THE NONLINEAR SCHRÖDINGER EQUATION GROUND STATES ON PRODUCT SPACES

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ABSTRACT. We study the nature of the Nonlinear Schrödinger equation ground states on the product spaces $\mathbb{R}^n \times M^k$, where M^k is a compact Riemannian manifold. We prove that for small L^2 masses the ground states coincide with the corresponding $\mathbb{R}^n$ ground states. We also prove that above a critical mass the ground states have nontrivial M^k dependence. Finally, we address the Cauchy problem issue which transform the variational analysis to dynamical stability results.

MSC: 35Q55, 37K45. **Keywords:** NLS, stability of solitons, rigidity.

1. INTRODUCTION

Our goal here is to study the nature of the Nonlinear Schrödinger equation ground states when the problem is posed on the product spaces $\mathbb{R}^n \times M^k$, where M^k is a compact Riemannian manifold. We thus consider the following Cauchy problems

$$\begin{cases} i\partial_t u - \Delta_{x,y} u - u|u|^\alpha = 0, \ (t,x,y) \in \mathbb{R} \times \mathbb{R}^n_x \times M^k_y \\ u(0,x,y) = \varphi(x,y) \end{cases} \tag{1.1}$$

where

$$\Delta_{x,y} = \sum_{j=1}^n \partial^2_{x_j} + \Delta_y$$

and Δ_y is the Laplace-Beltrami operator on M^k_y. Recall that the Laplace-Beltrami operator is defined in local coordinates as follows:

$$\frac{1}{\sqrt{det(g_{i,j}(y))}} \partial_{y_i} \sqrt{det(g_{i,j}(y))} g^{i,j}(y) \partial_{y_j}$$

where $g^{i,j}(y) = (g_{i,j}(y))^{-1}$ and $g_{i,j}(y)$ is the metric tensor.
We assume that $0 < \alpha < 4/(n+k)$ which corresponds to L^2 subcritical nonlinearity.
In this paper, we shall study the following two questions:

- the existence and stability of solitary waves for (1.1);
- the global well posedness of the Cauchy problem associated to (1.1).

The equation (1.1) has two (at least formal) conservation laws, the energy

$$\mathcal{E}_{n,M^k,\alpha}(u) = \int_{M^k_y} \int_{\mathbb{R}^d_x} \Big(\frac{1}{2} |\nabla_{x,y} u|^2 - \frac{1}{2+\alpha} |u|^{2+\alpha} \Big) dx dvol_{M^k_y} \tag{1.2}$$

and the L^2 mass,

$$\|u\|^2_{L^2(\mathbb{R}^n \times M^k)} = \int_{M^k_y} \int_{\mathbb{R}^n_x} |u|^2 dx dvol_{M^k_y} \tag{1.3}$$

Here we denote by $dvol_{M^k_y}$ the volume form on M^k. Recall that in local coordinates it can be written as $\sqrt{det(g_{i,j}(y))}dy$. Moreover the i-th component (in local coordinates) of the gradient $(\nabla_y u(y))$ is

$$g^{i,j}(y)\partial_{y_j} u$$

One has the classical Gagliardo-Nirenberg inequality

$$\|u\|^{2+\alpha}_{L^{2+\alpha}(\mathbb{R}^n\times M^k)} \leq C\|u\|^{\theta(\alpha)}_{H^1(\mathbb{R}^n\times M^k)}\|u\|^{2+\alpha-\theta(\alpha)}_{L^2(\mathbb{R}^n\times M^k)} \tag{1.4}$$

where $\theta(\alpha) = (n+k)\alpha/2$. Thus $\theta(\alpha) < 2$ under our assumption $0 < \alpha < 4/(n+k)$. This implies that the conservation laws (1.2) and (1.3) imply a control on the H^1 norm which excludes a L^2 self-focusing blow-up and thus one expects that (1.1) has a well-defined global dynamics. This problem seems quite delicate for a general M^k. However if we replace M^k with $\mathbb{R}^k$ it is well-known (see [11], [4] and the references therein) that (1.1) has a global strong solution for every $L^2(\mathbb{R}^{n+k})$ initial data.

Our argument to construct stable solutions to (1.1) follows the one proposed in [5]. Hence we shall look at the following minimization problems:

$$K^\rho_{n,M^k,\alpha} = \inf_{\substack{u\in H^1(\mathbb{R}^n\times M^k)\\ \|u\|_{L^2(\mathbb{R}^n\times M^k)}=\rho}} \mathcal{E}_{n,M^k,\alpha}(u) \tag{1.5}$$

and $\mathcal{E}_{n,M^k,\alpha}(u)$ is defined in (1.2). In the sequel we shall use the following notation:

$$\mathcal{M}^\rho_{n,M^k,\alpha} = \{v \in H^1(\mathbb{R}^n \times M^k)| \\ \|v\|_{L^2(\mathbb{R}^n\times M^k)} = \rho \text{ and } \mathcal{E}_{n,M^k,\alpha}(v) = K^\rho_{n,M^k,\alpha}\} \tag{1.6}$$

The first result we state concerns the compactness of minimizing sequences to (1.5).

Theorem 1.1. *Let M^k be a compact manifold and $0 < \alpha < 4/(n+k)$. Then we have the following:*

$$K^\rho_{n,M^k,\alpha} > -\infty \text{ and } \mathcal{M}^\rho_{n,M^k,\alpha} \neq \emptyset,\ \forall \rho > 0; \tag{1.7}$$

$$\forall u_j \in H^1(\mathbb{R}^n \times M^k) \text{ s.t. } \|u_j\|_{L^2(\mathbb{R}^n\times M^k)} = \rho, \lim_{j\to\infty} \mathcal{E}_{n,M^k,\alpha}(u_j) = K^\rho_{n,M^k,\alpha} \tag{1.8}$$

$$\exists \text{ a subsequence } u_{j_l} \text{ and } \tau_l \in \mathbb{R}^n_x \text{ s.t. } u_{j_l}(x+\tau_l, y) \text{ converges in } H^1(\mathbb{R}^n \times M^k).$$

The proof of Theorem 1.1 is based on the concentration compactness principle and it will be given in the appendix. Also the following stability theorem follows from a standard argument, hence its classical proof will be recalled in the appendix.

Theorem 1.2. *Let $\rho > 0$ be fixed and n, M^k, α as in Theorem 1.1. Assume moreover that*

(1.9) *the Cauchy problem (1.1) is globally well posed for any data $\varphi \in \mathcal{U}$ where $\mathcal{U}$ is a $H^1(\mathbb{R}^n \times M^k)$-neighborhood of $\mathcal{M}^\rho_{n,M^k,\alpha}$.*

Then the set $\mathcal{M}^\rho_{n,M^k,\alpha}$ is orbitally stable, i.e.:

$$\forall \epsilon > 0\ \exists \delta = \delta(\epsilon) > 0 \text{ s.t.}$$

$$\varphi \in \mathcal{U}, \inf_{v\in\mathcal{M}^\rho_{n,M^k,\alpha}} \|\varphi - v\|_{H^1(\mathbb{R}^n\times M^k)} < \delta(\epsilon)$$

$$\text{implies } \sup_{t\in\mathbb{R}} \Big(\inf_{v\in\mathcal{M}^\rho_{n,M^k,\alpha}} \|u_\varphi(t) - v\|_{H^1(\mathbb{R}^n\times M^k)} \Big) < \epsilon$$

where $u_\varphi(t,x,y)$ is the unique global solution to (1.1).

Let us emphasize that the stability result stated in Theorem 1.2 has two major defaults: the first one is that we don't have an explicit description of the minimizers $\mathcal{M}^\rho_{n,M^k,\alpha}$; the second one is that it is subordinated to (1.9), i.e. the global well posedness of the Cauchy problem (1.1). The main contributions of this paper concern a partial understanding of the aforementioned questions.
Notice that (see [4]) a special family of solutions to (1.1) is given by

$$u(t,x,y) = e^{-i\omega t}u_{n,\omega,\alpha}(x)$$

where $\omega > 0$ and $u_{n,\omega,\alpha}(x)$ is defined as the unique radial solution to:

$$-\Delta_x u_{n,\omega,\alpha} + \omega u_{n,\omega,\alpha} = u_{n,\omega,\alpha}|u_{n,\omega,\alpha}|^\alpha \tag{1.10}$$
$$u_{n,\omega,\alpha} \in H^1(\mathbb{R}^n_x),\ u_{n,\omega,\alpha}(x) > 0,\ x \in \mathbb{R}^n_x$$

Next, we set

$$\mathcal{N}_{n,\omega,\alpha} = \{e^{i\theta}u_{n,\omega,\alpha}(x+\tau) | \tau \in \mathbb{R}^n, \theta \in \mathbb{R}\} \tag{1.11}$$

Notice that there is a natural embedding $H^1(\mathbb{R}^n_x) \subset H^1(\mathbb{R}^n_x \times M^k_y)$. In fact every function in $H^1(\mathbb{R}^n_x)$ can be extended in a trivial way w.r.t. the y variable on $\mathbb{R}^n_x \times M^k_y$ and this extension will belong to $H^1(\mathbb{R}^n \times M^k)$. In particular since now on the set $\mathcal{N}_{n,\omega,\alpha}$ defined in (1.11), will be considered without any further comment in a twofold way: as a subset of $H^1(\mathbb{R}^n_x)$ and $H^1(\mathbb{R}^n_x \times M^k_y)$. By a rescaling argument one can prove that the function

$$(0,\infty) \ni \omega \to \|u_{n,\omega,\alpha}\|^2_{L^2(\mathbb{R}^n_x)} \in (0,\infty)$$

is strictly increasing for any $0 < \alpha < \frac{4}{n}$ and

$$\lim_{\omega\to\infty} \|u_{n,\omega,\alpha}\|_{L^2(\mathbb{R}^n_x)} = \infty \text{ and } \lim_{\omega\to 0} \|u_{n,\omega,\alpha}\|_{L^2(\mathbb{R}^n_x)} = 0$$

As a consequence for any fixed $0 < \alpha < \frac{4}{n}$ we have:

$$\forall \rho > 0\ \exists!\ \omega(\rho) > 0 \text{ s.t. } \|u_{n,\omega(\rho),\alpha}\|_{L^2(\mathbb{R}^n_x)} = \rho \tag{1.12}$$

In next theorem the set $\mathcal{N}_{n,\omega,\alpha}$ is the one defined in (1.11) and $\mathcal{M}^\rho_{n,M^k,\alpha}$ is defined in (1.6).

Theorem 1.3. *Let n, M^k, α as in Theorem 1.2. There exists $\rho^* \in (0,\infty)$ such that:*

$$\mathcal{M}^\rho_{n,M^k,\alpha} = \mathcal{N}_{n,\omega(\rho/\sqrt{vol(M^k)}),\alpha},\ \forall \rho < \rho^* \tag{1.13}$$

and

$$\mathcal{M}^\rho_{n,M^k,\alpha} \cap \mathcal{N}_{n,\omega(\rho/\sqrt{vol(M^k)}),\alpha} = \emptyset,\ \forall \rho > \rho^* \tag{1.14}$$

where $\omega(\rho/\sqrt{vol(M^k)})$ is uniquely defined in (1.12). *In particular for $\rho > \rho^*$ the elements of $\mathcal{M}^\rho_{n,M^k,\alpha}$ depend in a nontrivial way on the M^k variable.*

By the approach of Weinstein [13] one may expect that $\mathcal{N}_{n,\omega,\alpha}$ is stable under (1.1) for $\alpha < 4/n$ and ω small enough, see [9] for a recent related work. It should however be pointed out that in such a stability result one would not get the variational description of $\mathcal{N}_{n,\omega,\alpha}$ as is the case in Theorem 1.3 ($\alpha < 4/(n+k)$). We underline that by combining Theorem 1.2 and Theorem 1.3 we get a stable set for large values of the mass ρ, and in general it is independent of the solitary solitary waves associated to NLS in $\mathbb{R}^n$.

