Control of the multiclass G/G/1 queue in the moderate deviation regime*

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Abstract

A multi-class single-server system with general service time distributions is studied in a moderate deviation heavy traffic regime. In the scaling limit, an optimal control problem associated with the model is shown to be governed by a differential game, that can be explicitly solved. While the characterization of the limit by a differential game is akin to results at the large deviation scale, the analysis of the problem is closely related to the much studied area of control in heavy traffic at the diffusion scale.

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1 Introduction

Models of controlled queueing systems have been studied under various scaling limits. These include heavy traffic diffusion approximations, which are based on the central limit theorem (see [8], [5] and references therein) and large deviation (LD) asymptotics (see eg., [1], [2] and references therein). To the best of our knowledge, the intermediate, moderate deviation (MD) scale has not been considered before in relation to controlled queueing systems. In this paper we consider the multi-class G/G/1 model in a heavy traffic MD regime with a risk-sensitive type cost of a general form, characterize its asymptotic behavior in terms of a differential game (DG), and solve the game. In a special but important case, we also identify a simple policy that is asymptotically optimal (AO). The treatment in the MD regime shares important characteristics with both asymptotic regimes alluded to above. It is similar to analogous results in the LD regime, in that the limit behavior is indeed governed by a DG. The DG itself is

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closely related to Brownian control problems (BCP) that arise in diffusion approximations. In particular, the solution method by which BCP are transformed into problems involving the so-called workload process, turns out to be useful for solving these DG as well.

Treatments of queueing models in the MD regime without dynamic control aspects include the following. In [22], Puhalskii and Whitt prove LD and MD principles for renewal processes. Puhalskii [21] establishes LD and MD principles for queue length and waiting time processes for the single server queue and for single class queueing networks in heavy traffic (Puhalskii refers to this regime as *near* heavy traffic, to emphasize that the deviations from critical load are at a larger scale than under standard heavy traffic; we will use the term heavy traffic in this paper). Majewski [20] treats feedforward multi-class network models with priority. Wischik [25] (see also [17]) illuminates on various links between results on queueing problems in LD and MD regimes, as well as similarities between MD and diffusion scale results, particularly the validity of results such as the snapshot principle and state space collapse. Based on these similarities he conjectures that the well-established dynamic control theory for heavy traffic diffusion approximations should have a parallel at the MD scale (our treatment certainly confirms this expectation for the model under investigation). Cruise [9] considers LD and MD as a part of a broader parametrization framework for studying queueing systems.

In the model under consideration (see the next section for a complete description), customers of I different classes arrive at the system following renewal processes and are enqueued in buffers, one for each class. A server, that may offer simultaneous service to the various classes, divides its effort among the (at most) I customers waiting at the head of the line of each buffer. The service time distributions depend on the class. The problem is to control these fractions of effort so as to minimize a cost. MD scaling is obtained by considering a sequence b_n, where $b_n \to \infty$, $\sqrt{n}/b_n \to \infty$. The arrival and service time scales are set proportional to a large parameter n, with possible correction of order $b_n\sqrt{n}$. Denoting by $X_n^i(t)$, the number of class-i jobs in the n-th system at time t, a scaled version is given by $\tilde{X}_n = (b_n\sqrt{n})^{-1}X_n$. Moreover, a heavy traffic condition is assumed, namely that the limiting traffic intensity is one. The cost is given by

$$\frac{1}{b_n^2}\log\mathbb{E}\{e^{b_n^2[\int_0^T h(\tilde{X}_n(t))dt+g(\tilde{X}_n(T))]}\},$$

where $T > 0$, and h and g are given functions.

This type of cost is called *risk-sensitive* (see the book by Whittle [24]). The optimal control formulation of a dynamical system with small noise goes back to Fleming [14], who studies the associated Hamilton-Jacobi equations. The connection of risk-sensitive cost to DG was made by Jacobson [19]. The study of risk-sensitive control via LD theory and the formulation of the corresponding maximum principle are due to Whittle [23]. Various aspects of this approach have been studied for controlled stochastic differential equations, for example, [12], [15], [16]. For queueing networks, risk sensitive control in the LD regime has been studied in [10], [1], [2]. Operating a queueing system so as to avoid large queue length or waiting time is important in practice, for preventing buffer overflow and assuring quality of service. A risk-sensitive criterion penalizes such events heavily, and thus provides a natural way to address these considerations. Further motivation for this formulation is that the solution automatically leads to robustness properties of the policy (see Dupuis et al. [11]). Note that working in MD scale leads to some additional desired robustness properties. Namely, since the rate function in this case typically depends only on first and second moments of the underlying primitives, the characteristics of the problem are insensitive to distributional perturbations which preserve these moments. The price paid for working in MD scale is that a heavy traffic condition has to be assumed for the problem to be meaningful (as it is in diffusion approximations but not in LD analysis).

The DG governing the limit behavior can be solved explicitly, a fact that not only is useful in characterizing the limit in a concrete way, but also turns out to be of crucial importance when proving the convergence. To describe the game (see Section 2 for the precise definition), consider the dynamics

$$\varphi(t) = x + yt + \int_0^t (\tilde{\lambda}(s) - \tilde{\mu}(s))ds + \eta(t) \in \mathbb{R}_+^I.$$

Here x is an initial condition, y is a term capturing the order $b_n\sqrt{n}$ time scale correction alluded to above, and $\tilde{\lambda}$ and $\tilde{\mu}$ represent perturbations at scale $b_n/\sqrt{n}$ of arrival and service rates, respectively. These are functions mapping $[0,T] \to \mathbb{R}_+^I$, controlled by player 1. Next, $\eta : [0,\infty) \to \mathbb{R}_+^I$ is a function whose formal derivative represents deviations at scale $b_n/\sqrt{n}$ of the fraction of effort dedicated by the server to each class. This function is controlled by player 2, and is regarded admissible if (a) for all t, $\varphi(t) \in \mathbb{R}_+^I$, (b) $\theta \cdot \eta(0) \geq 0$, and (c) $\theta \cdot \eta$ is nondecreasing, where $\theta = (\frac{1}{\mu_1}, \ldots, \frac{1}{\mu_I})$ is what is often called the *workload vector* in the heavy traffic literature. The cost, which player 1 (resp., 2) attempts to maximize (minimize) is given by

$$\int_0^T h(\varphi(s))ds + g(\varphi(T)) - \int_0^T \sum [a_i\tilde{\lambda}_i(s)^2 + b_i\tilde{\mu}_i(s)^2]ds, \tag{1}$$

where a_i and b_i are positive constants.

