

N-enlarged Galilei Hopf algebra and its twist deformations

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Abstract

The N-enlarged Galilei Hopf algebra is constructed. Its twist deformations are considered and the corresponding twisted space-times are derived.

1 Introduction and preliminaries

Presently, it is well known, that in accordance with the Hopf-algebraic classification of all deformations of relativistic and nonrelativistic symmetries, one can distinguish three types of quantum spaces [1], [2] (for details see also [3]):

1) Canonical ($\theta^{\mu\nu}$ -deformed) type of quantum space [4]-[6]

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \quad (1)$$

2) Lie-algebraic modification of classical space-time [6]-[9]

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}^\rho \hat{x}_\rho, \quad (2)$$

and

3) Quadratic deformation of Minkowski and Galilei spaces [6], [9]-[11]

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}^{\rho\tau} \hat{x}_\rho \hat{x}_\tau, \quad (3)$$

with coefficients $\theta_{\mu\nu}$, $\theta_{\mu\nu}^\rho$ and $\theta_{\mu\nu}^{\rho\tau}$ being constants.

Besides, it has been demonstrated in [12], [13] and [3] that in the case of so-called standard, acceleration-enlarged and doubly enlarged Newton-Hooke Hopf algebras, the twist deformation provides the new space-time noncommutativity of the form^{1, 2}

$$4) \quad [t, \hat{x}_i] = 0 \quad , \quad [\hat{x}_i, \hat{x}_j] = if_\pm \left(\frac{t}{\tau} \right) \theta_{ij}(\hat{x}), \quad (4)$$

with time-dependent functions

$$f_+ \left(\frac{t}{\tau} \right) = f \left(\sinh \left(\frac{t}{\tau} \right), \cosh \left(\frac{t}{\tau} \right) \right) \quad , \quad f_- \left(\frac{t}{\tau} \right) = f \left(\sin \left(\frac{t}{\tau} \right), \cos \left(\frac{t}{\tau} \right) \right) \quad ,$$

$\theta_{ij}(\hat{x}) \sim \theta_{ij} = \text{const}$ or $\theta_{ij}(\hat{x}) \sim \theta_{ij}^k \hat{x}_k$ and τ denoting the time scale parameter. Moreover, in τ approaching infinity limit there have been obtained the following standard, acceleration-enlarged and doubly enlarged Galilei spaces [12], [13], [3]

$$5) \quad [\hat{x}_\mu, \hat{x}_\nu] = i\alpha_{\mu\nu}^{\rho_1 \dots \rho_n} \hat{x}_{\rho_1} \dots \hat{x}_{\rho_n}, \quad (5)$$

with $n = 0, 1, \dots, 6$. It should be also noted that different relations between all mentioned above quantum spaces-times (1), 2), 3), 4) and 5)) have been summarized in paper [3].

¹ $x_0 = ct$.

² The discussed space-times have been defined as the quantum representation spaces, so-called Hopf modules (see [4], [5], [14], [15]), for quantum acceleration-enlarged Newton-Hooke Hopf algebras.

In this article we construct the so-called N-enlarged Galilei Hopf algebra $\mathcal{U}_0^{(N)}(\mathcal{G})$, which generates the following transformations of classical nonrelativistic space

$$x_i \longrightarrow \omega_{ij}x_j + \sum_{n=0}^N a_{in}t^n, \quad (6)$$

$$t \longrightarrow t + t_0, \quad (7)$$

with ω_{ij} denoting the $O(d)$ rotations and with real parameters a_{in} ($n = 0, 1, \dots, N$). Particularly, for $N = 1, 2, 3$ the rules (6), (7) correspond respectively to the standard, acceleration-enlarged [16] and doubly enlarged [17] Galilei Hopf symmetries. In the next step of our investigations we provide the (Abelian) twist deformations of $\mathcal{U}_0^{(N)}(\mathcal{G})$ Hopf structure. Consequently, we get the following two types of space-time noncommutativity

$$[t, \hat{x}_i] = 0 \quad , \quad [\hat{x}_a, \hat{x}_b] = i\alpha^{ij}t^{n+m}(\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi}), \quad (8)$$

and

$$[t, \hat{x}_i] = 0 \quad , \quad [\hat{x}_a, \hat{x}_b] = 2i\alpha t^n [\delta_{ia}(\hat{x}_k\delta_{bl} - \hat{x}_l\delta_{bk}) - \delta_{ib}(\hat{x}_k\delta_{al} - \hat{x}_l\delta_{ak})], \quad (9)$$

which for $N = 1, 2, 3$ reproduce the relation (5).

