QUANTUM ERGODIC RESTRICTION FOR CAUCHY DATA: INTERIOR QUE AND RESTRICTED QUE

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Abstract. We prove a quantum ergodic restriction theorem for the Cauchy data of a sequence of quantum ergodic eigenfunctions on a hypersurface H of a Riemannian manifold (M, g). The technique of proof is to use a Rellich type identity to relate quantum ergodicity of Cauchy data on H to quantum ergodicity of eigenfunctions on the global manifold M. This has the interesting consequence that if the eigenfunctions are quantum unique ergodic on the global manifold M, then the Cauchy data is automatically quantum unique ergodic on H with respect to operators whose symbols vanish to order one on the glancing set of unit tangential directions to H.

1. Introduction

This article is concerned with the QER (quantum ergodic restriction) problem for hypersurfaces in compact Riemannian manifolds (M, g). We consider the eigenvalue problem on M

$$\begin{cases} -\Delta_g \varphi_j = \lambda_j^2 \varphi_j, \quad \langle \varphi_j, \varphi_k \rangle = \delta_{jk} \\ \\ B\varphi_j = 0 \text{ on } \partial M \end{cases},$$

where $\langle f, g \rangle = \int_M f \bar{g} dV$ (dV is the volume form of the metric) and where B is the boundary operator, e.g. $B\varphi = \varphi|_{\partial M}$ in the Dirichlet case or $B\varphi = \partial_\nu \varphi|_{\partial M}$ in the Neumann case. We also allow $\partial M = \emptyset$. We work with the semiclassical calculus as in the references [Bu, DZ, HaZe, TZ1], to which we refer for background. We set the Planck constant equal to $h_j = \lambda_j^{-1}$; for notational simplicity we often drop the subscript j. We then let φ_h be a corresponding orthonormal basis of eigenfunctions with eigenvalue h^{-2}, so that the eigenvalue problem takes the semi-classical form,

$$\begin{cases} (-h^2 \Delta_g - 1)\varphi_h = 0, \\ \\ B\varphi_h = 0 \text{ on } \partial M \end{cases}, \tag{1.1}$$

where $B = I$ or $B = hD_\nu$ in the Dirichlet or Neumann cases respectively.

Let $H \subset M$ be a smooth hypersurface which does not meet ∂M if $\partial M \neq \emptyset$. The main result (Theorem 1) is that the full semiclassical Cauchy data

$$CD(\varphi_h) := \{(\varphi_h|_H, \; hD_\nu \varphi_h|_H)\} \tag{1.2}$$

of eigenfunctions is *always* quantum ergodic along any hypersurface $H \subset M$ if the eigenfunctions are quantum ergodic on the global manifold M. The proof is a generalization of the boundary case where $H = \partial M$, which was proved in [HaZe] and in [Bu]. Our proof is modeled on that of [Bu], developing ideas of [GL] (see also [CTZ2] for an abstract microlocal approach). This automatic QER property of Cauchy data stands in contrast to the conditional nature of the QER property for the Dirichlet data alone, which requires an "asymmetry" condition on H with respect to geodesics [TZ1, TZ2, DZ]. Note that in the boundary case $H = \partial M$, the Dirichlet resp. Neumann boundary condition kills one of the two components of the Cauchy data, so that the Cauchy data QER theorem appears the same as the QER theorem for Neumann data (resp. Dirichlet data) alone.

The automatic QER property of Cauchy data has an interesting and possibly surprising consequence for QUE (quantum unique ergodicity). An orthonormal basis $\{\varphi_h\}$ of $-h^2\Delta_g$-eigenfunctions is called QUE on M if $\langle A\varphi_h, \varphi_h\rangle \to \omega(A)$ for all pseudodifferential operators $A \in \Psi^0(M)$. Here, $\omega(A) = \int_{S^*M} \sigma_A d\mu_L$ where $d\mu_L$ is normalized Liouville measure, $\Psi^0(M)$ is the space of 0-order semiclassical pseudodifferential operators on M, and σ_A is the principal symbol of A in $\Psi^0(M)$. Then QUE on (M, g) implies a certain QUER (quantum uniquely ergodic restriction) property of the Cauchy data on an embedded orientable separating hypersurface H. In Corollary 1.1 it is proved that QUE on (M, g) implies QUER with respect to the subalgebra of semiclassical pseudodifferential operators on H whose symbols vanish to order 1 along S^*H. This happens because the passage from QUE in the ambient manifold to QUER on the hypersurface involves multiplying the symbols by a certain factor which vanishes to order one on S^*H, i.e. the unit directions (co-)tangent to H. Therefore, QUE in the ambient manifold does not imply QUER for all pseudodifferential operators on H, and indeed the test operators damp out the possible modes which concentrate microlocally on H. We nevertheless refer to it as a QUER property because it holds for the entire sequence of eigenfunctions; there is no need to remove a subsequence of density zero for the subalgebra limits.

