer, Berlin, 1996, pp. 42–95.

[26] P. Del Moral and J. Garnier, *Genealogical particle analysis of rare events*, Ann. Appl. Probab., 15 (2005), pp. 2496–2534.

[27] P. Del Moral and A. Guionnet, *Large deviations for interacting particle systems: applications to non-linear filtering*, Stochastic Process. Appl., 78 (1998), pp. 69–95.

[28] P. Del Moral and E. Rio, *Concentration inequalities for mean field particle models*, Ann. Appl. Probab., 21 (2011), pp. 1017–1052.

[29] E. Nier, J. Yang, T. Yorulmazer, and A. Alentorn, *Network models and financial stability*, Journal of Economic Dynamics and Control, 31 (2007), pp. 2033 – 2060. Tenth Workshop on Economic Heterogeneous Interacting Agents.

[30] J. E. Stiglitz, *Risk and Global Economic Architecture: Why Full Financial Integration May Be Undesirable*, Working Paper 15718, National Bureau of Economic Research, February 2010.

[31] H. Tanaka, *Limit theorems for certain diffusion processes with interaction*, in Stochastic analysis (Katata/Kyoto, 1982), vol. 32 of North-Holland Math. Library, North-Holland, Amsterdam, 1984, pp. 469–488.