There are various motivations for present studies. First of all, we construct explicitly the most general Hopf algebra of nonrelativistic symmetries containing standard, acceleration-enlarged and doubly enlarged Galilei groups. Secondly, we get the completely new quantum (twist-deformed) space-times associated with $\mathcal{U}_\alpha^{(N)}(\mathcal{G})$ Hopf structure. Such a result seems to be quite interesting due to the fact, that it extends in natural way the mentioned above classification of the quantum spaces. Finally, it should be noted that the obtained results permit to consider the classical as well as quantum particle models defined on new noncommutative nonrelativistic space-times (8) and (9).

The paper is organized as follows. In first section we provide the N-enlarged Galilei Hopf algebra $\mathcal{U}_0^{(N)}(\mathcal{G})$. The second section is devoted to its twist deformations and to the derivation of corresponding quantum space-times. The comments on the N-enlarged Newton-Hooke Hopf algebra and final remarks are presented in the last section.

2 N-enlarged Galilei Hopf algebra $\mathcal{U}_0^{(N)}(\mathcal{G})$

In this section we construct the N-enlarged Galilei Hopf algebra which generates the space-time transformations (6) and (7). First of all, it is easy to see that the corresponding generators are represented on the space of functions as follows

$$M_{ij} \triangleright f(t, \bar{x}) = i(x_i\partial_j - x_j\partial_i) f(t, \bar{x}), \quad (10)$$

$$H \triangleright f(t, \bar{x}) = i\partial_t f(t, \bar{x}), \quad (11)$$

$$G_i^{(n)} \triangleright f(t, \bar{x}) = it^n\partial_i f(t, \bar{x}) \quad ; \quad n = 0, 1, \dots, N, \quad (12)$$

where M_{ij} , H , $G_i^{(0)} (= P_i)$ and $G_i^{(1)} (= K_i)$ can be identified with rotation, time translation, momentum and boost operators respectively. Next, by straightforward calculation one can find the following commutation relations

$$[M_{ij}, M_{kl}] = i(\delta_{il} M_{jk} - \delta_{jl} M_{ik} + \delta_{jk} M_{il} - \delta_{ik} M_{jl}) \quad , \quad [H, M_{ij}] = 0 \quad , \quad (13)$$

$$[M_{ij}, G_k^{(n)}] = i(\delta_{jk} G_i^{(n)} - \delta_{ik} G_j^{(n)}) \quad , \quad [G_i^{(n)}, G_j^{(m)}] = 0 \quad , \quad (14)$$

$$[G_i^{(n)}, H] = -inG_i^{(n-1)} \quad , \quad (15)$$

which together with classical coproduct and antipode

$$\Delta_0(a) = a \otimes 1 + 1 \otimes a \quad , \quad S_0(a) = -a \quad , \quad (16)$$

define the N-enlarged Galilei Hopf algebra $\mathcal{U}_0^{(N)}(\mathcal{G})$. It should be noted, that for $N = 1, 2, 3$ we get the standard $\mathcal{U}_0(\mathcal{G})$, acceleration-enlarged $\mathcal{U}_0(\widehat{\mathcal{G}})$ and doubly enlarged $\mathcal{U}_0(\widehat{\widehat{\mathcal{G}}})$ Galilei Hopf structures proposed in [16] and [17] respectively.