We introduce a hypersurface H, which we assume to be orientable, embedded, and separating in the sense that

$$M\backslash H = M_+ \cup M_-$$

where $M_\pm$ are domains with boundary in M. This is not a restrictive assumption since we can arrange that any hypersurface is part of the boundary of a domain.

We define the *microlocal lift* of the Neumann data by

$$\int_{B^*H} a\, d\Phi_h^N := \langle Op_H(a) hD_\nu \varphi_h|_H, hD_\nu \varphi_h|_H \rangle_{L^2(H)},$$

where $Op_H(a)$ is the semiclassical h-quantization of $a \in S^0(H)$, the space of zeroth-order semiclassical symbols on H. We also define the *renormalized microlocal lift* of the Dirichlet data by

$$\int_{B^*H} a\, d\Phi_h^{RD} := \langle Op_H(a)(1 + h^2\Delta_H)\varphi_h|_H, \varphi_h|_H\rangle_{L^2(H)}.$$

These distributions are asymptotically positive, but are not normalized to have mass one and may tend to infinity. To be concrete, recall that in this note, $-h^2\Delta_H$ is the non-negative tangential Laplacian, so that the operator $(1+h^2\Delta_H)$ is characteristic precisely on the glancing set S^*H of H. In this sense, we have renormalized the Dirichlet data by damping out the whispering gallery components.

For the full Cauchy data we define the microlocal lift $d\Phi_h^{CD}$ by

$$d\Phi_h^{CD} = d\Phi_h^N + d\Phi_h^{RD}. \tag{1.3}$$

Our first result is that the Cauchy data of a sequence of quantum ergodic eigenfunctions restricted to H is automatically QER for semiclassical pseudodifferential operators with symbols vanishing on the glancing set S^*H, i.e. that

$$d\Phi_h^{CD} \to \omega,$$

where

$$\omega(a) = \frac{4}{\mu(S^*M)} \int_{B^*H} a_0(x',\xi')(1-|\xi'|^2)^{1/2} d\sigma$$

is the limit state of Theorem 1. Here, a_0 is the principal symbol of the semi-classical pseudodifferential operator a^w quantizing the semiclassical symbol a. This was proved in a different way in [TZ1] in the case of piecewise smooth Euclidean domains. The assumption $H \cap \partial M = \emptyset$ is for simplicity of exposition and because the case $H = \partial M$ is already known.

Theorem 1. *Suppose $H \subset M$ is a smooth, codimension 1 embedded orientable separating hypersurface and assume $H \cap \partial M = \emptyset$ if $\partial M \neq \emptyset$. Assume that $\{\varphi_h\}$ is a quantum ergodic sequence. Then the appropriately renormalized Cauchy data $d\Phi_h^{CD}$ of φ_h is quantum ergodic in the sense that for any $a^w \in \Psi^0(H)$,*

$$\langle a^w hD_\nu \varphi_h|_H, hD_\nu\varphi_h|_H\rangle_{L^2(H)} + \left\langle a^w(1+h^2\Delta_H)\varphi_h|_H, \varphi_h|_H\right\rangle_{L^2(H)}$$

$$\to_{h\to 0^+} \tfrac{4}{\mu(S^*M)} \int_{B^*H} a_0(x',\xi')(1-|\xi'|^2)^{1/2} d\sigma,$$

where $a_0(x',\xi')$ is the principal symbol of a^w, $-h^2\Delta_H$ is the induced tangential (semiclassical) Laplacian with principal symbol $|\xi'|^2$, μ is the Liouville measure on S^*M, and $d\sigma$ is the standard symplectic volume form on B^*H.