It is instructive to compare this to the game obtained under LD scaling. The form presented here corresponds to the multiclass M/M/1 model, following [2] (the setting there includes multiple, heterogenous servers, but the presentation here is specialized to the case of a single server). One considers

$$\varphi = \Gamma[\psi], \qquad \psi(t) = x + \int_0^t (\bar{\lambda}(s) - u(s) \bullet \bar{\mu}(s))ds,$$

where Γ is the Skorohod map with normal reflection on the boundary of the positive orthant, $\bar{\lambda}$ and $\bar{\mu}$ are functions $[0,T] \to [0,\infty)^I$, representing perturbations at the LD scale, and controlled by a maximizing player; $u : [0,T] \to S$ where $S = \{s \in [0,1]^I : \sum s_i = 1\}$ is controlled by minimizing player representing fraction of effort per class, and $\bullet$ denotes the entrywise product of two vectors of the same dimension. The cost here takes the form

$$\int_0^T h(\varphi(s))ds + g(\varphi(T)) - \int_0^T [1 \cdot l(\bar{\lambda}(s)) + u(s) \cdot \hat{l}(\bar{\mu}(s))]ds, \tag{2}$$

where l and $\hat{l}$ represent LD cost associated with atypical behavior (see [2] for more details). The paper [2] provides a characterization of the game's value in terms of a Hamilton-Jacobi-Isaacs (HJI) equation. However, an explicit solution to the game is provided there only in the special case where h vanishes and g is linear. In contrast, the game associated with MD turns out to be explicitly solvable, as we show in this paper. The reason for this is that while in the LD game the last term of the cost (2) involves both $(\bar{\lambda}, \bar{\mu})$ and u, the corresponding term in (1) involves only $(\tilde{\lambda}, \tilde{\mu})$, not η. Hence this term plays no role when one computes the optimal response η to a given $(\tilde{\lambda}, \tilde{\mu})$ (it does when one optimizes over $(\tilde{\lambda}, \tilde{\mu})$). This optimal response is computed via projecting the dynamics in the direction of the workload vector, and using minimality considerations of the one-dimensional Skorohod problem. In fact, the optimal response η to $(\tilde{\lambda}, \tilde{\mu})$ is precisely the one that arises in the diffusion scale analysis of the model, used there to map the Brownian motion term to the optimal control for the BCP. Thus the link to diffusion approximations is strong.

In [2] (following the technique of [1]), the convergence is proved by establishing upper and lower bounds on the limiting risk-sensitive control problem's value in terms of the lower and, respectively, upper values of the DG. The existence of a limit is then argued via uniqueness of solutions to the HJI equation satisfied by both values. The arrival and service are assumed to follow Poisson processes and the convergence proof uses the form of the Markovian generator and martingale inequalities related to it. Since in the MD regime the performance depends only on the first two moments of the primitives, these moments carry all relevant information regarding the limit (under tail assumptions), and so in this paper we aim at general arrival and service processes. As a result, the tools based on the Markovian formulation mentioned above can not be used. The approach we take uses completely different considerations. The asymptotic behavior of the risk-sensitive control problem is estimated, above and below, directly by the DG lower value (the corresponding upper value is not dealt with at all in this paper). This is made possible thanks to the explicit solvability of the game. More precisely, the arguments by which the game's optimal strategy is found, including the workload formulation and the minimality property associated with the Skorohod map, give rise, when applied to the control problem, to the lower bound. The proof of the upper bound is by construction of a particular control which again uses the solution of the game and its properties. Note that this approach eliminates the need for any PDE analysis.

The control that is constructed in the proof of the upper bound is too complicated for practical implementation. A simple solution to the DG is available in the case where h and g are linear (see Section 5 for the precise linearity condition). It is a fixed priority policy according to the well-known $c\mu$ rule. As our final result shows, applying a priority policy in the queueing model, according to the same order of customer classes, is AO in this case.

To summarize the main contribution of the paper, we have (a) provided the first treatment of a queueing control problem at the MD scale (b) identified and solved the DG governing the scaling limit for quite a general setting, and (c) proved AO of a simple policy in the linear case. Finally, it is important to mention that our results strongly suggest that techniques such as the equivalent workload formulation, which have proven powerful for control problems at the diffusion scale, are likely to be useful at the MD scale in far greater generality than the present setting. We intend to study this in future work.

We will use the following notations. For a positive integer k and $a, b \in \mathbb{R}^k$, $a \cdot b$ denotes the usual scalar product, while $\|\cdot\|$ denotes Euclidean norm. For $T > 0$ and a function $f : [0,T] \to \mathbb{R}^k$, let $\|f\|_t^* = \sup_{s\in[0,t]} \|f(s)\|$, $t \in [0,T]$. When $k = 1$, we write $|f|_t^*$ for $\|f\|_t^*$ and $\|f\|^*$ for $\|f\|_T^*$. Denote by $C([0,T], \mathbb{R}^k)$ and $D([0,T], \mathbb{R}^k)$ the spaces of continuous functions $[0,T] \to \mathbb{R}^k$ and, respectively, functions that are right-continuous with finite left limits (RCLL). Endow the space $D([0,T], \mathbb{R}^k)$ with the Skorohod-Prohorov-Lindvall metric or

J_1 metric, defined as

$$d(\varphi,\varphi')=\inf_{f\in\Upsilon}\Big(\|f\|^\circ\vee\sup_{[0,T]}\|\varphi(t)-\varphi'(f(t))\|\Big),\qquad \varphi,\varphi'\in D([0,T],\mathbb{R}^k)$$

where Υ is the set of strictly increasing, continuous functions from $[0,T]$ onto itself, and

$$\|f\|^\circ=\sup_{0\le s<t\le T}\Big|\log\frac{f(t)-f(s)}{t-s}\Big|.$$

As is well known [6], $D([0,T],\mathbb{R}^k)$ is a Polish space under the induced topology.