3 Twist deformations of N-enlarged Galilei Hopf algebra and the corresponding quantum space-times

Let us now turn to the twist deformations of the Hopf structure provided in pervious section. First of all, in accordance with Drinfeld twist procedure [18]-[20], the algebraic sector of twisted N-enlarged Galilei Hopf algebra $\mathcal{U}_\alpha^{(N)}(\mathcal{G})$ remains undeformed (see (13)-(15)), while the coproducts and antipodes transform as follows (see formula (16))

$$\Delta_0(a) \rightarrow \Delta_\alpha(a) = \mathcal{F}_\alpha \circ \Delta_0(a) \circ \mathcal{F}_\alpha^{-1} \quad , \quad S_\alpha(a) = u_\alpha S_0(a) u_\alpha^{-1} \quad , \quad (17)$$

with $u_\alpha = \sum f_{(1)} S_0(f_{(2)})$ (we use Sweedler's notation $\mathcal{F}_\alpha = \sum f_{(1)} \otimes f_{(2)}$). Besides, it should be noted, that the twist factor $\mathcal{F}_\alpha \in \mathcal{U}_\alpha^{(N)}(\mathcal{G}) \otimes \mathcal{U}_\alpha^{(N)}(\mathcal{G})$ satisfies the classical cocycle condition

$$\mathcal{F}_{\alpha 12} \cdot (\Delta_0 \otimes 1) \mathcal{F}_\alpha = \mathcal{F}_{\alpha 23} \cdot (1 \otimes \Delta_0) \mathcal{F}_\alpha \quad , \quad (18)$$

and the normalization condition

$$(\epsilon \otimes 1) \mathcal{F}_\alpha = (1 \otimes \epsilon) \mathcal{F}_\alpha = 1 \quad , \quad (19)$$

with $\mathcal{F}_{\alpha 12} = \mathcal{F}_\alpha \otimes 1$ and $\mathcal{F}_{\alpha 23} = 1 \otimes \mathcal{F}_\alpha$.

It is well known, that the twisted algebra $\mathcal{U}_\alpha^{(N)}(\mathcal{G})$ can be described in terms of so-called classical r -matrix $r \in \mathcal{U}_\alpha^{(N)}(\mathcal{G}) \otimes \mathcal{U}_\alpha^{(N)}(\mathcal{G})$, which satisfies the classical Yang-Baxter equation (CYBE)

$$[[r_\alpha, r_\alpha]] = [r_{\alpha 12}, r_{\alpha 13} + r_{\alpha 23}] + [r_{\alpha 13}, r_{\alpha 23}] = 0 \quad , \quad (20)$$

where symbol $[[\cdot, \cdot]]$ denotes the Schouten bracket and for $r = \sum_i a_i \otimes b_i$

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1 \quad , \quad r_{13} = \sum_i a_i \otimes 1 \otimes b_i \quad , \quad r_{23} = \sum_i 1 \otimes a_i \otimes b_i \quad .$$

In this article we consider two types of Abelian twist deformation of N-enlarged Galilei Hopf algebra, described by the following r -matrices³

$$r^{(n,m)} = \frac{1}{2} \alpha^{ij} G_i^{(n)} \wedge G_i^{(m)} \quad [\alpha^{ij} = -\alpha^{ji}] \quad , \quad (21)$$

and

$$r^{(n)} = \alpha G_i^{(n)} \wedge M_{kl} \quad [i, k, l - \text{fixed}, \quad i \neq k, l] \quad , \quad (22)$$

where α^{ij} , α denote the deformation parameters. Due to the Abelian character of the above carriers (all of them contain the mutually commuting elements of the algebra), the corresponding twist factors can be obtained in a standard way [18]-[20], i.e. they take the form

$$\mathcal{F}^{(n,m)} = \exp(ir^{(n,m)}) \quad \text{and} \quad \mathcal{F}^{(n)} = \exp(ir^{(n)}) \quad . \quad (23)$$

Of course, for $N = 1, 2, 3$ we obtain the twist factors for $\mathcal{U}_\alpha(\mathcal{G})$, $\mathcal{U}_\alpha(\widehat{\mathcal{G}})$ and $\mathcal{U}_\alpha(\widehat{\widehat{\mathcal{G}}})$ Hopf structures discussed in articles [12], [13] and [3].