Next we shall focus on the question of the global well-posedness of the Cauchy problem associated to (1.1) in the particular case $n \geq 1$, $k = 1$. For every $n > 1$ we fix the numbers

$$p := p(n, \alpha) = \frac{4(2+\alpha)}{n\alpha} \text{ and } q := q(n, \alpha) = 2 + \alpha$$

and for every $T > 0$ we define the localized norms:

$$\|u(t,x,y)\|_{X_T} \equiv \|u(t,x,y)\|_{L^p((-T,T);L^q(\mathbb{R}^n_x;H^1(M^1_y))} \tag{1.15}$$

and

$$\|u(t,x,y)\|_{Y_T} \equiv \|\nabla_x u\|_{L^p((-T,T);L^q(\mathbb{R}^n_x;L^2(M^1_y))} \tag{1.16}$$

Theorem 1.4. *Let $n \geq 1$ be fixed and $\alpha < 4/(n+1)$, then for every initial data $\varphi \in H^1(\mathbb{R}^n \times M^1)$, the Cauchy problem* (1.1) *has a unique global solution $u(t,x,y)$ satisfying :*

$$u(t,x,y) \in \mathcal{C}((-T,T); H^1(\mathbb{R}^n \times M^1)) \cap X_T \cap Y_T, \ \forall T > 0$$

Remark 1.1. The main difficulty in the analysis of the Cauchy problem (1.1) (compared with the Cauchy problem in the euclidean space) is related with the fact that the propagator $e^{-it\Delta_{x,y}}$ on $\mathbb{R}^n \times M^1_y$ does not satisfies the Strichartz estimates which are available for the propagator $e^{-it\Delta_{\mathbb{R}^{n+k}}}$ on the euclidean space $\mathbb{R}^{n+k}$.

Let us now describe some other known cases when (1.1) is well-posed in $H^1(\mathbb{R}^n \times M^k)$ under the assumption $\alpha < 4/(n+k)$. Using the analysis of [2, 3] one may prove such a well-posedness result in the case $\mathbb{R} \times M^2$, i.e. $n = 1$ and $k = 2$. Moreover, using the analysis of the recent papers [6] and [7] one may also prove such a well-posedness result in the cases $\mathbb{R}^2 \times \mathbb{T}^2$ and $\mathbb{R} \times \mathbb{T}^3$ respectively.

Notation. Next we fix some notations. We denote by L^p_x and H^s_x respectively the space $L^p(\mathbb{R}^n_x)$ and $H^s(\mathbb{R}^n_x)$. We also use the notation $L^p_{x,y} = L^p(\mathbb{R}^n_x \times M^k_y)$ and $L^p_x L^q_y = L^p(\mathbb{R}^n_x; L^q(M^k_y))$. If $v(t)$ is a time dependent function defined on $\mathbb{R}_t$ and valued in a Banach space X, then we define

$$\|v\|^p_{L^p_t(X)} = \int_{\mathbb{R}} \|v(t)\|^p_X dt$$

For every $p \in [1, \infty]$ we denote by $p' \in [1, \infty]$ its conjugate Hölder exponent. We denote by $e^{-it\Delta_{x,y}}$ the free propagator associated to the Schrödinger equation on $\mathbb{R}^n_x \times M^k_y$.

2. Some useful results on the euclidean space $\mathbb{R}^n_x$ with $n \geq 1$

In this section we recall some well known facts (see [4]) related to the following minimization problem on $\mathbb{R}^n_x$:

$$I^\rho_{n,\alpha} = \inf_{\substack{u \in H^1_x \\ \|u\|_{L^2_x} = \rho}} \mathcal{E}_{n,\alpha}(u) \tag{2.1}$$

where for $\alpha < 4/n$

$$\mathcal{E}_{n,\alpha}(u) = \frac{1}{2} \int_{\mathbb{R}^n_x} |\nabla_x u|^2 dx - \frac{1}{2+\alpha} \int_{\mathbb{R}^n_x} |u|^{2+\alpha} dx \tag{2.2}$$

By an elementary rescaling argument we have

$$I^\rho_{n,\alpha} = \rho^{(8+4\alpha-2\alpha n)/(4-\alpha n)} I^1_{n,\alpha} \tag{2.3}$$

It is well–known that

$$-\infty < I^\rho_{n,\alpha} < 0, \ \forall \rho > 0 \tag{2.4}$$

and

$$\mathcal{M}^\rho_{n,\alpha} = \mathcal{N}_{n,\omega(\rho),\alpha} \tag{2.5}$$

where $\mathcal{N}_{n,\omega,\alpha}$ is defined in (1.11),

$$\mathcal{M}^\rho_{n,\alpha} = \{u \in H^1_x | \|u\|_{L^2_x} = \rho \text{ and } \mathcal{E}_{n,\alpha}(u) = I^\rho_{n,\alpha}\} \tag{2.6}$$

and $\omega(\rho)$ is defined uniquely (see (1.12)) by the relation

$$\|u_{n,\omega(\rho),\alpha}\|_{L^2_x} = \rho$$

We also recall that the functions $u_{n,\omega,\alpha}$ (defined as the unique radially symmetric and positive solution to (1.10)) satisfy the following Pohozaev type identity (for a proof of (2.7) see the proof of (3.21) in next section):

$$\int_{\mathbb{R}^n_x} |\nabla_x u_{n,\omega,\alpha}|^2 dx = \frac{\alpha n}{2(\alpha+2)} \int_{\mathbb{R}^n_x} |u_{n,\omega,\alpha}|^{2+\alpha} dx \tag{2.7}$$

On the other hand if we multiply (1.10) by $u_{n,\omega,\alpha}$ and we integrate by parts then we get

$$\int_{\mathbb{R}^n_x} |\nabla_x u_{n,\omega,\alpha}|^2 dx + \omega \|u_{n,\omega,\alpha}\|^2_{L^2_x} = \int_{\mathbb{R}^n_x} |u_{n,\omega,\alpha}|^{2+\alpha} dx$$

that in conjunction with (2.7) gives

$$\omega \|u_{n,\omega,\alpha}\|_2^2 = \frac{2\alpha + 4 - \alpha n}{\alpha n} \int_{\mathbb{R}^n_x} |\nabla_x u_{n,\omega,\alpha}|^2 dx \tag{2.8}$$

$$= \frac{4\alpha + 8 - 2\alpha n}{\alpha n - 4} \Big(\frac{1}{2} \int_{\mathbb{R}^n_x} |\nabla_x u_{n,\omega,\alpha}|^2 dx - \frac{1}{2+\alpha} \int_{\mathbb{R}^n_x} |u_{n,\omega,\alpha}|^{2+\alpha} dx \Big)$$

$$= \frac{4\alpha + 8 - 2\alpha n}{\alpha n - 4} I^{\|u_{n,\omega,\alpha}\|_{L^2_x}}_{n,\alpha}$$

(at the last step we have used the fact that due to (2.5) we have that $u_{n,\omega,\alpha}$ is a minimizer for $\mathcal{E}_{n,\alpha}$ on its associated constrained).
Finally notice that by (2.7) we deduce

$$I^{\|u_{n,\omega,\alpha}\|_{L^2_x}}_{n,\alpha} = \mathcal{E}_{n,\alpha}(u_{n,\omega,\alpha}) = \frac{\alpha n - 4}{2\alpha n} \int_{\mathbb{R}^n_x} |\nabla_x u_{n,\omega,\alpha}|^2 dx \tag{2.9}$$

3. An auxiliary problem

In this section we study the minimizers of the following minimization problems

$$J_{n,M^k,\alpha,\lambda} = \inf_{\substack{u \in H^1(\mathbb{R}^n \times M^k) \\ \|u\|_{L^2_{x,y}} = 1}} \mathcal{E}_{n,M^k,\alpha,\lambda}(u) \tag{3.1}$$

where

$$\mathcal{E}_{n,M^k,\alpha,\lambda}(u) = \int_{M^k_y} \int_{\mathbb{R}^n_x} \Big(\frac{\lambda}{2} |\nabla_y u|^2 + \frac{1}{2} |\nabla_x u|^2 - \frac{1}{2+\alpha} |u|^{2+\alpha} \Big) dx dvol_{M^k_y}$$

We also introduce the following sets:

$$\mathcal{M}_{n,M^k,\alpha,\lambda} = \{w \in H^1(\mathbb{R}^n \times M^k) | \|w\|_{L^2_{x,y}} = 1 \text{ and } \mathcal{E}_{n,M^k,\alpha,\lambda}(w) = J_{n,M^k,\alpha,\lambda}\}$$

Theorem 3.1. *Let n, M^k and $0 < \alpha < \frac{4}{n+k}$ be given. There exists $\lambda^* \in (0,\infty)$ such that:*

$$\mathcal{M}_{n,M^k,\alpha,\lambda} = \mathcal{N}_{n,\bar{\omega},\alpha}, \ \forall \lambda > \lambda^* \tag{3.2}$$

and

$$\mathcal{M}_{n,M^k,\alpha,\lambda} \cap \mathcal{N}_{n,\bar{\omega},\alpha} = \emptyset, \ \forall \lambda < \lambda^* \tag{3.3}$$

where $\bar{\omega}$ is defined by the condition

$$vol(M^k)\|u_{n,\bar{\omega},\alpha}\|^2_{L^2_x} = 1$$

We fix a sequence $\lambda_j \to \infty$ and a corresponding sequence of functions $u_{\lambda_j} \in \mathcal{M}_{n,M^k,\alpha,\lambda_j}$. In the sequel we shall assume that

$$u_{\lambda_j}(x,y) \geq 0, \ \forall (x,y) \in \mathbb{R}^n_x \times M^k_y \tag{3.4}$$

Indeed it is well-known that if u_{λ_j} is a minimizer, then also $|u_{\lambda_j}|$ is a minimizer. In particular there exists at least one minimizer which satisfies (3.4).
Notice that the functions u_{λ_j} depend in principle on the full set of variables (x,y). Our aim is to prove that for j large and up to subsequence, the functions u_{λ_j} will not depend explicitly on the variable y.
First we prove some a priori bounds satisfied by $u_{\lambda_j}(x,y)$. Recall that the quantities $I^\rho_{n,\alpha}$ are defined in (2.1).