The proof simply relates the interior and restricted microlocal lifts and reduces the QER property along H to the QE property of the ambient manifold. If we assume that QUE holds in the ambient manifold, we automatically get QUER, which is our first Corollary:

Corollary 1.1. *Suppose that $\{\varphi_h\}$ is QUE on M. Then the distributions $\{d\Phi_h^{CD}\}$ have a unique weak* limit*

$$\omega(a) := \frac{4}{\mu(S^*M)} \int_{B^*H} a_0(x',\xi')(1-|\xi'|^2)^{1/2} d\sigma.$$

We note that $d\Phi_h^{CD}$ involves the renormalized microlocal lift $d\Phi_h^{RD}$ rather than the microlocal lift of the Dirichlet data. However, in Theorem 2, we see that the analogue of Theorem 1 holds for a *density one subsequence* if we use the further renormalized distributions $d\Phi_h^D + d\Phi_h^{RN}$ where the microlocal lift $d\Phi_h^D \in \mathcal{D}'(B^*H)$ of the Dirichlet data of φ_h is defined by

$$\int_{B^*H} a\, d\Phi_h^D := \langle Op_H(a)\varphi_h|_H, \varphi_h|_H\rangle_{L^2(H)},$$

and

$$\int_{B^*H} a\, d\Phi_h^{RN} := \langle (1+h^2\Delta_H + i0)^{-1} Op_H(a) hD_\nu\varphi_h|_H, hD_\nu\varphi_h|_H\rangle_{L^2(H)}.$$

Theorem 2. *Suppose $H \subset M$ is a smooth, codimension 1 embedded orientable separating hypersurface and assume $H \cap \partial M = \emptyset$ if $\partial M \neq \emptyset$. Assume that $\{\varphi_h\}$*

is a quantum ergodic sequence. Then, there exists a sub-sequence of density one as $h \to 0^+$ *such that for all* $a^w \in \Psi^0(H)$,

$$\left\langle (1 + h^2\Delta_H + i0)^{-1} a^w hD_\nu \varphi_h|_H, hD_\nu \varphi_h|_H \right\rangle_{L^2(H)} + \langle a^w \varphi_h|_H, \varphi_h|_H \rangle_{L^2(H)}$$

$$\to_{h\to 0^+} \tfrac{4}{\mu(S^*M)} \textstyle\int_{B^*H} a_0(x', \xi')(1 - |\xi'|^2)^{-1/2} d\sigma,$$

where $a_0(x', \xi')$ *is the principal symbol of* a^w.

The additional step in the proof is a pointwise local Weyl law as in [TZ1] section 8.4 showing that only a sparse set of eigenfunctions could scar on the glancing set S^*H. This is precisely the step which is not allowed in the QUER problem. Therefore, QUER for all $Op_H(a)$ might fail for this rescaled problem; to determine whether it holds for all $Op_H(a)$ we would need a new idea. However, the following is a direct consequence of Theorem 2

Corollary 1.2. *Suppose that* $\{\varphi_h\}$ *is QUE on* M. *Then the distributions* $\{d\Phi_h^D + d\Phi_h^{RN}\}$ *have a unique weak* limit*

$$\omega(a) := \frac{4}{\mu(S^*M)} \int_{B^*H} a_0(x', \xi')(1 - |\xi'|^2)^{-1/2} d\sigma$$

with respect to the subclass of symbols which vanish on S^*H.

We prove Theorem 1 by means of a Rellich identity adapted from [GL, Bu]. It is also possible to prove the theorem using the layer potential approach in Step 2 (Proof of (7.4)) in section 7 of [HaZe]. To adapt this proof, one would need to introduce a semi-classical Green's function in place of the Euclidean Green's function, verify that it has the properties of the latter in section 4 of [HaZe], and then go through the proof of Step 2. Despite the authors' fondness for the layer potential approach, this proof is much longer than the infinitesimal Rellich identity approach and we have decided to omit the details.

Acknowledgements. The first version of this article was written at the same time as [TZ1, TZ2] but its completion was post-poned while the authors proved the QER phenomenon for Dirichlet data alone. We were further stimulated to complete the article by a discussion with Peter Sarnak at the Spectral Geometry conference at Dartmouth in July, 2010 in which we debated whether QUE in the ambient domain implies QUER along H. We said 'yes', Sarnak said 'no'; Corollary 1.1 explains the sense in which both answers are right.