The organization of the paper is as follows. The next section introduces the model and an associated differential game and states the main result. In Section 3 we find a solution to the game and describe properties of it that are useful in the sequel. Section 4 gives the proof of the main theorem. In Section 5 we discuss the case of linear cost and identify an AO policy. Finally, the appendix gives the proof of a proposition stated in Section 2.

2 Model and results

2.1 The model

The model consists of I customer classes and a single server. A buffer with infinite room is dedicated to each customer class, and upon arrival, customers are queued in the corresponding buffers. Within each class, customers are served at the order of arrival. The server may only serve the customer at the head of each line. Moreover, processor sharing is allowed, and so the server is capable of serving up to I customers (of distinct classes) simultaneously.

The parameters and processes that we now introduce will depend on an index $n\in\mathbb{N}$, that will serve as a scaling parameter. Arrivals occur according to independent renewal processes, and service times are independent and identically distributed across each class. Let $\mathcal{I}=\{1,2,\dots,I\}$. Let $\lambda_n^i>0, n\in\mathbb{N}, i\in\mathcal{I}$ be given parameters, representing the *reciprocal mean inter-arrival times* of class-i customers. Given are I independent sequence $\{IA^i(l):l\in\mathbb{N}\}_{i\in\mathcal{I}}$, of positive i.i.d. random variables with mean $\mathbb{E}[IA^i(1)]=1$ and variance $\sigma_{i,IA}^2=\mathrm{Var}(IA^i(1))\in(0,\infty)$. With $\sum_1^0=0$, the number of arrivals of class-i customers up to time t, for the n-th system, is given by

$$A_n^i(t)=\sup\Big\{l\ge 0\ :\sum_{k=1}^{l}\frac{IA^i(k)}{\lambda_n^i}\le t\Big\},\quad t\ge 0.$$

Similarly we consider another class of parameters $\mu_n^i>0, n\in\mathbb{N}, i\in\mathcal{I}$, representing *reciprocal mean service times*. We are also given I independent sequence $\{ST^i(l):l\in\mathbb{N}\}_{i\in\mathcal{I}}$ of positive i.i.d. random variables (independent also of the sequences $\{IA^i\}$) with mean $\mathbb{E}[ST^i(1)]=1$ and variance $\sigma_{i,ST}^2=\mathrm{Var}(ST^i(1))\in(0,\infty)$. The time required to complete the service of the l-th class-i customer is given by $ST^i(l)/\mu_n^i$, and the *potential service time* processes are defined as

$$S_n^i(t)=\sup\Big\{l\ge 0\ :\sum_{k=1}^{l}\frac{ST^i(k)}{\mu_n^i}\le t\Big\},\quad t\ge 0.$$

We also consider the *moderate deviations rate parameters* $\{b_n\}$, that form a sequence, fixed throughout, with the property that $\lim b_n = \infty$ while $\lim \frac{b_n}{\sqrt{n}} = 0$, as $n \to \infty$. The arrival and service parameters are assumed to satisfy the following conditions. As $n \to \infty$,

- $\frac{\lambda_n^i}{n} \to \lambda^i > 0$ and $\frac{\mu_n^i}{n} \to \mu^i \in (0, \infty)$,
- $\tilde{\lambda}_n^i := \frac{1}{b_n \sqrt{n}}(\lambda_n^i - n\lambda^i) \to \tilde{\lambda}^i \in (-\infty, \infty)$,
- $\tilde{\mu}_n^i := \frac{1}{b_n \sqrt{n}}(\mu_n^i - n\mu^i) \to \tilde{\mu}^i \in (-\infty, \infty)$.

Also the system is assumed to be critically loaded in the sense that $\sum_1^I \rho^i = 1$ where $\rho^i = \frac{\lambda^i}{\mu^i}$ for $i \in \mathcal{I}$.

For $i \in \mathcal{I}$, let X_n^i be a process representing the number of class-i customers in the n-th system. With $\mathbb{S} = \{x \in [0,1]^I : \sum x_i \le 1\}$, let B_n be a process taking values in $\mathbb{S}$, whose i-th component represents the fraction of effort devoted by the server to the served class-i customer. The number of service completions of class-i jobs during the time interval $[0, t]$ is assumed to be given by

$$D_n^i(t) := S_n^i(T_n^i(t)), \tag{3}$$

where

$$T_n^i(t) = \int_0^t B_n^i(s)ds. \tag{4}$$

The following equation follows from foregoing verbal description

$$X_n^i(t) = X_n^i(0) + A_n^i(t) - S_n^i(T_n^i(t)). \tag{5}$$

For simplicity, the initial conditions $X_n^i(0)$ are assumed to be deterministic. Note that, by construction, the arrival and potential service processes have RCLL paths, and accordingly, so do D_n and X_n.

The process B_n is regarded a control, that is determined based on observations from the past (and present) events in the system. A precise definition is as follows. Fix $T > 0$ throughout. Given n, the process B_n is said to be an *admissible control* if its sample paths lie in $D([0,T], \mathbb{R}^I)$, and

- It is adapted to the filtration

  $$\sigma\{A_n^i(s), S_n^i(T_n^i(s)), i \in \mathcal{I}, s \le t\},$$

  where T_n is given by (4);

- For $i \in \mathcal{I}$ and $t \ge 0$, one has

  $$X_n^i(t) = 0 \quad \text{implies} \quad B_n^i(t) = 0, \tag{6}$$

  where X_n is given by (5).