Lemma 3.1. *Assume the same assumptions as in Theorem 3.1, then we have:*

$$\lim_{j\to\infty} J_{n,M^k,\alpha,\lambda_j} = vol(M^k) I_{n,\alpha}^{1/\sqrt{vol(M^k)}} \tag{3.5}$$

and

$$\lim_{j\to\infty} \lambda_j \int_{M^k_y}\int_{\mathbb{R}^n_x} |\nabla_y u_{\lambda_j}|^2 dx dvol_{M^k_y} = 0 \tag{3.6}$$

Proof. First notice that

$$J_{n,M^k,\alpha,\lambda_j} \leq vol(M^k) I_{n,\alpha}^{1/\sqrt{vol(M^k)}} \tag{3.7}$$

In fact let $w(x) \in H^1_x$ be such that $\|w\|_{L^2_x} = \frac{1}{\sqrt{vol(M^k)}}$ and $\mathcal{E}_{n,\alpha}(w) = I_{n,\alpha}^{1/\sqrt{vol(M^k)}}$. Then we get easily:

$$J_{n,M^k,\alpha,\lambda_j} \leq \mathcal{E}_{n,M^k,\alpha,\lambda_j}(w(x))$$
$$= vol(M^k)\Big(\frac{1}{2}\int_{\mathbb{R}^n_x} |\nabla_x w|^2 dx - \frac{1}{2+\alpha}\int_{\mathbb{R}^n_x} |w|^{2+\alpha} dx\Big) = vol(M^k) I_{n,\alpha}^{1/\sqrt{vol(M^k)}}$$

and this concluded the proof of (3.7).
Next we claim that

$$\lim_{j\to\infty} \int_{M^k_y}\int_{\mathbb{R}^n_x} |\nabla_y u_{\lambda_j}|^2 dx dvol_{M^k_y} = 0 \tag{3.8}$$

In order to prove this fact assume by the absurd that it is false then there exists a subsequence of λ_j (that we still denote by λ_j) such that

$$\lim_{j\to\infty}\lambda_j=\infty \text{ and } \int_{M^k_y}\int_{\mathbb{R}^n_x}|\nabla_y u_{\lambda_j}|^2 dx dvol_{M^k_y}\geq \epsilon_0>0$$

and in particular

$$\lim_{j\to\infty}(\lambda_j-1)\int_{M^k_y}\int_{\mathbb{R}^n_x}|\nabla_y u_{\lambda_j}|^2 dx dvol_{M^k_y}=\infty \tag{3.9}$$

On the other hand by the classical Gagliardo Nirenberg inequality (see (1.4)) we deduce the existence of $0<\mu<2$ such that:

$$\frac{1}{2}\int_{M^k_y}\int_{\mathbb{R}^n_x}(|\nabla_y v|^2+|\nabla_x v|^2+|v|^2)dx dvol_{M^k_y}-\frac{1}{2+\alpha}\int_{M^k_y}\int_{\mathbb{R}^n_x}|v|^{2+\alpha}dx dvol_{M^k_y}$$
$$\geq \frac{1}{2}\int_{M^k_y}\int_{\mathbb{R}^n_x}(|\nabla_y v|^2+|\nabla_x v|^2+|v|^2)dx dvol_{M^k_y}$$
$$-C\left[\int_{M^k_y}\int_{\mathbb{R}^n_x}(|\nabla_y v|^2+|\nabla_x v|^2+|v|^2)dx dvol_{M^k_y}\right]^{\mu}$$
$$\geq \inf_{t>0}(1/2t^2-Ct^{\mu})=C(\mu)>-\infty$$
$$\forall v\in H^1(\mathbb{R}^n\times M^k) \text{ s.t. } \|v\|_{L^2_{x,y}}=1$$

By the previous inequality we get

$$\mathcal{E}_{n,M^k,\alpha,\lambda_j}(v)-\frac{1}{2}(\lambda_j-1)\int_{M^k_y}\int_{\mathbb{R}^n_x}|\nabla_y v|^2\geq -\frac{1}{2}+C(\mu)$$
$$\forall v\in H^1(\mathbb{R}^n\times M^k) \text{ s.t. } \|v\|_{L^2_{x,y}}=1$$

In particular if we choose $v=u_{\lambda_j}$ then we get

$$J_{n,M^k,\alpha,\lambda_j}=\mathcal{E}_{n,M^k,\alpha,\lambda_j}(u_{\lambda_j})$$
$$\geq \frac{1}{2}(\lambda_j-1)\int_{M^k_y}\int_{\mathbb{R}^n_x}|\nabla_y u_{\lambda_j}|^2 dx dvol_{M^k_y}-\frac{1}{2}+C(\mu)$$

By (3.9) this implies $\lim_{n\to\infty}J_{n,M^k,\alpha,\lambda_j}=\infty$ and this is in contradiction with (3.7). Hence (3.8) is proved.

Next we introduce the functions

$$w_j(y)=\|u_{\lambda_j}(x,y)\|^2_{L^2_x}$$

Notice that

$$\|w_j(y)\|_{L^1_y}=1 \tag{3.10}$$

and moreover

$$\int_{M^k_y}|\nabla_y w_j(y)|dvol_{M^k_y}\leq C\int_{M^k_y}\int_{\mathbb{R}^n_x}|u_{\lambda_j}(x,y)||\nabla_y u_{\lambda_j}(x,y)|dx dvol_{M^k_y}$$
$$\leq C\|u_{\lambda_j}\|_{L^2_{x,y}}\|\nabla_y u_{\lambda_j}\|_{L^2_{x,y}}$$

Hence due to (3.8) we get

$$\lim_{j\to\infty}\|\nabla_y w_j\|_{L^1_y}=0 \tag{3.11}$$

By combining (3.10) and (3.11) with the Rellich compactness theorem and with the Sobolev embedding $W^{1,1}(M^1) \subset L^\infty(M^1)$ and $W^{1,1}(M^2) \subset L^2(M^2)$ we deduce respectively in the case $k = 1$ and $k = 2$ that (up to a subsequence)

$$\lim_{j\to\infty} \|w_j(y) - 1/vol(M^1)\|_{L^r_y} = 0, \ \forall 1 \le r < \infty \tag{3.12}$$

and

$$\lim_{j\to\infty} \|w_j(y) - 1/vol(M^2)\|_{L^r_y} = 0, \ \forall 1 \le r < 2 \tag{3.13}$$

For $k > 2$ we use the Sobolev embedding $H^1(M^k) \subset L^{2k/(k-2)}(M^k)$ and we get

$$\sup_j \|u_{\lambda_j}\|_{L^2_x L^{2k/(k-2)}_y} \le C \sup_j \|u_{\lambda_j}\|_{L^2_x H^1(M^k_y)} < \infty$$

(where at the last step we have used the fact $\sup_j \left(\|u_{\lambda_j}\|_{L^2_{x,y}} + \|\nabla_y u_{\lambda_j}\|_{L^2_{x,y}}\right) < \infty$). By the Minkowski inequality the bound above implies $\sup_j \|u_{\lambda_j}\|_{L^{2k/(k-2)}_y L^2_x}$ which is equivalent to the condition

$$\sup_j \|w_j(y)\|_{L^{k/(k-2)}_y} < \infty \text{ for } k > 2 \tag{3.14}$$

By combining (3.10) and (3.11) with the Rellich compactness theorem we deduce that up to a subsequence

$$\|w_j(y) - 1/vol(M^k)\|_{L^1_y} = 0 \text{ for } k > 2$$

and hence by interpolation with (3.14) we get

$$\|w_j(y) - 1/vol(M^k)\|_{L^r_y} = 0 \text{ for } k > 2, 1 \le r < k/(k-2) \tag{3.15}$$

By the definition of $I^\rho_{n,\alpha}$ (see (2.1)) and (2.3) we get

$$\frac{1}{2}\int_{\mathbb{R}^n_x} |\nabla_x u_{\lambda_j}(x,y)|^2 dx - \frac{1}{2+\alpha}\int_{\mathbb{R}^n_x} |u_{\lambda_j}(x,y)|^{2+\alpha} dx \tag{3.16}$$

$$\ge I_{n,\alpha}^{\|u_{\lambda_j}(\cdot,y)\|_{L^2_x}} = I^1_{n,\alpha}\|u_{\lambda_j}(\cdot,y)\|_{L^2_x}^{(8+4\alpha-2\alpha n)/(4-\alpha n)} = I^1_{n,\alpha} w_j(y)^{(4+2\alpha-\alpha n)/(4-\alpha n)}$$
$$\forall y \in M^k, \ \forall j \in \mathbb{N}$$

Next notice that by definition

$$J_{n,M^k,\alpha,\lambda_j} = \mathcal{E}_{n,M^k,\alpha,\lambda_j}(u_{\lambda_j}) \tag{3.17}$$
$$= \frac{1}{2}\int_{M^k_y}\int_{\mathbb{R}^n_x} (\lambda_j|\nabla_y u_{\lambda_j}|^2 + |\nabla_x u_{\lambda_j}|^2) dxdy - \frac{1}{2+\alpha}\int_{M^k_y}\int_{\mathbb{R}^n_x} |u|^{2+\alpha} dx dvol_{M^k_y}$$

and we can continue

$$... \ge \int_{M^k_y}\Big(\frac{1}{2}\int_{\mathbb{R}^n_x} |\nabla_x u_{\lambda_j}(x,y)|^2 dx - \frac{1}{2+\alpha}\int_{\mathbb{R}^n_x} |u_{\lambda_j}(x,y)|^{2+\alpha} dx\Big) dvol_{M^k_y} \tag{3.18}$$
$$\ge I^1_{n,\alpha}\int_{M^k_y} w_j(y)^{(4+2\alpha-\alpha n)/(4-\alpha n)} dvol_{M^k_y}$$
$$= I^1_{n,\alpha} vol(M^k) vol(M^k)^{-(4+2\alpha-\alpha n)/(4-\alpha n)} + o(1)$$

where $o(1) \to 0$ as $j \to \infty$ and at the last step we have combined (3.12), (3.13) and (3.15) respectively for $k = 1$, $k = 2$ and $k > 2$ and we used our assumption on α. By combining this fact with (2.3) we have

$$\liminf_{j\to\infty} J_{n,M^k,\alpha,\lambda_j} \ge vol(M^k) I_{n,\alpha}^{1/\sqrt{vol(M^k)}} \tag{3.19}$$

Hence (3.5) follows by combining (3.7) with (3.19).

Next we prove (3.6). For that purpose, it suffices to keep the term $\lambda_j|\nabla_y u_{\lambda_j}|^2$ in the previous analysis. Namely, by combining (3.5) with (3.17) and (3.18) we get

$$vol(M^k)I_{n,\alpha}^{1/\sqrt{vol(M^k)}} + g(j) \geq \frac{1}{2}\lambda_j \int_{M_y^k}\int_{\mathbb{R}_x^n} |\nabla_y u_{\lambda_j}|^2 dxdvol_{M_y^k} + h(j) \tag{3.20}$$

where

$$\lim_{j\to\infty} g(j) = 0$$

and

$$\liminf_{j\to\infty} h(j) \geq vol(M^k)I_{n,\alpha}^{1/\sqrt{vol(M^k)}}$$

Hence (3.6) follows by (3.20).