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2. Rellich approach: Proof of Theorem 1

We have assumed H is a separating hypersurface, so that H is the boundary of a smooth open submanifold of M, $H = \partial M_+ \subset M$. There is no loss of generality in this assumption, since we may always use a cutoff to a subset of H. We then use a Rellich type identity to write the integral of a commutator over M_+ as a sum of integrals over the boundary (of course the same argument would apply on $M_- = M \setminus M_+$). The argument is partially motivated by Burq's proof of boundary quantum ergodicity (ie. the case $H = \partial M$).

Let $x = (x', x_n)$ be normal coordinates in a small tubular neighbourhood $H(\epsilon)$ of H defined near a point $x_0 \in H$. In these coordinates we can locally write

$$H(\epsilon) := \{(x', x_n) \in U \times \mathbb{R}, \, |x_n| < \epsilon\}.$$

Here $U \subset \mathbb{R}^{n-1}$ is a coordinate chart containing $x_0 \in H$ and $\epsilon > 0$ is arbitrarily small but for the moment, fixed. We let $\chi \in C_0^\infty(\mathbb{R})$ be a cutoff with $\chi(x) = 0$ for $|x| \geq 1$ and $\chi(x) = 1$ for $|x| \leq 1/2$. In terms of the normal coordinates,

$$-h^2 \Delta_g = \frac{1}{g(x)} hD_{x_n} g(x) hD_{x_n} + R(x_n, x', hD_{x'})$$

where, R is a second-order h-differential operator along H with coefficients that depend on x_n, and $R(0, x', hD_{x'}) = -h^2 \Delta_H$ is the induced tangential semiclassical Laplacian on H.

By Green's formula and (1.1) we get the Rellich identity

$$\begin{aligned} \frac{i}{h} \int_{M_+} & \left([-h^2 \Delta_g, \, A(x, hD_x)] \, \varphi_h(x) \right) \overline{\varphi_h(x)} \, dx \\ &= \int_H (hD_\nu \, A(x', x_n, hD_x) \varphi_h|_H) \, \overline{\varphi_h}|_H \, d\sigma_H \\ &\quad + \int_H (\, A(x', x_n, hD_x) \, \varphi_h|_H) \, \overline{hD_\nu \varphi_h}|_H \, d\sigma_H. \end{aligned} \tag{2.1}$$

Here, $D_{x_j} = \frac{1}{i} \frac{\partial}{\partial x_j}$, $D_\nu = \frac{1}{i} \partial_\nu$ etc., where ∂_ν is the interior unit normal to M_+. Also, $A(x, hD_x)$ is a semiclassical pseudodifferential operator on M.

We then chooe $A(x, hD_x)$ to be

$$A(x', x_n, hD_x) = \chi(\frac{x_n}{\epsilon}) \, hD_{x_n} a(x', hD').$$

Since $\chi(0) = 1$ it follows that the second term on the right side of (2.1) is just

$$\langle a(x', hD') hD_{x_n} \varphi_h|_H, hD_{x_n} \varphi_h|_H \rangle \, . \tag{2.2}$$

The first term on right hand side of (2.1) equals

(2.3)

$$\begin{aligned} &\int_H (hD_n(\chi(x_n/\epsilon) hD_n a(x', hD') \varphi_h))|_{x_n=0} \overline{\varphi_h}|_{x_n=0} \, d\sigma_H \\ &= \int_H \left(\chi(x_n/\epsilon) a(x', hD') (hD_n)^2 \varphi_h + \frac{h}{i\epsilon} \chi'(x_n/\epsilon) hD_n a(x', hD') \varphi_h \right) \Big|_{x_n=0} \overline{\varphi_h}|_{x_n=0} d\sigma_H \\ &= \int_H (\chi(x_n/\epsilon) a(x', hD') (1 - R(x_n, x', hD')) \varphi_h)|_{x_n=0} \overline{\varphi_h}|_{x_n=0} d\sigma_H, \end{aligned}$$

since $\chi'(0) = 0$ and $((hD_n)^2 + R)\varphi_h = \varphi_h$ in these coordinates.