Denote the class of all admissible processes B_n by $\mathfrak{U}_n$. Note that this class depends on the processes A_n and S_n, but we consider these processes as fixed.

We next introduce centered and scaled versions of the processes. For $i \in \mathcal{I}$ let

$$\tilde{A}^i_n(t) = \frac{1}{b_n\sqrt{n}}(A^i_n(t) - \lambda^i_n t), \quad \tilde{S}^i_n(t) = \frac{1}{b_n\sqrt{n}}(S^i_n(t) - \mu^i_n t), \quad \tilde{X}^i_n(t) = \frac{1}{b_n\sqrt{n}} X^i_n(t). \tag{7}$$

It is easy to check from (5) that

$$\tilde{X}^i_n(t) = \tilde{X}^i_n(0) + y^i_n t + \tilde{A}^i_n(t) - \tilde{S}^i_n(T^i_n(t)) + Z^i_n(t), \tag{8}$$

where we denote

$$Z^i_n(t) = \frac{\mu^i_n}{n}\frac{\sqrt{n}}{b_n}(\rho^i t - T^i_n(t)), \qquad y^i_n = \tilde{\lambda}^i_n - \rho^i \tilde{\mu}^i_n. \tag{9}$$

Note that these processes have the property

$$\sum_i \frac{n}{\mu^i_n} Z^i_n \quad \text{starts from zero and is nondecreasing,} \tag{10}$$

thanks to the fact that $\sum_i B^i_n \le 1$ while $\sum_i \rho_i = 1$. Clearly $\tilde{X}^i_n$ is nonnegative, i.e.,

$$\tilde{X}^i_n(t) \ge 0 \qquad t \ge 0, \ i \in \mathcal{I}. \tag{11}$$

We impose the following condition on the initial values:

$$\tilde{X}_n(0) \to x \in \mathbb{R}^I_+ \text{ as } n \to \infty.$$

The scaled processes $(\tilde{A}^n, \tilde{S}^n)$ are assumed to satisfy a *moderate deviation principle*. To express this assumption, let $\mathbb{I}_k, k = 1, 2$, be functions on $D([0,T], \mathbb{R}^I)$ defined as follows. For $\psi = (\psi_1, \ldots, \psi_I) \in D([0,T], \mathbb{R}^I)$,

$$\mathbb{I}_1(\psi) = \begin{cases} \frac{1}{2}\sum_{i=1}^I \frac{1}{\lambda^i \sigma^2_{i,IA}} \int_0^T \dot{\psi}^2_i(s)ds & \text{if all } \psi_i \text{ are absolutely continuous and } \psi(0) = 0, \\ \infty & \text{otherwise,} \end{cases}$$

and

$$\mathbb{I}_2(\psi) = \begin{cases} \frac{1}{2}\sum_{i=1}^I \frac{1}{\mu^i \sigma^2_{i,ST}} \int_0^T \dot{\psi}^2_i(s)ds & \text{if all } \psi_i \text{ are absolutely continuous and } \psi(0) = 0, \\ \infty & \text{otherwise.} \end{cases}$$

Let $\mathbb{I}(\psi) = \mathbb{I}_1(\psi^1) + \mathbb{I}_2(\psi^2)$ for $\psi = (\psi^1, \psi^2) \in D([0,T], \mathbb{R}^{2I})$.

Condition 2.1 (Moderate deviation principle) *The sequence*

$$(\tilde{A}_n, \tilde{S}_n) = (\tilde{A}^1_n, \ldots, \tilde{A}^I_n, \tilde{S}^1_n, \ldots, \tilde{S}^I_n),$$

satisfies the LDP with rate parameters b_n and rate function $\mathbb{I}$ in $D([0,T], \mathbb{R}^{2I})$; i.e.,

- *For any closed set $F \subset D([0,T], \mathbb{R}^{2I})$*

$$\limsup \frac{1}{b_n^2} \log \mathbb{P}((\tilde{A}_n, \tilde{S}_n) \in F) \le -\inf_{\psi \in F} \mathbb{I}(\psi),$$

- *For any open set $G \subset D([0,T],\mathbb{R}^{2I})$*

$$\liminf \frac{1}{b_n^2} \log \mathbb{P}((\tilde{A}_n, \tilde{S}_n) \in G) \geq - \inf_{\psi \in G} \mathbb{I}(\psi).$$

Remark 2.1 *It is shown in [22] that each one of the following statements is sufficient for Condition 2.1 to hold:*

- *There exist constants $u_0 > 0$, $\beta \in (0,1]$ such that $E[e^{u_0(IA^i)^\beta}], E[e^{u_0(ST^i)^\beta}] < \infty$, $i \in \mathcal{I}$, and $b_n^{\beta-2} n^{\beta/2} \to \infty$;*
- *For some $\varepsilon > 0$, $E[(IA^i)^{2+\varepsilon}], E[(ST^i)^{2+\varepsilon}] < \infty$, $i \in \mathcal{I}$, and $b_n^{-2} \log n \to \infty$.*

To present the risk-sensitive control problem, let h and g be nonnegative, continuous functions from $\mathbb{R}_+^I$ to $\mathbb{R}$, monotone nondecreasing with respect to the usual partial order on $\mathbb{R}_+^I$. Assume that h, g have at most linear growth, i.e., there exist constants c_1, c_2 such that

$$g(x) + h(x) \leq c_1 \|x\| + c_2.$$

Given n, the cost associated with the initial condition $\tilde{X}_n(0)$ and control $B_n \in \mathfrak{U}_n$ is given by

$$J^n(\tilde{X}_n(0), B_n) = \frac{1}{b_n^2} \log \mathbb{E}\Big[e^{b_n^2[\int_0^T h(\tilde{X}_n(s))ds + g(\tilde{X}_n(T))]}\Big]. \tag{12}$$

The value function of interest is given by

$$V^n(\tilde{X}_n(0)) = \inf_{B_n \in \mathfrak{U}_n} J^n(\tilde{X}_n(0), B_n).$$