□

Lemma 3.2. *We have the following identity:*

$$\int_{M_y^k}\int_{\mathbb{R}_x^n} |\nabla_x u_{\lambda_j}|^2 dxdvol_{M_y^k} = \frac{\alpha n}{2(2+\alpha)}\int_{M_y^k}\int_{\mathbb{R}_x^n} |u_{\lambda_j}|^{2+\alpha} dxdvol_{M_y^k} \tag{3.21}$$

Moreover there exist $J\in\mathbb{N}$ such that

$$\forall j > J\ \exists \omega(\lambda_j) > 0\ s.t.$$

$$-\lambda_j\Delta_y u_{\lambda_j} - \Delta_x u_{\lambda_j} + \omega(\lambda_j)u_{\lambda_j} = u_{\lambda_j}|u_{\lambda_j}|^\alpha \tag{3.22}$$

and the following limit exists

$$\lim_{j\to\infty}\omega(\lambda_j) = \bar\omega \in (0,\infty) \tag{3.23}$$

Proof. Since u_{λ_j} is a constrained minimizer for $\mathcal{E}_{n,M^k,\alpha,\lambda_j}$ on the ball of size 1 in $L^2(\mathbb{R}^n\times M^k)$, then we get

$$\frac{d}{d\epsilon}\Big[\mathcal{E}_{n,M^k,\alpha,\lambda_j}(\epsilon^{\frac{n}{2}}u_{\lambda_j}(\epsilon x, y)\Big]_{\epsilon=1} = 0$$

which is equivalent to

$$\frac{d}{d\epsilon}\Big[\frac{1}{2}\lambda_j\int_{M_y^k}\int_{\mathbb{R}_x^n}|\nabla_y u_{\lambda_j}|^2 dxdvol_{M_y^k}$$

$$+\frac{1}{2}\epsilon^2\int_{M_y^k}\int_{\mathbb{R}_x^n}|\nabla_x u_{\lambda_j}|^2 dxdvol_{M_y^k} - \frac{1}{2+\alpha}\epsilon^{\alpha n/2}\|u_{\lambda_j}\|_{L_{x,y}^{2+\alpha}}^{2+\alpha}\Big]_{\epsilon=1} = 0$$

By computing explicitly the derivative (in ϵ) we deduce (3.21).

Next notice that by using the Lagrange multiplier technique we get (3.22) for a suitable $\omega(\lambda_j)\in\mathbb{R}$. On the other hand by (3.22) we get

$$\int_{M_y^k}\int_{\mathbb{R}_x^n}(\lambda_j|\nabla_y u_{\lambda_j}|^2 + |\nabla_x u_{\lambda_j}|^2)dxdvol_{M_y^k} + \omega(\lambda_j)\|u_{\lambda_j}\|_{L_{x,y}^2}^2$$

$$= \int_{M_y^k}\int_{\mathbb{R}_x^n}|u_{\lambda_j}|^{2+\alpha}dxdvol_{M_y^k}$$

that by (3.21) gives

$$\omega(\lambda_j) = \frac{-\alpha n + 4 + 2\alpha}{\alpha n} \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_x u_{\lambda_j}|^2 dx dvol_{M_y^k} - \lambda_j \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_y u_{\lambda_j}|^2 dx dvol_{M_y^k}$$

and hence by (3.6) we get

$$\omega(\lambda_j) = \frac{-\alpha n + 4 + 2\alpha}{\alpha n} \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_x u_{\lambda_j}|^2 dx dvol_{M_y^k} + o(1) \tag{3.24}$$

where $\lim_{j\to\infty} o(1) = 0$.

On the other hand notice that by (3.21) we get

$$J_{n,M^k,\alpha,\lambda_j} = \mathcal{E}_{n,M^k,\alpha,\lambda_j}(u_{\lambda_j})$$

$$= \frac{\alpha n - 4}{2\alpha n} \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_x u_{\lambda_j}|^2 dx dvol_{M_y^k} + \frac{1}{2} \int_{M_y^k} \int_{\mathbb{R}_x^n} \lambda_j |\nabla_y u_{\lambda_j}|^2 dx dvol_{M_y^k}$$

and by (3.6)

$$\int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_x u_{\lambda_j}|^2 dx dvol_{M_y^k} = \frac{2\alpha n}{\alpha n - 4} J_{n,M^k,\alpha,\lambda_j} + o(1) \tag{3.25}$$

By (3.5) it implies

$$\int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_x u_{\lambda_j}|^2 dx dvol_{M_y^k} = \frac{2\alpha n}{\alpha n - 4} vol(M^k) I_{n,\alpha}^{1/\sqrt{vol(M^k)}} + o(1) \tag{3.26}$$

that in conjunction with (3.24) and (2.4) implies $\omega(\lambda_j) > 0$ for j large enough. Moreover (3.23) follows by (3.24) and (3.26).

□

Next recall that the sets $\mathcal{M}_{n,\alpha}^{\rho}$ are the ones defined in (2.6).

Lemma 3.3. *Let $\bar{\omega}$ be as in* (3.23) *and let $v(x) \in \mathcal{M}_{n,\alpha}^{1/\sqrt{vol(M^k)}}$ be such that $v(x) > 0$. Then*

$$-\Delta_x v + \bar{\omega} v = v|v|^\alpha$$

Proof. It is well-known that

$$-\Delta_x v + \omega_1 v = v|v|^\alpha$$

for a suitable $\omega_1 > 0$. More precisely we can assume that up to translation $v = u_{n,\omega_1,\alpha}$. Our aim is to prove that $\omega_1 = \bar{\omega}$. Notice that by (2.8)

$$\omega_1 \frac{1}{vol(M^k)} = \frac{4\alpha + 8 - 2\alpha n}{\alpha n - 4} I_{n,\alpha}^{\|v\|_{L_x^2}} = \frac{4\alpha + 8 - 2\alpha n}{\alpha n - 4} I_{n,\alpha}^{1/\sqrt{vol(M^k)}} \tag{3.27}$$

On the other hand by (3.24) and (3.26) we get

$$\omega(\lambda_j) = \frac{-2\alpha n + 8 + 4\alpha}{\alpha n - 4} vol(M^k) I_{n,\alpha}^{1/\sqrt{vol(M^k)}} + o(1)$$

and hence passing to the limit in j we get

$$\bar{\omega} = \frac{-2\alpha n + 8 + 4\alpha}{\alpha n - 4} vol(M^k) I_{n,\alpha}^{1/\sqrt{vol(M^k)}} \tag{3.28}$$

By combining (3.27) and (3.28) we get $\bar{\omega} = \omega_1$.

□

Lemma 3.4. *There exist a subsequence of λ_j (that we shall denote still by λ_j) and a sequence $\tau_j \in \mathbb{R}^n_x$ such that*

$$\lim_{j\to\infty} \|u_{\lambda_j}(x+\tau_j, y) - u_{\bar\omega}\|_{H^1(\mathbb{R}^n\times M^k)} = 0$$

where $u_{\bar\omega} \in \mathcal{N}_{n,\bar\omega,\alpha}$, $u_{\bar\omega} > 0$ and $\bar\omega$ is defined in (3.23).

Proof. By combining (3.6) and (3.26), and since $\|u_{\lambda_j}\|_{L^2_{x,y}} = 1$, we deduce that u_{λ_j} is bounded in $H^1(\mathbb{R}^n \times M^k)$. Moreover by combining (3.5) with the fact that $I_{n,\alpha}^{1/\sqrt{vol(M^k)}} < 0$ (see (2.4)) then we get

$$\inf_j \|u_{\lambda_j}\|_{L^{2+\alpha}_{x,y}} > 0$$

By using the localized version of the Gagliardo Nirenberg inequality (6.5) (in the same spirit as in the appendix) we get the existence (up to subsequence) of $\tau_j \in \mathbb{R}^n_x$ such that

$$u_{\lambda_j}(x+\tau_j, y) \rightharpoonup w \neq 0 \text{ in } H^1(\mathbb{R}^n \times M^k)$$

Moreover due to (3.4) we can assume that

$$w(x,y) \geq 0 \text{ a.e. } (x,y) \in \mathbb{R}^n_x \times M^k_y$$

and by (3.6) we get $\nabla_y w = 0$. In particular w is y-independent.
By combining (3.6) and (3.23) we pass to the limit in (3.22) in the distribution sense and we get

$$-\Delta_x w + \bar\omega w = w|w|^\alpha \text{ in } \mathbb{R}^n_x,\ w(x) \geq 0,\ w \neq 0 \tag{3.29}$$

We claim that

$$\|w\|_{L^2_x} = \frac{1}{\sqrt{vol(M^k)}} \tag{3.30}$$

If not then we can assume $\|w\|_{L^2_x} = \beta < \frac{1}{\sqrt{vol(M^k)}}$ and since w solves (3.29) by (2.5) we get

$$w \in \mathcal{M}^\beta_{n,\alpha} \tag{3.31}$$

On the other hand by Lemma 3.3 the equation (3.29) is satisfied by any $v \in \mathcal{M}_{n,\alpha}^{\frac{1}{\sqrt{vol(M^k)}}}$. Hence again by (2.5) and by the injectivity of the map $\rho \to \omega(\rho)$ (see (1.12)) we deduce that necessarily $\beta = \frac{1}{\sqrt{vol(M^k)}}$.
In particular by (3.30) we deduce

$$\lim_{j\to\infty} \|u_{\lambda_j}(x+\tau_j, y) - w\|_{L^2_{x,y}} = 0$$

Next notice that by (3.6) and since we have already proved that $\nabla_y w = 0$ we deduce that

$$\lim_{j\to\infty} \|\nabla_y u_{\lambda_j}(x+\tau_j, y)\|_{L^2_{x,y}} = 0 = \|\nabla_y w\|_{L^2_{x,y}}$$

Hence in order to conclude that $u_{\lambda_j}(x+\tau_j, y)$ converges strongly to w in $H^1(\mathbb{R}^n \times M^k)$ it is sufficient to prove that

$$\lim_{j\to\infty} \|\nabla_x u_{\lambda_j}(x+\tau_j, y)\|_{L^2_{x,y}} = \sqrt{vol(M^k)}\|\nabla_x w\|_{L^2_x} = \|\nabla_x w\|_{L^2_{x,y}}$$

This last fact follows by combining (2.9) (where we use the fact that $w \in \mathcal{N}_{n,\bar\omega,\alpha}$ by (3.29) and $\|w\|_{L^2_x} = \frac{1}{\sqrt{vol(M^k)}}$ by (3.30)) and (3.26).