It follows from (2.1)-(2.3) that

$$\langle a^w hD_\nu \varphi_h|_H, hD_\nu \varphi_h|_H \rangle_{L^2(H)} + \left\langle a^w (1 + h^2 \Delta_H) \varphi_h|_H, \varphi_h|_H \right\rangle_{L^2(H)} \tag{2.4}$$

$$= \left\langle Op_h \left(\left\{ \xi_n^2 + R(x_n, x', \xi'), \, \chi(\frac{x_n}{\epsilon}) \xi_n a(x', \xi') \right\} \right) \varphi_h, \, \varphi_h \right\rangle_{L^2(M_+)} + \mathcal{O}_\epsilon(h). \tag{2.5}$$

We now assume that φ_h is a sequence of quantum ergodic eigenfunctions, and take the $h \to 0^+$ limit on both sides of (2.4). We apply interior quantum ergodicity to the term on the right side of (2.4). We compute

$$\begin{aligned}\left\{\xi_n^2 + R(x_n, x', \xi'),\, \chi(\tfrac{x_n}{\epsilon})\xi_n a(x',\xi')\right\} &= \tfrac{2}{\epsilon}\chi'(\tfrac{x_n}{\epsilon})\xi_n^2 a(x',\xi') \\ &\quad + \chi(\tfrac{x_n}{\epsilon})R_2(x', x_n, \xi'),\end{aligned} \tag{2.6}$$

where R_2 is a zero order symbol. Let $\chi_2 \in \mathcal{C}^\infty$ satisfy $\chi_2(t) = 0$ for $t \leq -1/2$, $\chi_2(t) = 1$ for $t \geq 0$, and $\chi_2'(t) > 0$ for $-1/2 < t < 0$, and let ρ be a boundary defining function for M_+. Then $\chi_2(\rho/\delta)$ is 1 on M_+ and 0 outside a $\delta/2$ neighbourhood. Now the assumptions that the sequence φ_h is quantum ergodic implies that the matrix element of the second term on the right side of (2.6) is bounded by

$$\begin{aligned}&\left|\langle Op_h(\chi(x_n/\epsilon)R_2(x,\xi'))\varphi_h, \varphi_h\rangle_{L^2(M_+)}\right| \\ &\quad \leq \|\chi_2(\rho/\delta)\chi(x_n/\epsilon)\varphi_h\|_{L^2(M)}\|\tilde{\chi}_2(\rho/\delta)\tilde{\chi}(x_n/\epsilon)\varphi_h\|_{L^2(M)} \\ &\quad = \mathcal{O}_\delta(\epsilon) + o_{\delta,\epsilon}(1),\end{aligned}$$

where $\tilde{\chi}$ and $\tilde{\chi}_2$ are smooth, compactly supported functions which are one on the support of χ and χ_2 respectively. Here, the last line follows from interior quantum ergodicity of the φ_h since the volume of the supports of $\chi(x_n/\epsilon)$ and $\tilde{\chi}(x_n/\epsilon)$ is comparable to ϵ.

To handle the matrix element of the first term on the right side of (2.6), we note that $\chi'(x_n/\epsilon)|_{M_+} = \tilde{\chi}'(x_n/\epsilon)$ for a smooth function $\tilde{\chi} \in \mathcal{C}^\infty(M)$ satisfying $\tilde{\chi} = 1$ in a neighbourhood of $M \setminus M_+$ and zero inside a neighbourhood of H. Then, again by interior quantum ergodicity, we have

$$\begin{aligned}&2\left\langle Op_h\left(\frac{1}{\epsilon}\chi'(\frac{x_n}{\epsilon})\,\xi_n^2 a(x',\xi')\right)\varphi_h,\ \varphi_h\right\rangle_{L^2(M_+)} \\ &= 2\left\langle Op_h\left(\frac{1}{\epsilon}\tilde{\chi}'(\frac{x_n}{\epsilon})\,\xi_n^2 a(x',\xi')\right)\varphi_h,\ \varphi_h\right\rangle_{L^2(M)} \\ &= \frac{2}{\mu(S^*M)}\int_{S^*M}\frac{1}{\epsilon}\tilde{\chi}'(\frac{x_n}{\epsilon})(1 - R(x',x_n,\xi'))a(x',\xi')\,d\mu + O(\epsilon) + o_\epsilon(1) \\ &= \frac{2}{\mu(S^*M)}\int_{S^*M_+}\frac{1}{\epsilon}\chi'(\frac{x_n}{\epsilon})(1 - R(x',x_n,\xi'))a(x',\xi')\,d\mu + O(\epsilon) + o_\epsilon(1),\end{aligned} \tag{2.7}$$

since $\tilde{\chi}'$ and χ' are supported inside M_+. Combining the above calculations yields

$$\begin{aligned}&\langle a^w hD_\nu\varphi_h|_H, hD_\nu\varphi_h|_H\rangle_{L^2(H)} + \left\langle a^w(1 + h^2\Delta_H)\varphi_h|_H, \varphi_h|_H\right\rangle_{L^2(H)} \\ &= \frac{2}{\mu(S^*M)}\int_{S^*M_+}\frac{1}{\epsilon}\chi'(\frac{x_n}{\epsilon})(1 - R(x',x_n,\xi'))a(x',\xi')\,d\mu + O_\delta(\epsilon) + o_{\delta,\epsilon}(1).\end{aligned} \tag{2.8}$$