The corresponding quantum space-times are defined as the representation spaces (Hopf modules) for N-enlarged Galilei Hopf algebra $\mathcal{U}_\alpha^{(N)}(\mathcal{G})$, with action of the generators M_{ij} , H and $G_i^{(n)}$ given by (10)-(12) (see [4], [5], [14], [15]). Besides, the \star -multiplication of arbitrary two functions covariant under $\mathcal{U}_\alpha^{(N)}(\mathcal{G})$ is defined as follows

$$f(t, \bar{x}) \star^{(n,m)} g(t, \bar{x}) := \omega \circ ((\mathcal{F}^{(n,m)})^{-1} \triangleright f(t, \bar{x}) \otimes g(t, \bar{x})) \quad , \quad (24)$$

$$f(t, \bar{x}) \star^{(n)} g(t, \bar{x}) := \omega \circ ((\mathcal{F}^{(n)})^{-1} \triangleright f(t, \bar{x}) \otimes g(t, \bar{x})) \quad , \quad (25)$$

where symbols $\mathcal{F}^{(n,m)}$, $\mathcal{F}^{(n)}$ denote the twist factors (see (23)) and $\omega \circ (a \otimes b) = a \cdot b$. Consequently, we get

$$[t, x_a]_{\star^{(n,m)}} = 0 \quad , \quad [x_a, x_b]_{\star^{(n,m)}} = i\alpha^{ij} t^{n+m} (\delta_{ai} \delta_{bj} - \delta_{aj} \delta_{bi}) \quad , \quad (26)$$

and

$$[t, x_a]_{\star^{(n)}} = 0 \quad , \quad [x_a, x_b]_{\star^{(n)}} = 2i\alpha t^n [\delta_{ia} (x_k \delta_{bl} - x_l \delta_{bk}) - \delta_{ib} (x_k \delta_{al} - x_l \delta_{ak})] \quad , \quad (27)$$

respectively. Obviously, for $N = 1, 2, 3$ we obtain the quantum spaces provided in [12], [13] and [3]. It should be also noted that for deformation parameters α^{ij} and α approaching zero, the above quantum space-times become classical.

³ $a \wedge b = a \otimes b - b \otimes a$.

4 Comments on the N-enlarged Newton-Hooke Hopf algebra and final remarks

In this short paper we provide the (new) N-enlarged Galilei Hopf structure $\mathcal{U}_0^{(N)}(\mathcal{G})$. Besides, we discuss its twist deformations and derive the corresponding quantum space-times. However, it should be noted that there still remain the few open problems. Firstly, one can ask about the relativistic counterpart of $\mathcal{U}_0^{(N)}(\mathcal{G})$ quantum group related by contraction scheme, which connects both relativistic and nonrelativistic Hopf algebras. Moreover, one should better understand the meaning of the parameters a_{in} in the formula (6). This aim can be achieved by the analyzing of the classical and quantum dynamical models defined on the twisted spaces (8) and (9). Finally, it seems to be quite interesting to find the corresponding N-enlarged Newton-Hooke transformations, which in the limit of cosmological constant (τ) approaching infinity reproduce the rules (6) and (7). Unfortunately, the solution of such a problem seems to be computationally nontrivial for arbitrary value of N . For example, if $N = 6$ one can derive⁴

$$\begin{aligned}
 x_i &\longrightarrow \omega_{ij}x_j + a_{i0}C_{\pm}\left(\frac{t}{\tau}\right) + a_{i1}\tau S_{\pm}\left(\frac{t}{\tau}\right) \pm 2a_{i2}\tau^2\left(C_{\pm}\left(\frac{t}{\tau}\right) - 1\right) + \\
 &\pm 6a_{i3}\tau^3\left(S_{\pm}\left(\frac{t}{\tau}\right) - \frac{t}{\tau}\right) + 24a_{i4}\tau^4\left(C_{\pm}\left(\frac{t}{\tau}\right) \mp \frac{1}{2}\left(\frac{t}{\tau}\right)^2 - 1\right) + \quad (28) \\
 &+ 120a_{i5}\tau^5\left(S_{\pm}\left(\frac{t}{\tau}\right) \mp \frac{1}{6}\left(\frac{t}{\tau}\right)^3 - \frac{t}{\tau}\right) + \\
 &+ 720a_{i6}\tau^6\left(\pm C_{\pm}\left(\frac{t}{\tau}\right) \mp \frac{1}{24}\left(\frac{t}{\tau}\right)^4 - \frac{1}{2}\left(\frac{t}{\tau}\right)^2 \mp 1\right), \\
 t &\longrightarrow t + t_0, \quad (29)
 \end{aligned}$$