□

Lemma 3.5. *There exists* $j_0 > 0$ *such that*

$$\nabla_y u_{\lambda_j} = 0, \ \forall j > j_0$$

Proof. By Lemma 3.4 we can assume that

$$u_{\lambda_j} \to u_{\bar\omega} \text{ in } H^1(\mathbb{R}^n \times M^k) \tag{3.32}$$

We introduce $w_j = \sqrt{-\Delta_y} u_{\lambda_j}$. Notice that due to (3.22) the functions w_j satisfy

$$-\lambda_j \Delta_y w_j - \Delta_x w_j + \omega(\lambda_j) w_j = \sqrt{-\Delta_y}(u_{\lambda_j}|u_{\lambda_j}|^\alpha) \tag{3.33}$$

that after multiplication by w_j implies

$$\int_{M_y^k}\int_{\mathbb{R}_x^n} \Big[\lambda_j |\nabla_y w_j|^2 + |\nabla_x w_j|^2 + \omega(\lambda_j)|w_j|^2 \tag{3.34}$$
$$-\sqrt{-\Delta_y}(u_{\lambda_j}|u_{\lambda_j}|^\alpha) w_j\Big] dx dvol_{M_y^k} = 0$$

In turn it gives

$$0 = \int_{M_y^k}\int_{\mathbb{R}_x^n} (\lambda_j - 1)|\nabla_y w_j|^2 - (\alpha+1)\sqrt{-\Delta_y}(u_{\lambda_j}|u_{\bar\omega}|^\alpha) w_j dx dvol_{M_y^k} + \tag{3.35}$$
$$\int_{M_y^k}\int_{\mathbb{R}_x^n} (|\nabla_y w_j|^2 + |\nabla_x w_j|^2 + \bar\omega |w_j|^2 + \sqrt{-\Delta_y}(u_{\lambda_j}((\alpha+1)|u_{\bar\omega}|^\alpha - |u_{\lambda_j}|^\alpha)) w_j dx dvol_{M_y^k}$$
$$+ \int_{M_y^k}\int_{\mathbb{R}_x^n} (\omega(\lambda_j) - \bar\omega)|w_j|^2 dx dy \equiv$$
$$I_j + II_j + III_j$$

Next we fix an orthonormal basis of eigenfunctions for $-\Delta_y$, i.e. $-\Delta_y \varphi_k = \mu_k \varphi_k$ and $\varphi_0 = \text{const}$. We can write the following development

$$w_j(x,y) = \sum_{k \in \mathbb{N}\setminus\{0\}} a_{j,k}(x)\varphi_k(y) \tag{3.36}$$

(where the eigenfunction φ_0 does not enter in the development). By using the representation in (3.36) we get

$$I_j \geq \sum_{k\neq 0} (\lambda_j - 1)|\mu_k|^2 \int_{\mathbb{R}_x^n} |a_{j,k}(x)|^2 dx - (\alpha+1)\sum_{k\neq 0}\int_{\mathbb{R}_x^n} |u_{\bar\omega}(x)|^\alpha |a_{j,k}(x)|^2 dx \tag{3.37}$$

and by (3.23) we get

$$III_j = o(1)\|w_j\|^2_{L^2_{x,y}} \tag{3.38}$$

By combining (3.37) with (3.38) we get

$$I_j + III_j \geq 0 \tag{3.39}$$

for j large enough. In order to estimate II_j notice that by the Cauchy-Schwartz inequality we get

$$\Big|\int_{M_y^k}\int_{\mathbb{R}_x^n} \sqrt{-\Delta_y}(u_{\lambda_j}((\alpha+1)|u_{\bar\omega}|^\alpha - |u_{\lambda_j}|^\alpha)) w_j dx dvol_{M_y^k}\Big| \tag{3.40}$$
$$\leq \|\sqrt{-\Delta_y}(u_{\lambda_j}((\alpha+1)|u_{\bar\omega}|^\alpha - |u_{\lambda_j}|^\alpha))\|_{L_x^{\frac{2(n+k)}{n+k+2}} L_y^{\frac{2(n+k)}{n+k+2}}} \|w_j\|_{L_{x,y}^{\frac{2(n+k)}{n+k-2}}}$$

$$\leq C\|\nabla_y(u_{\lambda_j}((\alpha+1)|u_{\bar\omega}|^\alpha - |u_{\lambda_j}|^\alpha))\|_{L_x^{\frac{2(n+k)}{n+k+2}} L_y^{\frac{2(n+k)}{n+k+2}}} \|w_j\|_{L_{x,y}^{\frac{2(n+k)}{n+k-2}}}$$

where at the last step we have used the following estimate

$$\forall p \in (1,\infty)\ \exists c(p), C(p) > 0 \text{ s.t.} \tag{3.41}$$

$$c(p)\|\sqrt{-\Delta_y} f\|_{L_y^p} \leq \|\nabla_y f\|_{L_y^p} \leq C(p)\|\sqrt{-\Delta_y} f\|_{L_y^p}$$

Indeed, using [10, Theorem 3.3.1], we have that $\sqrt{-\Delta_y}$ is a first order classical pseudo differential operator on M with a principle symbol $(g^{i,j}(y)\xi_i\,\xi_j)^{1/2}$. Observe that

$$C_1 \sum_{i,j} g^{i,j}(y)\xi_i\,\xi_j \leq \sum_i |\sum_j g^{i,j}(y)\xi_j|^2 \leq C_2|\xi|^2 \leq C_3 \sum_{i,j} g^{i,j}(y)\xi_i\,\xi_j$$

Moreover one can assume that in (3.41) f has no zero frequency. Then one can deduce (3.41) by working in local coordinates, introducing a classical angular partition of unity according to the index $l \in [1,\cdots,k]$ such that

$$\sum_{i,j} g^{i,j}(y)\xi_i\,\xi_j \leq c|\sum_j g^{l,j}(y)\xi_j|^2$$

and, most importantly, using the L^p boundedness of zero order pseudo differential operators on $\mathbb{R}^k$ (for the proof of this fact we refer to [10, Theorem 3.1.6]).

Next, by the chain rule we get

$$\nabla_y\Big(u_{\lambda_j}((\alpha+1)|u_{\bar\omega}|^\alpha - |u_{\lambda_j}|^\alpha)\Big)$$
$$= (\alpha+1)\nabla_y u_{\lambda_j}\Big(|u_{\bar\omega}|^\alpha - |u_{\lambda_j}|^\alpha\Big)$$

and by the Hölder inequality we can continue the estimate (3.40) as follows

$$... \leq C\Big\|\|\nabla_y u_{\lambda_j}\|_{L_y^q}\||u_{\bar\omega}|^\alpha - |u_{\lambda_j}|^\alpha\|_{L_y^r}\Big\|_{L_x^{\frac{2(n+k)}{n+k+2}}} \|w_j\|_{L_{x,y}^{\frac{2(n+k)}{n+k-2}}}$$

where

$$\frac{1}{q} + \frac{1}{r} = \frac{n+k+2}{2(n+k)}$$

and again by the Hölder inequality in the x-variable we can continue

$$... \leq C\|\nabla_y u_{\lambda_j}\|_{L_{x,y}^q}\||u_{\bar\omega}|^\alpha - |u_{\lambda_j}|^\alpha\|_{L_{x,y}^r}\|w_j\|_{L_{x,y}^{\frac{2(n+k)}{n+k-2}}}$$

Notice that if we fix

$$q = \frac{2(n+k)}{n+k-2} \text{ and } r = \frac{n+k}{2}$$

then by combining the Sobolev embedding

$$H^1_{x,y} \subset L_{x,y}^{\frac{2(n+k)}{n+k-2}} \tag{3.42}$$

with (3.32) and (3.41), we can continue the estimate

$$... \leq o(1)\|\sqrt{-\Delta_y} u_{\lambda_j}\|_{L_{x,y}^q}\|w_j\|_{H^1_{x,y}} = o(1)\|w_j\|^2_{H^1_{x,y}}$$

where $\lim_{j\to\infty} o(1) = 0$. By combining this information in conjunction with the structure of II_j we get

$$II_j \geq \|w_j\|^2_{H^1_{x,y}}(1 - o(1)) \geq 0 \text{ for } j > j_0 \tag{3.43}$$

By combining (3.35), (3.39) and (3.43) we deduce $w_j = 0$ for j large enough.

□

Proof of Theorem 3.1 By using the diamagnetic inequality we deduce that (up to a remodulation factor $e^{i\theta}$) we can assume that $v \in \mathcal{M}_{n,M^k,\alpha,\lambda}$ is real valued. Moreover if $v \in \mathcal{M}_{n,M^k,\alpha,\lambda}$ then also $|v| \in \mathcal{M}_{n,M^k,\alpha,\lambda}$. By a standard application of the strong maximum principle we finally deduce that it is not restrictive to assume that $v \in \mathcal{M}_{n,M^k,\alpha,\lambda}$ and $v(x,y)>0, \ \forall (x,y) \in \mathbb{R}^n_x \times M^k_y$.

First step: $\exists \tilde\lambda > 0$ *s.t.* $\forall v \in \mathcal{M}_{n,M^k,\alpha,\lambda}, v(x,y)>0$ *we have* $\nabla_y v = 0, \ \forall \lambda > \tilde\lambda$

Assume that the conclusion is false then there exists $\lambda_j \to \infty$ such that $u_{\lambda_j}(x,y) \in M_{n,M^k,\alpha,\lambda_j}, u_{\lambda_j}(x,y) > 0$ and $\nabla_y u_{\lambda_j} \neq 0$. This is absurd due to Lemma 3.5.

Second step: conclusion

We define

$$\lambda^* = \inf_\lambda \{\lambda > 0 | \nabla_y v = 0 \ \forall v \in \mathcal{M}_{n,M^k,\alpha,\lambda}\}$$

By the first step $\lambda^* < \infty$. Moreover it is easy to deduce that if $\lambda > \lambda^*$ then the minimizers of the problem $J_{n,M^k,\alpha,\lambda}$ are precisely the same minimizers of the problem $I_{n,\alpha}^{1/\sqrt{vol(M^k)}}$, which in turn are characterized in section 2 (hence we get (3.2)).

Next we prove that $\lambda^* > 0$. It is sufficient to show that

$$\lim_{\lambda \to 0} J_{n,M^k,\alpha,\lambda} < vol(M^k) I_{n,\alpha}^{1/\sqrt{vol(M^k)}} \tag{3.44}$$

(see (2.1) and (3.1) for a definition of the quantities involved in the inequality above). Let us fix $\rho(y) \in C^\infty(M^k)$ such that

$$\int_{M^k} |\rho|^2 dvol_{M^k_y} = 1$$

and $\rho^2(y_0) \neq \frac{1}{vol(M^k)}$ for some $y_0 \in M^k$ (i.e. $\rho(y)$ is not identically constant). Then we introduce the functions

$$\psi(x,y) = \rho(y)^{4/(4-\alpha n)} Q(\rho(y)^{(2\alpha)/(4-\alpha n)} x)$$

where $Q(x)$ is the unique radially symmetric minimizer for $I_{n,\alpha}^{1/\sqrt{vol(M^k)}}$. Then we get

$$\|\psi(x,y)\|^2_{L^2_x} = (\rho(y))^2 \text{ and } \mathcal{E}_{n,\alpha}(\psi(x,y)) = I^1_{n,\alpha}(\rho(y))^{\frac{8+4\alpha-2\alpha n}{4-\alpha n}}$$

and as a consequence we deduce

$$\int_{M^k_y} \int_{\mathbb{R}^n_x} \Big(\frac{1}{2} |\nabla_x \psi(x,y)|^2 - \frac{1}{2+\alpha} |\psi(x,y)|^{2+\alpha} \Big) dx dvol_{M^k_y}$$

$$= I^1_{n,\alpha} \int_{M^k_y} (\rho(y))^{\frac{8+4\alpha-2\alpha n}{4-\alpha n}} dvol_{M^k_y}$$

$$< I^1_{n,\alpha} \Big(\int_{M^k} (\rho(y))^2 dvol_{M^k_y} \Big)^{\frac{4-\alpha n+2\alpha}{4-\alpha n}} vol(M^k)^{-\frac{2\alpha}{4-\alpha n}} = I^1_{n,\alpha} vol(M^k)^{-\frac{2\alpha}{4-\alpha n}}$$

where at the last inequality we have used the fact that $I^1_{n,\alpha} < 0$ in conjunction with the Hölder inequality (moreover we get the inequality $<$ since by hypothesis $\rho(y)$ is not identically constant). As a byproduct we get

$$\lim_{\lambda\to 0} \mathcal{E}_{n,M^k,\alpha,\lambda}(\psi(x,y)) < I^1_{n,\alpha} vol(M^k)^{-\frac{2\alpha}{4-\alpha n}} = vol(M^k) I_{n,\alpha}^{1/\sqrt{vol(M^k)}}$$

(where we have used (2.3)) which in turn implies (3.44).