2.2 A differential game

We next develop a differential game for the limit behavior of the above control problem. Let $\theta = (\frac{1}{\mu^1}, \ldots, \frac{1}{\mu^I})$ and $y = (y_1, \ldots, y_I)$ where $y_i = \tilde{\lambda}^i - \rho^i \tilde{\mu}^i$. Denote $P = C_0([0,T],\mathbb{R}^{2I})$ (the subset of $C([0,T],\mathbb{R}^{2I})$ of functions starting from zero) and

$$E = \{\zeta \in C([0,T],\mathbb{R}^I) : \theta \cdot \zeta \text{ starts from zero and is nondecreasing}\}.$$

Endow both spaces with the uniform topology. Let R be the mapping from $D([0,T],\mathbb{R}^I)$ into itself defined by

$$R[\psi]_i(t) = \psi_i(\rho_i t), \qquad t \in [0,T],\ i \in \mathcal{I}.$$

Given $\psi = (\psi^1, \psi^2) \in P$ and $\zeta \in E$, the *dynamics associated with initial condition x and data* ψ, ζ is given by

$$\varphi_i(t) = x_i + y_i t + \psi_i^1(t) - R[\psi^2]_i(t) + \zeta_i(t), \qquad i \in \mathcal{I}. \tag{13}$$

Note the analogy between the above equation and equation (8), and between the condition $\theta \cdot \zeta$ nondecreasing and property (10). The following condition, analogous to property (11), will also be used,

$$\varphi_i(t) \geq 0, \qquad t \geq 0,\ i \in \mathcal{I}. \tag{14}$$

The game is defined in the sense of Elliott and Kalton [13], for which we need the notion of strategies. A measurable mapping $\alpha : P \to E$ is called a *strategy for the minimizing player* if it satisfies a causality property. Namely, for every $\psi = (\psi^1, \psi^2), \tilde{\psi} = (\tilde{\psi}^1, \tilde{\psi}^2) \in P$ and $t \in [0, T]$,

$$(\psi^1, R[\psi^2])(s) = (\tilde{\psi}^1, R[\tilde{\psi}^2])(s) \text{ for all } s \in [0,t] \quad \text{implies} \quad \alpha[\psi](s) = \alpha[\tilde{\psi}](s) \text{ for all } s \in [0,t]. \tag{15}$$

Given an initial condition x, a strategy α is said to be *admissible* if, whenever $\psi \in P$ and $\zeta = \alpha[\psi]$, the corresponding dynamics (13) satisfies the nonnegativity constraint (14). The set of all admissible strategies for the minimizing player is denoted by A (or, when the dependence on the initial condition is important, A_x). Given x and $(\psi, \zeta) \in P \times E$, we define the cost by

$$c(\psi, \zeta) = \int_0^T h(\varphi(t))dt + g(\varphi(T)) - \mathbb{I}(\psi),$$

where φ is the corresponding dynamics. The value of the game is defined by

$$V(x) = \inf_{\alpha \in A_x} \sup_{\psi \in P} c(\psi, \alpha[\psi]).$$

2.3 Main result

For $w \in \mathbb{R}_+$, denote

$$h^*(w) = \inf\{h(x) : x \in \mathbb{R}_+^I, \theta \cdot x = w\}, \quad g^*(w) = \inf\{g(x) : x \in \mathbb{R}_+^I, \theta \cdot x = w\}.$$

We need the following condition. It is similar to the one imposed in [4], [3], where an analogous many-server model is treated in a diffusion regime.

Condition 2.2 (Existence of a continuous minimizing curve) *There exists a continuous map $f : \mathbb{R}_+ \to \mathbb{R}_+^I$ such that for all $w \in \mathbb{R}_+$,*

$$\theta \cdot f(w) = w, \quad h^*(w) = h(f(w)), \quad g^*(w) = g(f(w)).$$

Example 2.1 a. *The linear case: $h(x) = \sum c_i x_i$ and $g(x) = \sum d_i x_i$, for some nonnegative constants c_i, d_i. If we require that $c_I \mu_I = \min_i c_i \mu_i$ and $d_I \mu_I = \min_i d_i \mu_i$ then the condition holds with $f(w) = (0, \ldots, 0, w\mu_I)$. This is the case considered in Section 5.*
b. *If h is homogeneous of degree $\alpha, 0 < \alpha \le 1$, and $x^* \in argmin\{h(x) : \theta \cdot x = 1\}$, it is easy to check that $f(w) = wx^*$ satisfies the above condition provided $g = dh$ for some non-negative constant d.*

Condition 2.3 (Exponential moments) *Denote*

$$\Lambda_T(\psi^1, \psi^2) = \sum_{i=1}^I \sup_{[0,T]} |\psi_i^1(t)| + \sum_{i=1}^I \sup_{[0,T]} |\psi_i^2(t)|.$$

Then for any constant K,

$$\limsup_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{E}[e^{b_n^2 K \Lambda_T(\tilde{A}_n, \tilde{S}_n)}] < \infty.$$

A sufficient condition for the above is as follows (see the appendix for a proof).

Proposition 2.1 *If there exists $u_0 > 0$ such that $\mathbb{E}[e^{u_0 \, IA^i}]$ and $\mathbb{E}[e^{u_0 \, ST^i}]$, $i \in \mathcal{I}$, are finite then Condition 2.3 holds.*

Note that taking $\beta = 1$ in Remark 2.1 shows that the hypothesis of Proposition 2.1 is sufficient for Condition 2.1 as well.