Finally, we take the $h \to 0^+$-limit in (2.8) followed by the $\epsilon \to 0^+$-limit, and finally the $\delta \to 0^+$ limit. The result is that, since the left-hand side in (2.8) is independent of ϵ and δ,

$$\begin{aligned}&\lim_{h\to 0^+}\langle a^w hD_\nu\varphi_h|_H, hD_\nu\varphi_h|_H\rangle_{L^2(H)} + \left\langle a^w(1 + h^2\Delta_H)\varphi_h|_H, \varphi_h|_H\right\rangle_{L^2(H)} \\ &\quad = \frac{2}{\mu(S^*M)}\int_{S_H^*M}(1 - R(x', x_n = 0, \xi'))\,d\tilde{\sigma} \\ &\quad = \frac{4}{\mu(S^*M)}\int_{B^*H}(1 - |\xi'|^2)^{1/2}a(x',\xi'))\,d\sigma,\end{aligned} \tag{2.9}$$

where $d\tilde{\sigma}$ is the symplectic volume form on $S^*_H M$, and $d\sigma$ is the symplectic volume form on B^*H.

□

3. Proof of Theorem 2

The proof follows as in Theorem 1 with a few modifications. For fixed $\epsilon_1 > 0$ we choose the test operator

$$A(x', x_n, hD_x) = (I + h^2\Delta_H(x', hD') + i\epsilon_1)^{-1}\chi(\frac{x_n}{\epsilon})\, hD_{x_n} a(x', hD') \tag{3.1}$$

and since $WF_h'(\varphi_h|_H) \subset B^*H$ (see [TZ2] section 11) it suffices to assume that $a \in C_0^\infty(T^*H)$ with

$$\operatorname{supp} a \subset B^*_{1+\epsilon_1^2}(H).$$

Let $\chi_{\epsilon_1}(x', \xi') \in C_0^\infty(B^*_{1+\epsilon_1^2} \setminus B^*_{1-2\epsilon_1^2}; [0,1])$ be a cutoff near the glancing set S^*H with $\chi_{\epsilon_1}(x', \xi') = 1$ when $(x', \xi') \in B^*_{1+\epsilon_1^2} \setminus B^*_{1-\epsilon_1^2}$. Then, with $A(x, hD_x)$ in (3.1), the same Rellich commutator argument as in Theorem 1 gives

$$\begin{aligned} &\big\langle (1 + h^2\Delta_H + i\epsilon_1)^{-1} a^w (1-\chi_{\epsilon_1})^w hD_\nu \varphi_h|_H, hD_\nu\varphi_h|_H \big\rangle_{L^2(H)} \\ &\quad + \left\langle a^w(1-\chi_{\epsilon_1})^w \left(\frac{1-|\xi'|^2}{1-|\xi'|^2 + i\epsilon_1}\right)^w \varphi_h|_H, \varphi_h|_H \right\rangle_{L^2(H)} \\ &\quad \to \frac{4}{\mu(S^*M)} \int_{B^*H} a_0(x', \xi')(1-\chi_{\epsilon_1}(x', \xi')) \left(\frac{(1-|\xi'|^2)^{1/2}}{1-|\xi'|^2 + i\epsilon_1}\right) d\sigma. \end{aligned} \tag{3.2}$$