Let us finally prove (3.3). It is sufficient to show that if $v \in \mathcal{M}_{n,M^k,\alpha,\lambda}$ for $\lambda < \lambda^*$ then $\nabla_y v \neq 0$. Assume by the absurd that it is false, then we get $\lambda_1 < \lambda^*$ and $v_1 \in \mathcal{M}_{n,M^k,\alpha,\lambda_1}$ such that $\nabla_y v_1 = 0$. Arguing as above it implies that

$$J_{n,M^k,\alpha,\lambda_1} = vol(M^k) I_{n,\alpha}^{1/\sqrt{vol(M^k)}} \tag{3.45}$$

On the other hand by definition of λ^* there exists $\lambda_2 \in (\lambda_1, \lambda^*]$ and $v_2 \in \mathcal{M}_{n,M^k,\alpha,\lambda_2}$ such that $\nabla_y v_2 \neq 0$. As a consequence we deduce that

$$J_{n,M^k,\alpha,\lambda_1} < \mathcal{E}_{n,M^k,\alpha,\lambda_2}(v_2) = J_{n,M^k,\alpha,\lambda_2} \leq vol(M^k) I_{n,\alpha}^{1/\sqrt{vol(M^k)}}$$

where at the last step we have used (3.7). Hence we get a contradiction with (3.45).

□

4. Proof of theorem 1.3

In the sequel the homogeneity of the euclidean space $\mathbb{R}^n$ will play a key role. Due to this property we shall be able to reduce the proof of Theorem 1.3 to the problem studied in the previous section.

In view of section 2 it is sufficient to prove that there exists $\rho^* > 0$ such that

$$v \in \mathcal{M}^\rho_{n,M^k,\alpha} \text{ implies } \nabla_y v = 0 \text{ for } \rho < \rho^* \tag{4.1}$$

and

$$v \in \mathcal{M}^\rho_{n,M^k,\alpha} \text{ implies } \nabla_y v \neq 0 \text{ for } \rho > \rho^* \tag{4.2}$$

By an elementary computation we have that the map

$$S_1 \ni u \to \rho^{4/(4-\alpha n)} u(\rho^{2\alpha/(4-\alpha n)} x, y) \in S_\rho$$

where

$$S_\lambda = \{v \in H^1(\mathbb{R}^n \times M^k) | \|v\|_{L^2_{x,y}} = \lambda\}$$

is a bijection. Moreover we have

$$\mathcal{E}_{n,M_k,\alpha}(\rho^{4/(4-\alpha n)} u(\rho^{2\alpha/(4-\alpha n)} x, y)) = \rho^{(8-2\alpha n)/(4-\alpha n)} \int_{M^k_y}\int_{\mathbb{R}^n_x} |\nabla_y u|^2 dx dvol_{M^k_y}$$

$$+\rho^{(8-2\alpha n+4\alpha)/(4-\alpha n)} \int_{M^k_y}\int_{\mathbb{R}^n_x} |\nabla_x u|^2 dx dvol_{M^k_y}$$

$$-\rho^{(8-2\alpha n+4\alpha)/(4-\alpha n)} \frac{1}{2+\alpha} \int_{M^k_y}\int_{\mathbb{R}^n_x} |u|^{2+\alpha} dx dvol_{M^k_y}$$

$$= \rho^{(8-2\alpha n+4\alpha)/(4-\alpha n)} \Big(\frac{1}{2}\rho^{-4\alpha/(4-\alpha n)} \int_{M^k_y}\int_{\mathbb{R}^n_x} |\nabla_y u|^2 dx dvol_{M^k_y}$$

$$+\frac{1}{2}\int_{M^k_y}\int_{\mathbb{R}^n_x} |\nabla_x u|^2 - \frac{1}{2+\alpha}|u|^{2+\frac{4}{d}} dx dvol_{M^k_y}\Big)$$

In particular (4.1) and (4.2) are satisfied provided that there exists $\rho^* > 0$ such that

$$v \in \mathcal{M}_{n,M^k,\alpha,\rho^{-4\alpha/(4-\alpha n)}} \text{ implies } \nabla_y v = 0 \text{ for } \rho < \rho^* \tag{4.3}$$

and

$$v \in \mathcal{M}_{n,M^k,\alpha,\rho^{-4\alpha/(4-\alpha n)}} \text{ implies } \nabla_y v \neq 0 \text{ for } \rho > \rho^* \tag{4.4}$$

that in turn follow by Theorem 3.1.

5. Proof of Theorem 1.4

The main tool we use is the following Strichartz type estimates (whose proof follows by [12]).

Proposition 5.1. *For every manifold M^k_y, $n \geq 1$ and $p, q \in [2, \infty]$ such that:*

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \ (p,n) \neq (2,2)$$

there exists $C > 0$ such that

$$\|e^{-it\Delta_{x,y}} f\|_{L^p_t L^q_x H^1_y} + \Big\| \int_0^t e^{-i(t-s)\Delta_{x,y}} F(s) ds \Big\|_{L^p_t L^q_x H^1_y} \tag{5.1}$$
$$\leq C \Big(\|f\|_{L^2_x H^1_y} + \|F\|_{L^{p'}_t L^{q'}_x H^1_y} \Big);$$

$$\|\nabla_x e^{-it\Delta_{x,y}} f\|_{L^p_t L^q_x L^2_y} + \Big\| \nabla_x \int_0^t e^{-i(t-s)\Delta_{x,y}} F(s) ds \Big\|_{L^p_t L^q_x L^2_y} \tag{5.2}$$
$$\leq C \Big(\|\nabla_x f\|_{L^2_x L^2_y} + \|\nabla_x F\|_{L^{p'}_t L^{q'}_x L^2_y} \Big)$$

and

$$\Big\| \int_0^t e^{-i(t-s)\Delta_{x,y}} F(s) ds \Big\|_{L^p_t L^q_x L^2_y} \leq C \|F\|_{L^{p'}_t L^{q'}_x L^2_y} \tag{5.3}$$

Moreover

$$\|e^{-it\Delta_{x,y}} f\|_{L^\infty_t L^2_x H^1_y} + \Big\| \int_0^t e^{-i(t-s)\Delta_{x,y}} F(s) ds \Big\|_{L^\infty_t L^2_x H^1_y} \tag{5.4}$$
$$\leq C \Big(\|f\|_{L^2_x H^1_y} + \|F\|_{L^{p'}_t L^{q'}_x H^1_y} \Big)$$

and

$$\|\nabla_x e^{-it\Delta_{x,y}} f\|_{L^\infty_t L^2_x L^2_y} + \Big\| \nabla_x \int_0^t e^{-i(t-s)\Delta_{x,y}} F(s) ds \Big\|_{L^\infty_t L^2_x L^2_y} \tag{5.5}$$
$$\leq C \Big(\|\nabla_x f\|_{L^2_x L^2_y} + \|\nabla_x F\|_{L^{p'}_t L^{q'}_x L^2_y} \Big)$$

Next we shall use the norms $\|.\|_{X_T}$ and $\|.\|_{Y_T}$ introduced in (1.15) and (1.16) for time dependent functions. We also introduce the space Z_T whose norm is defined by

$$\|v\|_{Z_T} \equiv \|v\|_{X_T} + \|v\|_{Y_T}$$

and the nonlinear operator associated to the Cauchy problem (1.1):

$$\mathcal{T}_\varphi(u) \equiv e^{-it\Delta_{x,y}} \varphi + \int_0^t e^{-i(t-s)\Delta_{x,y}} u(s) |u(s)|^\alpha ds$$

We split the proof of Theorem 1.4 in several steps.

5.1. **Local Well Posedness.** This subsection is devoted to the proof of the following fact:

$$\forall \varphi \in H^1(\mathbb{R}^n \times M^1)\ \exists T = T(\|\varphi\|_{H^1(\mathbb{R}^n\times M^1)}) > 0$$
$$\text{and } \exists! v(t,x) \in Z_T \cap \mathcal{C}((-T,T); H^1(\mathbb{R}^n \times M^1))$$
$$\text{s.t. } \mathcal{T}_\varphi v(t) = v(t) \quad \forall t \in (-T,T)$$

First step:

$\forall \varphi \in H^1(\mathbb{R}^n \times M^1)\ \exists T = T(\|\varphi\|_{H^1(\mathbb{R}^n\times M^1)}) > 0, R = R(\|\varphi\|_{H^1(\mathbb{R}^n\times M^1)}) > 0$ *s.t.*

$$\mathcal{T}_\varphi(B_{Z_{\tilde{T}}}(0,R)) \subset B_{Z_{\tilde{T}}}(0,R)\ \forall \tilde{T} < T$$

First we estimate the nonlinear term:

$$\|u|u|^\alpha\|_{L_t^{p'}L_x^{q'}H_y^1} \leq \left\| \|u^\alpha(t,x,.)\|_{L_y^\infty} \|u(t,x,.)\|_{H_y^1} \right\|_{L_t^{p'}L_x^{q'}}$$

(where (p,q) is the couple in (1.15) and (1.16)) and after application of the Hölder inequality in (t,x) we get

$$\begin{aligned} ... &\leq \|u\|_{L_t^pL_x^qH_y^1}\|u\|^\alpha_{L_t^{\alpha\tilde{p}}L_x^{\alpha\tilde{q}}L_y^\infty} \\ &\leq C\|u\|_{L_t^pL_x^qH_y^1}\|u\|^\alpha_{L_t^{\alpha\tilde{p}}L_x^{\alpha\tilde{q}}H_y^1} \end{aligned}$$

where we have used the embedding $H_y^1 \subset L_y^\infty$ and we have chosen

$$\frac{1}{\tilde{p}} + \frac{1}{p} = 1 - \frac{1}{p}$$
$$\frac{1}{\tilde{q}} + \frac{1}{q} = 1 - \frac{1}{q}$$

By direct computation we have:

$$\alpha\tilde{q} = q \text{ and } \alpha\tilde{p} < p \tag{5.6}$$

By combining the nonlinear estimate above with (5.1), (5.6) and the Hölder inequality (in the time variable) we get:

$$\|\mathcal{T}_\varphi u\|_{X_T} \leq C(\|\varphi\|_{L_x^2H_y^1} + T^{a(d)}\|u\|^{1+\alpha}_{X_T}) \tag{5.7}$$

with $a(d) > 0$.
Arguing as above get

$$\begin{aligned} \|\nabla_x(u|u|^\alpha)\|_{L_t^{p'}L_x^{q'}L_y^2} &\leq C\|\nabla_x u\|_{L_t^pL_x^qL_y^2}\|u^\alpha\|_{L_t^{\tilde{p}}L_x^{\tilde{q}}L_y^\infty} \\ &\leq C\|u\|_{Y_T}\|u\|^\alpha_{L_t^{\alpha\tilde{p}}L_x^{\alpha\tilde{q}}H_y^1} \end{aligned}$$

where $\tilde{p}$ and $\tilde{q}$ are as above and we have used the embedding $H_y^1 \subset L_y^\infty$. As a consequence of this estimate and (5.2) we get:

$$\|\mathcal{T}_\varphi u\|_{Y_T} \leq C(\|\nabla_x\varphi\|_{L^2_{x,y}} + T^{a(d)}\|u\|_{Y_T}\|u\|^\alpha_{X_T}) \tag{5.8}$$

with $a(d) > 0$.
By combining (5.7) with (5.8) we get

$$\|\mathcal{T}_\varphi u\|_{Z_T} \leq C(\|\varphi\|_{H^1(\mathbb{R}^n\times M^1)} + T^{a(d)}\|u\|_{Z_T}\|u\|^\alpha_{Z_T})$$

The proof follows by a standard continuity argument.