Our main result is the following:

Theorem 2.1 *Let Conditions 2.1, 2.2 and 2.3 hold. Then $\lim_{n\to\infty} V^n(\tilde{X}_n(0)) = V(x)$.*

Remark 2.2 *While the game formulation given above is natural to work with in the proofs, there is a simpler, equivalent formulation which avoids the use of the time scaling operator R. Define a functional $\bar{\mathbb{I}}(\psi) = \bar{\mathbb{I}}_1(\psi^1) + \bar{\mathbb{I}}_2(\psi^2)$ on $D([0,T],\mathbb{R}^{2I})$, where $\bar{\mathbb{I}}_k, k = 1,2$, are functionals on $D([0,T],\mathbb{R}^I)$ given by $\bar{\mathbb{I}}_1 = \mathbb{I}_1$, and, for $\psi = (\psi_1,\dots,\psi_I) \in D([0,T],\mathbb{R}^I)$,*

$$\bar{\mathbb{I}}_2(\psi) = \begin{cases} \frac{1}{2}\sum_{i=1}^{I} \frac{1}{\rho^i\mu^i\sigma_{i,ST}^2} \int_0^T \dot{\psi}_i^2(s)ds & \text{if all } \psi_i \text{ are absolutely continuous and } \psi(0) = 0, \\ \infty & \text{otherwise.} \end{cases}$$

The dynamics of the game $\bar{\varphi}$ are now

$$\bar{\varphi}_i(t) = x_i = y_i t + \psi^1(t) - \psi^2(t) + \zeta^i(t) \geq 0.$$

A strategy α should now satisfy the following version of the causality property:

$$\psi(s) = \psi(s) \text{ for all } s \in [0,t] \ \text{ implies } \alpha[\psi](s) = \alpha[\tilde{\psi}](s) \text{ for all } s \in [0,t].$$

Denote the set of all such strategies by $\bar{A}_x$. Given x and $(\psi,\zeta) \in P \times E$, let

$$\bar{c}(\psi,\zeta) = \int_0^T h(\bar{\varphi}(t))dt + g(\bar{\varphi}(T)) - \bar{\mathbb{I}}(\psi),$$

where $\bar{\varphi}$ is as above. Then it is easy to see that the value of the game can also be defined as

$$V(x) = \inf_{\alpha \in \bar{A}_x} \sup_{\psi \in P} \bar{c}(\psi, \alpha[\psi]).$$

3 Solution of the game

In this section we find a minimizing strategy for V, under Condition 2.2, following an idea from [18]. Throughout this section, the initial condition x is fixed. Consider the one-dimensional *Skorohod map Γ* from $D([0,T],\mathbb{R})$ to itself given by

$$\Gamma[z](t) = z(t) - \inf_{s\in[0,t]} [z(s) \wedge 0], \quad t \in [0,T]. \tag{16}$$

Clearly, $\Gamma[z](t) \geq 0$ for all t. Let also

$$\bar{\Gamma}[z](t) = - \inf_{s\in[0,t]} [z(s) \wedge 0], \quad t \in [0,T].$$

It is clear from the definition that, for $z, w \in D([0,T],\mathbb{R})$

$$\sup_{[0,T]} |\Gamma[z] - \Gamma[w]| \leq 2 \sup_{[0,T]} |z - w|. \tag{17}$$

The construction below is based on the mapping Γ and the function f from Condition 2.2. For $\psi = (\psi^1, \psi^2) \in P$, let $\bar{R}[\psi]$ be defined by

$$\bar{R}[\psi](t) = x + yt + \psi^1(t) - R[\psi^2](t), \qquad t \in [0,T]. \tag{18}$$

Let

$$\varphi_\theta[\psi] = \Gamma[\theta \cdot \bar{\psi}], \tag{19}$$

$$\alpha_\theta[\psi](t) = f(\varphi_\theta[\psi](t)) - \bar{\psi}(t), \qquad t \in [0,T], \tag{20}$$

where $\bar{\psi} : [0,T] \to \mathbb{R}^I$ is given by $\bar{R}[\psi]$. Sometimes we also use the notation $\hat{\alpha}_\theta$ for the mapping defined by

$$\hat{\alpha}_\theta[\psi](t) = f(\hat{\varphi}_\theta[\psi](t)) - \hat{\psi}(t), \qquad t \in [0,T], \tag{21}$$

where $\hat{\varphi}_\theta[\psi] = \Gamma[\theta \cdot \hat{\psi}]$ and

$$\hat{\psi}(t) = x + yt + \psi^1(t) - \psi^2(t), \qquad t \in [0,T].$$

Note that $\alpha_\theta[\psi^1, \psi^2] = \hat{\alpha}_\theta[\psi^1, R[\psi^2]]$.

Let us show that α_θ is an admissible strategy. Let $\psi \in P$ be given and denote $\zeta = \alpha_\theta[\psi]$. Note that the dynamics corresponding to (ψ, ζ) is given by

$$\varphi = \bar{\psi} + \zeta = f(\varphi_\theta[\psi]).$$

Multiplying (20) by θ,

$$\theta \cdot \zeta = \varphi_\theta[\psi] - \theta \cdot \bar{\psi} = \bar{\Gamma}[\theta \cdot \bar{\psi}]. \tag{22}$$

Since $\theta \cdot \bar{\psi}(0) = \theta \cdot x \geq 0$, it follows that $\theta \cdot \zeta(0) = 0$. Moreover, by definition of $\bar{\Gamma}$, $\theta \cdot \zeta$ is nondecreasing. This shows $\zeta \in E$. The causality property (15) follows directly from an analogous property of Γ. Next, $\varphi_\theta(t) \geq 0$ for all t, and, by definition, f maps $\mathbb{R}_+$ to $\mathbb{R}^I_+$, whence $\varphi(t) \in \mathbb{R}^I_+$ for all t. This shows that α_θ is an admissible strategy.