with $C_+[\frac{t}{\tau}] = \cosh[\frac{t}{\tau}]$, $S_+[\frac{t}{\tau}] = \sinh[\frac{t}{\tau}]$ in the case of "de-Sitter" as well as with $C_-[\frac{t}{\tau}] = \cos[\frac{t}{\tau}]$, $S_-[\frac{t}{\tau}] = \sin[\frac{t}{\tau}]$ in the case of "anti-de-Sitter" algebra. One can also check that in such a situation the corresponding Hopf algebra is given by the commutation relations (13)-(15) with additional relation $[H, G_i^{(0)}(P_i)] = \pm \frac{i}{\tau}G_i^{(1)}(K_i)$ and classical coproduct (16). The proper (N=)6-enlarged Newton-Hooke twisted spaces can be calculated as well, and, for example, one of them takes the form

$$\begin{aligned}
 [t, x_a]_{\star(6,6)} &= 0, \\
 [x_a, x_b]_{\star(6,6)} &= i\alpha^{ij}518400\tau^{12}\left(\pm C_{\pm}\left(\frac{t}{\tau}\right) \mp \frac{1}{24}\left(\frac{t}{\tau}\right)^4 - \frac{1}{2}\left(\frac{t}{\tau}\right)^2 \mp 1\right)^2 \times \quad (30) \\
 &\times (\delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi}).
 \end{aligned}$$

The work in the mentioned above directions already started and are in progress.

⁴For $N = 1, 2, 3$ the above transformations contain the symmetries proposed in [21], [16] and [17] respectively.

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References

- [1] S. Zakrzewski; q-alg/9602001
- [2] Y. Brihaye, E. Kowalczyk, P. Maslanka; math/0006167
- [3] M. Daszkiewicz, Mod. Phys. Lett. B (in press)
- [4] R. Oeckl, J. Math. Phys. 40, 3588 (1999)
- [5] M. Chaichian, P.P. Kulish, K. Nashijima, A. Tureanu, Phys. Lett. B 604, 98 (2004)
- [6] M. Daszkiewicz, Mod. Phys. Lett. A 23, 505 (2008)
- [7] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy, Phys. Lett. B 264, 331 (1991)
- [8] S. Giller, P. Kosinski, M. Majewski, P. Maslanka and J. Kunz, Phys. Lett. B 286, 57 (1992)
- [9] J. Lukierski and M. Woronowicz, Phys. Lett. B 633, 116 (2006)
- [10] O. Ogievetsky, W.B. Schmidke, J. Wess, B. Zumino, Comm. Math. Phys. 150, 495 (1992)
- [11] P. Aschieri, L. Castellani, A.M. Scarfone, Eur. Phys. J. C 7, 159 (1999)
- [12] M. Daszkiewicz, Mod. Phys. Lett. A 24, 1325 (2009)
- [13] M. Daszkiewicz, Acta Phys. Pol. B 41, 1889 (2010)
- [14] C. Blohmann, J. Math. Phys. 44, 4736 (2003)
- [15] J. Wess; hep-th/0408080
- [16] J. Lukierski, P.C. Stichel, W.J. Zakrzewski; arXiv: 0710.3093 [hep-th]
- [17] J. Gomis, J. Lukierski, Phys. Lett. B 664, 107 (2008)
- [18] V.G. Drinfeld, Soviet Math. Dokl. 32, 254 (1985); Algebra i Analiz (in Russian) 1, Fasc. 6, p. 114 (1989)
- [19] V. Chari, A. Pressley, *"A Guide to Quantum Groups"*, Cambridge University Press, Cambridge, 1994

- [20] S. Majid, "*Foundations of quantum group theory*", Cambridge University Press, 2000
- [21] H. Bacry, J.M. Levy-Leblond, J. Math. Phys. 9, 1605 (1968)