It remains to determine the contribution of the glancing set S^*H. As in [Bu, DZ, HaZe, TZ1] we use a local Weyl law to do this. Because of the additional normal derivative term the argument is slightly different than in the cited articles and so we give some details. For the rest of this proof, we need to recall that $h \in \{\lambda_j^{-1}\}$, and we write h_j for this sequence to emphasize that it is a discrete sequence of values $h_j \to 0$. Since $\|a^w(x', hD')\|_{L^2 \to L^2} = O(1)$, it follows that for $h \in (0, h_0(\epsilon_1)]$ with $h_0(\epsilon_1) > 0$ sufficiently small,

$$\begin{aligned} &\frac{1}{N(h)} \sum_{h_j \geq h} |\langle a^w \chi^w_{\epsilon_1} \varphi_{h_j}|_H, \varphi_{h_j}|_H \rangle_{L^2(H)}| \\ &\leq C \frac{1}{N(h)} \sum_{h_j \geq h} \left(|\langle \chi^w_{\epsilon_1} \varphi_{h_j}|_H, \chi^w_{2\epsilon_1}\varphi_{h_j}|_H\rangle_{L^2(H)}| + \mathcal{O}(h_j^\infty)\right) \\ &\leq \frac{C}{2} \frac{1}{N(h)} \sum_{h_j \geq h} \left(\|\chi^w_{\epsilon_1}\varphi_{h_j}|_H\|^2_{L^2(H)} + \|\chi^w_{2\epsilon_1}\varphi_{h_j}|_H\|^2_{L^2(H)} + \mathcal{O}(h_j^\infty)\right) \\ &= \mathcal{O}(\epsilon_1^2). \end{aligned} \tag{3.3}$$

By a Fourier Tauberian argument [TZ1] section 8.4, it follows that for $h \in (0, h_0(\epsilon_1)]$

$$\frac{1}{N(h)} \sum_{h_j \geq h} |\chi^w_{\epsilon_1, 2\epsilon_1} \varphi_{h_j}|_H(x')|^2 = \mathcal{O}(\epsilon_1^2) \tag{3.4}$$

uniformly for $x' \in H$. The last estimate in (3.3) follows from (3.4) by integration over H.

To estimate the normal derivative terms, we first recall the standard resolvent estimate

$$\|(1+h^2\Delta_H+i\epsilon_1)^{-1}u\|_{H^2_h(H)} \leq C\epsilon_1^{-1}\|u\|_{L^2(H)},$$

where H^2_h is the semiclassical Sobolev space of order 2 (see [Zw] Lemma 13.6). Applying the obvious embedding $H^2_h(H) \subset L^2(H)$, we recover

$$\begin{aligned}\|(1+h^2\Delta_H+i\epsilon_1)^{-1}u\|_{L^2(H)} &\leq C\|(1+h^2\Delta_H+i\epsilon_1)^{-1}u\|_{H^2_h(H)}\\ &\leq C\epsilon_1^{-1}\|u\|_{L^2(H)}\end{aligned}$$

to get that

$$\begin{aligned}(3.5)\quad & \frac{1}{N(h)}\sum_{h_j\geq h}|\langle(1+h^2\Delta_H+i\epsilon_1)^{-1}a^w\chi^w_{\epsilon_1}h_jD_{x_n}\varphi_{h_j}|_H, h_jD_{x_n}\varphi_{h_j}|_H\rangle_{L^2(H)}|\\ &\leq C'\epsilon_1^{-1}\frac{1}{N(h)}\sum_{h_j\geq h}\|\chi^w_{\epsilon_1}hD_{x_n}\varphi_{h_j}|_H\|_{L^2(H)}\,\|\chi^w_{2\epsilon_1}hD_{x_n}\varphi_{h_j}|_H\|_{L^2(H)}\\ &\leq \frac{C'\epsilon_1^{-1}}{2}\frac{1}{N(h)}\sum_{h_j\geq h}\Big(\|\chi^w_{\epsilon_1}hD_{x_n}\varphi_{h_j}|_H\|^2_{L^2(H)}+\|\chi^w_{2\epsilon_1}hD_{x_n}\varphi_{h_j}|_H\|^2_{L^2(H)}\Big)\\ &= \mathcal{O}(\epsilon_1^{-1}\epsilon_1^2)\\ &= \mathcal{O}(\epsilon_1).\end{aligned}$$

The last estimate follows again from the Fourier Tauberian argument in [TZ1] section 8.4, which gives

$$\frac{1}{N(h)}\sum_{h_j\geq h}|\chi^w_{\epsilon_1,2\epsilon_1}h_jD_{x_n}\varphi_{h_j}|_H(x')|^2 = \mathcal{O}(\epsilon_1^2) \tag{3.6}$$

uniformly for $x' \in H$.

Since $\epsilon_1 > 0$ is arbitrary, Theorem 2 follows from (3.3) and (3.5) by letting $\epsilon_1 \to 0^+$ in (3.2), followed $h \to 0^+$ and $\epsilon \to 0^+$ last.

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