Now we check that α_θ is indeed a minimizing strategy. This is based on the minimality property of the Skorohod map (see e.g. [7, Section 2]). Namely, if $z, w \in D([0,T] : \mathbb{R})$, w is nonnegative and nondecreasing, and $z(t) + w(t) \geq 0$ for all t, then

$$z(t) + w(t) \geq \Gamma[z](t), \qquad t \in [0,T].$$

Let $\alpha \in A$ be any admissible strategy and consider $\psi = (\psi^1, \psi^2) \in P$. Then the dynamics corresponding to ψ and $\zeta := \alpha[\psi]$ is given by $\varphi = \bar{\psi} + \zeta$. Since α is an admissible strategy, we have that

$$\theta \cdot \varphi = \theta \cdot \bar{\psi} + \theta \cdot \zeta \geq 0,$$

and $\theta \cdot \zeta$ is nonnegative and nondecreasing. Thus by the above minimality property,

$$\theta \cdot \varphi(t) \geq \Gamma[\theta \cdot \bar{\psi}](t) = \varphi_\theta[\psi](t), \qquad t \in [0,T].$$

Therefore using monotonicity of h we have, denoting $\varphi_\theta = \varphi_\theta[\psi]$,

$$\begin{aligned} h(\varphi(t)) &\geq \inf\{h(q) \;:\; \theta \cdot q = \theta \cdot \varphi(t)\} \\ &\geq \inf\{h(q) \;:\; \theta \cdot q = \varphi_\theta(t)\} = h(f(\varphi_\theta(t))). \end{aligned} \tag{23}$$

A similar estimate holds for g, namely

$$g(\varphi(T)) \geq g(f(\varphi_\theta(T))). \tag{24}$$

As a result,

$$\sup_{\psi \in P} c(\psi, \alpha[\psi]) \geq \sup_{\psi \in P} c(\psi, \alpha_\theta[\psi]).$$

This proves that α_θ is a minimizing strategy, namely

$$V(x) = \sup_{\psi \in P} c(\psi, \alpha_\theta[\psi]). \tag{25}$$

Extension and some properties of α_θ. As a strategy, α_θ is defined on P. We extend $\hat{\alpha}_\theta$ (and so α_θ) to

$$\bar{P} = D([0,T], \mathbb{R}^{2I}),$$

using the same definition (20). The argument leading to (23) and (24) is seen to be applicable for this extended map. Namely,

$$\begin{gathered} \psi, \zeta \in D([0,T], \mathbb{R}),\ \varphi(t) = \psi(t) + \zeta(t) \in \mathbb{R}_+^I,\ \theta \cdot \zeta \text{ nonnegative and nondecreasing} \\ \text{implies} \\ j(\varphi(t)) \geq j(f(\Gamma[\theta \cdot \psi](t))), \text{ for } j = h, g. \end{gathered} \tag{26}$$

Next, denote $\theta_* = \min_{i \in \mathcal{I}} \theta_i$ and $\theta^* = \max_{i \in \mathcal{I}} \theta_i$. Then Condition 2.2 implies that $\|f(w)\| \leq \frac{1}{\theta_*} w$ for $w \geq 0$. Let $\gamma_1 = (\frac{2\theta^*}{\theta_*} + 1)[\sum_{i=1}^I (x_i + T|y_i|)]$ Then for $t \in [0,T]$, using (21),

$$\|\hat{\alpha}_\theta[\psi]\|(t) \leq \Big(\frac{2\theta^*}{\theta_*} + 1\Big) \Lambda_t(\psi^1, \psi^2) + \gamma_1. \tag{27}$$

For $\kappa > 0$, we define

$$D(\kappa) = \{\psi = (\psi^1, \psi^2) \in D([0,T], \mathbb{R}^{2I}) :\ \|\psi^1\|^* + \|\psi^2\|^* \leq \kappa \text{ and } \bar{\psi}(0) \in \mathbb{R}_+^I\}, \tag{28}$$

where $\bar{\psi}$ is defined as above. Then using (21) and (27), for every κ there exists a constant $\beta = \beta(\kappa)$ such that, for all $\psi \in D(\kappa)$,

$$\|\hat{\alpha}_\theta[\psi]\|^* \leq \beta(\kappa), \tag{29}$$

$$|\varphi_\theta[\psi]|^* \leq \beta(\kappa). \tag{30}$$

Thus given $\varepsilon > 0$ we can find $\delta = \delta(\kappa, \varepsilon)$ such that

$$\|f(w_1) - f(w_2)\| < \frac{\varepsilon}{2} \text{ if } |w_1 - w_2| \leq \delta \text{ and } w_i \in [0, \beta(\kappa)].$$

Also using the relation $\hat{\varphi}_\theta[\psi] = \Gamma[\theta \cdot \hat{\psi}]$ where $\hat{\psi}$ is defined above, we have for any $\psi, \tilde{\psi} \in D([0,T], \mathbb{R}^{2I})$

$$|\hat{\varphi}_\theta[\psi] - \hat{\varphi}_\theta[\tilde{\psi}]|^* \leq c_1(\|\psi^1 - \tilde{\psi}^2\|^* + \|\psi^2 - \tilde{\psi}^2\|^*), \tag{31}$$

for some constant c_1. Choosing $\delta_1 = \delta_1(\kappa, \varepsilon)$ sufficiently small, for $\psi, \tilde{\psi} \in D(\kappa)$ we have, with φ and $\tilde{\varphi}$ denoting the dynamics corresponding to $(\psi, \hat{\alpha}_\theta[\psi])$ and resp., $(\tilde{\psi}, \hat{\alpha}_\theta[\tilde{\psi}])$,

$$\|\varphi - \tilde{\varphi}\|^* \leq \frac{\varepsilon}{2} \text{ if } \|\psi^1 - \tilde{\psi}^2\|^* + \|\psi^2 - \tilde{\psi}^2\|^* \leq \delta_1.$$

Therefore using the above estimate and (21) we have for $\psi, \tilde{\psi} \in D(\kappa)$ and δ_1 sufficiently small,

$$\|\hat{\alpha}_\theta[\psi] - \hat{\alpha}_\theta[\tilde{\psi}]\|^* \leq \varepsilon \text{ if } \|\psi^1 - \tilde{\psi}^2\|^* + \|\psi^2 - \tilde{\psi}^2\|^* \leq \delta_1. \tag{32}$$

This gives the continuity of the map $\hat{\alpha}_\theta$(and so of α_θ) on P.