Next we introduce the norm

$$\|w(t,x,y)\|_{\tilde{Z}_T} \equiv \|w(t,x,y)\|_{L^p((-T,T);L^q_x L^2_y)}$$

and we shall prove the following.

Second step: let $T, R > 0$ as in the previous step then

$$\exists T' = T'(\|\varphi\|_{H^1(\mathbb{R}^n\times M^1)}) < T \text{ s.t. } \mathcal{T}_\varphi$$

$$\text{is a contraction on } B_{Z_{T'}}(0,R) \text{ endowed with the norm } \|.\|_{\tilde{Z}_{T'}}$$

It is sufficient to prove:

$$\|\mathcal{T}_\varphi v_1 - \mathcal{T}_\varphi v_2\|_{\tilde{Z}_T} \leq CT^{a(d)}\|v_1 - v_2\|_{\tilde{Z}_T} \sup_{i=1,2}\{\|v_i\|_{Z_T}\}^\alpha \tag{5.9}$$

with $a(d) > 0$. Notice that we have

$$\|v_1|v_1|^\alpha - v_2|v_2|^\alpha\|_{L^{p'}((-T,T);L^{q'}_x L^2_y)}$$

$$\leq C\Big\|\|v_1 - v_2\|_{L^2_y}(\|v_1\|_{L^\infty_y} + \|v_2\|_{L^\infty_y})^\alpha\Big\|_{L^{p'}((-T,T);L^{q'}_x)}$$

$$\leq CT^{a(d)}\|v_1 - v_2\|_{\tilde{Z}_T} \sup_{i=1,2}\{\|v_i\|_{Z_T}\}^\alpha$$

where we have used the Sobolev embedding $H^1_y \subset L^\infty_y$ and the Hölder inequality in the same spirit as in the proof of (5.7) and (5.8). We conclude by combining the estimate above with the Strichartz estimate (5.3).

Third step: existence and uniqueness of solution in $Z_{T'}$ where T' is as in the previous step

We apply the contraction principle to the map $\mathcal{T}_\varphi$ defined on the complete space $B_{Z_{T'}}(0,R)$ endowed with the topology induced by $\|.\|_{\tilde{Z}_{T'}}$. It is well-known that this space is complete.

Fourth step: regularity of the solution

By combining the previous steps with the fixed point argument we get the existence of a solution $v \in Z_{T'}$. In order to get the regularity $v \in \mathcal{C}((-T',T'); H^1(\mathbb{R}^n \times M^1))$ it is sufficient to argue as in the first step (to estimate the nonlinearity) in conjugation with the Strichartz estimates (5.4) and (5.5).

5.2. **Global Well Posedness.** Next we prove that the local solution (whose existence has been proved above) cannot blow-up in finite time. The argument is standard and follows from the conservation laws:

$$\|u(t)\|_{L^2_{x,y}} \equiv \|\varphi\|_{L^2_{x,y}} \tag{5.10}$$

$$\mathcal{E}_{n,M^1,\alpha}(u(t)) + \frac{1}{2}\|u(t)\|^2_{L^2_{x,y}} \equiv \mathcal{E}_{n,M^1,\alpha}(\varphi) + \frac{1}{2}\|\varphi\|^2_{L^2_{x,y}} \tag{5.11}$$

where $\mathcal{E}_{n,M^1,\alpha}$ is defined in (1.2). By the Gagliardo Nirenberg inequality we deduce

$$\mathcal{E}_{n,M^1,\alpha}(u(t))+\frac{1}{2}\|u(t)\|^2_{L^2_{x,y}} \geq \frac{1}{2}\|u(t)\|^2_{H^1(\mathbb{R}^n\times M^1)} - C\|u(t)\|^{2+\alpha-\mu}_{L^2_{x,y}}\|u(t)\|^{\mu}_{H^1(\mathbb{R}^n\times M^1)}$$

for a suitable $\mu\in(0,2)$. By combining the estimate above with (5.10) and (5.11) we get

$$\frac{1}{2}\|u(t)\|^2_{H^1(\mathbb{R}^n\times M^1)} - C\|\varphi\|^{2+\alpha-\mu}_{L^2_{x,y}}\|u(t)\|^{\mu}_{H^1(\mathbb{R}^n\times M^1)} \leq \mathcal{E}_{n,M^1,\alpha}(\varphi)+\frac{1}{2}\|\varphi\|^2_{L^2_{x,y}}$$

Since $\mu\in(0,2)$ it implies that $\|u(t)\|_{H^1(\mathbb{R}^n\times M^1)}$ cannot blow-up in finite time.

6. Appendix

For the sake of completeness we prove in this appendix Theorems 1.1 and 1.2. Our argument is heavily inspired by the work [5] even if, in our opinion, the following presentation of Theorem 1.1 is simpler compared with the original one.

Proof of Theorem 1.1 For any given $\rho>0$ we shall denote by $u_{j,\rho}\in H^1(\mathbb{R}^n\times M^k)$ any constrained minimizing sequence, i.e.:

$$\|u_{j,\rho}\|_{L^2_{x,y}}=\rho \text{ and } \lim_{j\to\infty}\mathcal{E}_{n,M^k,\alpha}(u_{j,\rho})=K^{\rho}_{n,M^k,\alpha} \tag{6.1}$$

Next we split the proof in many steps.

First step: $K^{\rho}_{n,M^k,\alpha}>-\infty$ *and* $\sup_j\|u_{j,\rho}\|_{H^1_{x,y}}<\infty$, $\forall\rho>0$

By the classical Gagliardo Nirenberg inequality (see (1.4)) we get the existence of $\mu\in(0,2)$ such that

$$\mathcal{E}_{n,M^k,\alpha}(u_{j,\rho})+\frac{1}{2}\rho^2$$

$$\geq \frac{1}{2}\int_{M^k_y}\int_{\mathbb{R}^n}(|\nabla_{x,y}u_{j,\rho}|^2+|u_{j,\rho}|^2)dxdvolM^k_y - C(\rho)\|u_{j,\rho}\|^{\mu}_{H^1(\mathbb{R}^m\times M^k)}$$

$$\geq \inf_{t>0}(1/2t^2-C(\rho)t^{\mu})>-\infty$$

The conclusion follows by a standard argument.

Second step: the map $(0,\infty)\ni\rho\to K^{\rho}_{n,M^k,\alpha}$ *is continuous*

Fix $\rho\in(0,\infty)$ and let $\rho_j\to\rho$. Then we have

$$K^{\rho_j}_{n,M^k,\rho}\leq\mathcal{E}_{n,M^k,\alpha}\Big(\frac{\rho_j}{\rho}u_{j,\rho}\Big)=$$

$$\Big(\frac{\rho_j}{\rho}\Big)^2\Big(\frac{1}{2}\|\nabla_{x,y}u_{j,\rho}\|^2_{L^2_{x,y}}-\frac{1}{2+\alpha}\Big(\frac{\rho_j}{\rho}\Big)^{\alpha}\|u_{j,\rho}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\Big)$$

$$=\Big(\frac{\rho_j}{\rho}\Big)^2\Big(\frac{1}{2}\|\nabla_{x,y}u_{j,\rho}\|^2_{L^2_{x,y}}-\frac{1}{2+\alpha}\|u_{j,\rho}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\Big)$$

$$+\frac{1}{2+\alpha}\Big(\frac{\rho_j}{\rho}\Big)^2\Big(1-\Big(\frac{\rho_j}{\rho}\Big)^{\alpha}\Big)\|u_{j,\rho}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}$$

$$=\Big(\frac{1}{2}\|\nabla_{x,y}u_{j,\rho}\|^2_{L^2_{x,y}}-\frac{1}{2+\alpha}\|u_{j,\rho}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\Big)$$

$$+\Big(\Big(\frac{\rho_j}{\rho}\Big)^2-1\Big)\Big(\frac{1}{2}\|\nabla_{x,y}u_{j,\rho}\|^2_{L^2_{x,y}}-\frac{1}{2+\alpha}\|u_{j,\rho}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\Big)$$
$$+\frac{1}{2+\alpha}\Big(\frac{\rho_j}{\rho}\Big)^2\Big(1-\Big(\frac{\rho_j}{\rho}\Big)^\alpha\Big)\|u_{j,\rho}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}$$

Since we are assuming that $\rho_j\to\rho$ and $\sup_n\|u_{j,\rho}\|_{H^1(\mathbb{R}^n\times M^k)}<\infty$ (see the first step) we get

$$\limsup_{j\to\infty}K^{\rho_j}_{n,M^k,\alpha}\leq K^\rho_{n,M^k,\alpha}$$

To prove the opposite inequality let us fix $u_j\in H^1(\mathbb{R}^n\times M^k)$ such that

$$\|u_j\|_{L^2_{x,y}}=\rho_j \text{ and } \mathcal{E}_{n,M^k,\alpha}(u_j)<K^{\rho_j}_{n,M^k,\alpha}+\frac{1}{j} \tag{6.2}$$

By looking at the proof of the first step we also deduce that u_j can be chosen in such a way that

$$\sup_j\|u_j\|_{H^1(\mathbb{R}^n\times M^k)}<\infty \tag{6.3}$$

Then we can argue as above an we get

$$K^\rho_{n,M^k,\alpha}\leq\mathcal{E}_{n,M^k,\alpha}\left(\frac{\rho}{\rho_j}u_j\right)$$
$$=\Big(\frac{1}{2}\|\nabla_{x,y}u_j\|^2_{L^2_{x,y}}-\frac{1}{2+\alpha}\|u_j\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\Big)$$
$$+\Big(\Big(\frac{\rho}{\rho_j}\Big)^2-1\Big)\Big(\frac{1}{2}\|\nabla_{x,y}u_j\|^2_{L^2_{x,y}}-\frac{1}{2+\alpha}\|u_j\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\Big)$$
$$+\frac{1}{2+\alpha}\Big(\frac{\rho}{\rho_j}\Big)^2\Big(1-\left(\frac{\rho}{\rho_j}\right)^\alpha\Big)\|u_j\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}$$

By using (6.2), (6.3) and the assumption $\rho_j\to\rho$ we get

$$K^\rho_{n,M^k,\alpha}\leq\liminf_{j\to\infty}K^{\rho_j}_{n,M^k,\alpha}$$

Third step: for every $\rho>0$ we have (up to subsequence) $\inf_j\|u_{j,\rho}\|_{L^{2+\alpha}_{x,y}}>0$

It is sufficient to prove that $K^\rho_{n,M^k,\alpha}<0$. In fact we have

$$K^\rho_{n,M^k,\alpha}\leq vol(M^k)\mathcal{E}_{n,\alpha}(u_{n,\omega,\alpha})=vol(M^k)I^{\rho/\sqrt{vol(M^k)}}_{n,\alpha}<0 \tag{6.4}$$

where $\mathcal{E}_{n,\alpha}$ is the energy defined in (2.2) and ω is chosen in such a way that $\|u_{n,\omega,\alpha}\|_{L^2_x}=\frac{\rho}{\sqrt{vol(M^k)}}$. Notice that in (6.4) we have used (2.4) and (2.5).

Fourth step: for any minimizing sequence $u_{j,\rho}$ there exists $\tau_j\in\mathbb{R}^n$ s.t. (up to subsequence) $u_{j,\rho}(x+\tau_j,y)$ has a weak limit $\bar{u}\neq0$

We have the following localized Gagliardo Nirenberg inequality:

$$\|v\|_{L^{2+4/(n+k)}_{x,y}}\leq C\sup_{x\in\mathbb{R}^n}\left(\|v\|_{L^2_{Q^n_x\times M^k}}\right)^{2/(n+k+2)}\|v\|^{(n+k)/(n+k+2)}_{H^1(\mathbb{R}^n\times M^k)} \tag{6.5}$$

where

$$Q^n_x=x+[0,1]^n\ \forall x\in\mathbb{R}^n$$

The estimate above can be proved as follows (see [8] for a similar argument on the flat space $\mathbb{R}^{d+k}$). We fix $x_h \in \mathbb{R}^n$ in such a way that $\bigcup_h Q^n_{x_h} = \mathbb{R}^n$ and $meas_n(Q^n_{x_i} \cap Q^n_{x_j}) = 0$ for $i \neq j$ where $meas_n$ denotes the Lebesgue measure in $\mathbb{R}^n$. By the classical Gagliardo Nirenberg inequality we get:

$$\|v\|^{2+4/(n+k)}_{L^{2+4/(n+k)}_{Q^n_{x_h}\times M^k}} \leq C\|v\|^{4/(n+k)}_{L^2_{Q^n_{x_h}\times M^k}}\|v\|^2_{H^1(Q^n_{x_h}\times M^k)}$$

The proof of (6.5) follows by taking the sum of the previous estimates on $h \in \mathbb{N}$. Due to the boundedness of $u_{j,\rho}$ in $H^1(\mathbb{R}^m \times M^k)$ (see the first step) we deduce by (6.5) that

$$0 < \epsilon_0 = \inf_j \|u_{j,\rho}\|_{L^{2+4/(n+k)}_{x,y}} \leq C \sup_{x\in\mathbb{R}^n} \|u_{j,\rho}\|^{2/(n+k+2)}_{L^2_{Q^n_x\times M^k}} \tag{6.6}$$

(the l.h.s. above follows by combining the Hölder inequality with the third step). The proof can be concluded by the Rellich compactness theorem once we choose a sequence $\tau_j \in \mathbb{R}^n_x$ in such a way that

$$\inf_j \|u_{j,\rho}\|_{L^2_{Q^n_{\tau_j}\times M^k}} > 0$$

(the existence of such a sequence τ_j follows by (6.6)).

Fifth step: the map $(0,\bar\rho) \ni \rho \to \rho^{-2}K^\rho_{n,M^k,\alpha}$ is strictly decreasing

Let us fix $\rho_1 < \rho_2$ and u_{j,ρ_1} a minimizing sequence for $K^{\rho_1}_{n,M^k,\alpha}$. Then we have

$$K^{\rho_2}_{n,M^k,\alpha} \leq \mathcal{E}_{n,M^k,\alpha}\Big(\frac{\rho_2}{\rho_1}u_{j,\rho_1}\Big)$$

$$= \Big(\frac{\rho_2}{\rho_1}\Big)^2\Big(\frac{1}{2}\|\nabla_{x,y}u_{j,\rho_1}\|^2_{L^2_{x,y}} - \frac{1}{2+\alpha}\Big(\frac{\rho_2}{\rho_1}\Big)^\alpha\|u_{j,\rho_1}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\Big)$$

$$= \Big(\frac{\rho_2}{\rho_1}\Big)^2\Big(\frac{1}{2}\|\nabla_{x,y}u_{j,\rho_1}\|^2_{L^2_{x,y}} - \frac{1}{2+\alpha}\|u_{j,\rho_1}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\Big)$$

$$+\frac{1}{2+\alpha}\Big(\frac{\rho_2}{\rho_1}\Big)^2\Big(1-\Big(\frac{\rho_2}{\rho_1}\Big)^\alpha\Big)\|u_{j,\rho_1}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}$$

$$\leq \Big(\frac{\rho_2}{\rho_1}\Big)^2\Big(\frac{1}{2}\|\nabla_{x,y}u_{j,\rho_1}\|^2_{L^2_{x,y}} - \frac{1}{2+\alpha}\|u_{j,\rho_1}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\Big)$$

$$+\frac{1}{2+\alpha}\Big(\frac{\rho_2}{\rho_1}\Big)^2\Big(1-\Big(\frac{\rho_2}{\rho_1}\Big)^\alpha\Big)\inf_j\|u_{j,\rho_1}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}$$

By recalling (see the third step) that $\inf_j \|u_{j,\rho_1}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}} > 0$ we get

$$K^{\rho_2}_{n,M^k,\alpha} < \Big(\frac{\rho_2}{\rho_1}\Big)^2 K^{\rho_1}_{n,M^k,\alpha}$$

Sixth step: let $\bar u$ be as in the fourth step, then $\|\bar u\|_{L^2_{x,y}} = \rho$

Up to subsequence we get:

$$u_{j,\rho}(x+\tau_j,y) \to \bar u(x,y) \neq 0 \text{ a.e. } (x,y) \in \mathbb{R}^n_x \times M^k_y$$

and hence by the Brezis-Lieb lemma (see [1]) we get

$$\|u_{j,\rho}(x+\tau_j,y) - \bar u(x,y)\|^{2+\alpha}_{L^{2+\alpha}_{x,y}} \tag{6.7}$$

$$= \|u_{j,\rho}(x+\tau_j,y)\|^{2+\alpha}_{L^{2+\alpha}_{x,y}} - \|\bar{u}(x,y)\|^{2+\alpha}_{L^{2+\alpha}_{x,y}} + o(1)$$

Assume that $\|\bar{u}\|_{L^2_{x,y}} = \theta$, our aim is to prove $\theta = \rho$. Since $\bar{u} \neq 0$ necessarily $\theta > 0$. Notice that since $L^2_{x,y}$ is an Hilbert space we have

$$\rho^2 = \|u_{j,\rho}(x+\tau_j,y)\|^2_{L^2_{x,y}} \tag{6.8}$$

$$= \|u_{j,\rho}(x+\tau_j,y) - \bar{u}(x,y)\|^2_{L^2_{x,y}} + \|\bar{u}(x,y)\|^2_{L^2_{x,y}} + o(1)$$

and hence

$$\|u_{j,\rho}(x+\tau_j,y) - \bar{u}(x,y)\|^2_{L^2_{x,y}} = \rho^2 - \theta^2 + o(1) \tag{6.9}$$

By a similar argument

$$\int_{M^k_y}\int_{\mathbb{R}^n_x} |\nabla_x(u_{j,\rho}(x+\tau_j,y)) - \nabla_x\bar{u}(x,y)|^2 dxdy \tag{6.10}$$

$$+ \int_{M^k_y}\int_{\mathbb{R}^n_x} |\nabla_y(u_{j,\rho}(x+\tau_j,y)) - \nabla_y\bar{u}(x,y)|^2 dxdvol_{M^k_y}$$

$$+ \int_{M^k_y}\int_{\mathbb{R}^n_x} (|\nabla_x\bar{u}(x,y)|^2 + |\nabla_y\bar{u}(x,y)|^2) dxdvol_{M^k_y}$$

$$= \int_{M^k_y}\int_{\mathbb{R}^n_x} (|\nabla_x(u_{j,\rho}(x+\tau_j,y)|^2 + |\nabla_y u_{j,\rho}(x+\tau_j,y)|^2) dxdvol_{M^k_y} + o(1)$$

By combining (6.10) with (6.7) we get:

$$K^\rho_{n,M^k,\alpha} = \lim_{j\to\infty} \mathcal{E}_{n,M^k,\alpha}(u_{j,\rho}(x+\tau_j,y)) = \tag{6.11}$$

$$\lim_{j\to\infty} \mathcal{E}_{n,M^k,\alpha}(u_{j,\rho}(x+\tau_j,y) - \bar{u}(x,y)) + \mathcal{E}_{n,M^k,\alpha}(\bar{u})$$

and we can continue the estimate as follows

$$... \geq K^{\sqrt{\rho^2-\theta^2}+o(1)}_{n,M^k,\alpha} + K^\theta_{n,M^k,\alpha}$$

where we have used (6.9). Hence by using the second step we get

$$K^\rho_{n,M^k,\alpha} \geq K^{\sqrt{\rho^2-\theta^2}}_{n,M^k,\alpha} + K^\theta_{n,M^k,\alpha}$$

Assume that $\theta < \rho$, then by using the monotonicity proved in fifth step we get

$$K^\rho_{n,M^k,\alpha} > \frac{\rho^2-\theta^2}{\rho^2} K^\rho_{n,M^k,\alpha} + \frac{\theta^2}{\rho^2} K^\rho_{n,M^k,\alpha} = K^\rho_{n,M^k,\alpha}$$

and we have an absurd.

□

Proof of Theorem 1.2 Assume by the absurd that the conclusion is false, then there exists ρ and two sequences $\varphi_j \in H^1(\mathbb{R}^n \times M^k)$ and $t_j \in \mathbb{R}$ such that

$$\lim_{j\to\infty} dist_{H^1(\mathbb{R}^n\times M^k)}(\varphi_j, \mathcal{M}^\rho_{n,M^k,\alpha}) = 0 \tag{6.12}$$

and

$$\liminf_{j\to\infty} dist_{H^1(\mathbb{R}^n\times M^k)}(u_{\varphi_j}(t_j), \mathcal{M}^\rho_{n,M^k,\alpha}) > 0 \tag{6.13}$$

where u_{φ_j} is the solution to (1.1) with Cauchy data φ_j. By (6.12) we deduce the following informations:

$$\lim_{j\to\infty}\|\varphi_j\|_{L^2_{x,y}}=\rho \text{ and } \lim_{j\to\infty}\mathcal{E}_{n,M^k,\alpha}(\varphi_j)=K^\rho_{n,M^k,\alpha}$$

and hence due to the conservation laws satisfied by solutions to (1.1) we get

$$\lim_{j\to\infty}\|u_{\varphi_j}(t_j)\|_{L^2_{x,y}}=\rho \text{ and } \lim_{j\to\infty}\mathcal{E}_{n,M^k,\alpha}(u_{\varphi_j}(t_j))=K^\rho_{n,M^k,\alpha}$$

In turn by an elementary computation we get:

$$\|\tilde u_j\|_{L^2_{x,y}}=\rho \text{ and } \lim_{j\to\infty}\mathcal{E}_{n,M^k,\alpha}(\tilde u_j)=K^\rho_{n,M^k,\alpha}$$

(more precisely $\tilde u_j$ is constrained minimizing sequence for $K^\rho_{n,M^k,\alpha}$) where

$$\tilde u_j=\rho\frac{u_{\varphi_j}(t_j)}{\|u_{\varphi_j}(t_j)\|_{L^2_{x,y}}}$$

Moreover by (6.13) it is easy to deduce

$$\liminf_{j\to\infty} dist_{H^1(\mathbb{R}^n\times M^k)}(\tilde u_j,\mathcal{M}^\rho_{n,M^k,\alpha})>0$$

and it is in contradiction with the compactness of minimizing sequences for $K^\rho_{n,M^k,\alpha}$ stated in Theorem 1.1.

□

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