Let a positive integer k and a map $\varphi : [0,T] \to \mathbb{R}^k$ be given. Given also a constant $\eta > 0$, we define the η-oscillation of φ as

$$\mathrm{osc}_\eta(\varphi) = \sup\{\|\varphi(s) - \varphi(t)\| \ : \ |s-t| \leq \eta, \ s,t \in [0,T]\}.$$

Then, as follows directly from the definition of $\hat{\alpha}_\theta$ and the continuity of f, for any $\psi \in D(\kappa)$, given $\varepsilon > 0$ there exist $\delta > 0$ and $\eta > 0$ such that

$$\mathrm{osc}_\eta(\hat{\alpha}_\theta[\psi]) \leq \varepsilon \text{ provided } \mathrm{osc}_\eta(\psi) \leq \delta. \tag{33}$$

4 Proof of Theorem 2.1

4.1 Lower bound

Theorem 4.1 *Assume Conditions 2.1 and 2.2 to hold. Then* $\liminf V^n(\tilde{X}_n(0)) \geq V(x)$.

Proof: Fix $\tilde{\psi} = (\tilde{\psi}^1, \tilde{\psi}^2) \in P$. Let $d(\cdot,\cdot)$ be a metric on $D([0,T], \mathbb{R}^{2I})$ which induces the J_1 topology. Define, for $r > 0$,

$$\mathcal{A}_r = \{\psi \in D([0,T], \mathbb{R}^{2I}) \ : \ d(\psi, \tilde{\psi}) < r\}.$$

Since $\tilde{\psi}$ is continuous, for any $r_1 \in (0,1)$ there exists $r > 0$ such that

$$\psi \in \mathcal{A}_r \quad \text{implies} \quad \|\psi - \tilde{\psi}\|^* < r_1. \tag{34}$$

Define $\theta_n = (\frac{n}{\mu_n^1}, \frac{n}{\mu_n^2}, \ldots, \frac{n}{\mu_n^I})$. Then $\theta_n \to \theta$ as $n \to \infty$. Now, given $0 < \varepsilon < 1$, choose a sequence of policies $\{B_n\}$ such that

$$V^n(\tilde{X}_n(0)) + \varepsilon > J(\tilde{X}_n(0), B_n) \text{ and } B_n \in \mathfrak{U}_n \text{ for all } n.$$

Recall that

$$J(\tilde{X}_n(0), B_n) = \frac{1}{b_n^2} \log \mathbb{E}[e^{b_n^2[\int_0^T h(\tilde{X}_n(s))ds + g(\tilde{X}_n(T))]}], \tag{35}$$

where

$$\tilde{X}_n^i(t) = \tilde{X}_n^i(0) + y_n^i t + \tilde{A}_n^i(t) - \tilde{S}_n^i(T_n^i(t)) + Z_n^i(t), \tag{36}$$

$$Z_n^i(t) = \frac{\mu_n^i}{n}\frac{\sqrt{n}}{b_n}(\rho^i t - T_n^i(t)), \qquad T_n^i(t) = \int_0^t B_n^i(s)ds. \tag{37}$$

For $G > 0$, define a random variable τ_n by

$$\tau_n = \inf\{t \geq 0 : \theta_n \cdot Z_n(t) > G\} \wedge T \equiv \inf\Big\{t \geq 0 \ : \ \frac{\sqrt{n}}{b_n}\Big(t - \sum_{i=1}^I T_n^i(t)\Big) > G\Big\} \wedge T.$$

By (10), $\theta_n \cdot Z_n$ is nondecreasing and hence

$$\begin{aligned} \theta_n \cdot Z_n(t) &\leq G \text{ for } t \leq \tau_n \\ \theta_n \cdot Z_n(t) &> G \text{ for } t > \tau_n. \end{aligned}$$

Consider the event $(\tilde{A}_n, \tilde{S}_n) \in \mathcal{A}_r$. Under this event, for $t > \tau_n$,

$$\theta_n \cdot \tilde{X}_n(t) \geq -\|\theta_n\|(\kappa_0 + 2\|\tilde{\psi}\|^*) + G,$$

where κ_0 is a constant (not depending on n or G) and we used (34) and the boundedness of $\tilde{X}_n(0)$ and $\tilde{\lambda}_n^i - \rho^i\tilde{\mu}_n^i$. Since also θ_n converges, we can choose a constant κ_1 such that, on the indicated event,

$$\theta_n \cdot \tilde{X}_n(t) \geq -\kappa_1 + G, \qquad t > \tau_n. \tag{38}$$

Next, let $\varphi : [0,T] \to \mathbb{R}^I$ be the dynamics corresponding to $(\tilde{\psi}, \zeta)$, where $\zeta = \alpha_\theta[\tilde{\psi}]$, namely

$$\varphi_i(t) = x^i + y^i t + \tilde{\psi}_i^1(t) - \tilde{\psi}_i^2(\rho^i t) + \zeta^i(t). \tag{39}$$

Then $\varphi(t) = f(\varphi_\theta[\tilde{\psi}](t))$ (20). For any $\kappa > 0$ define a compact set $Q(\kappa)$ as

$$Q(\kappa) = \{q \in \mathbb{R}_+^I \ : \ 2q \cdot \theta \leq \kappa\}.$$

Now choose κ large enough so that

$$h(z) \geq \sup_t \|h(\varphi(t))\| \text{ and } g(z) \geq g(\varphi(T))$$

for all $z \in Q^c(\kappa)$. To see that this is possible note that $h(f(\cdot))$ is nondecreasing, and for $z \in Q^c(\kappa)$

$$h(z) \geq \min\{h(q) \ : \ \theta \cdot q = \theta \cdot z\} = h(f(\theta \cdot z)) \geq h(f(\kappa/2)),$$

whereas $\varphi(t)$, $t \in [0,T]$, is bounded. A similar argument applies for g. Since $\theta_* := \min_i \theta_i > 0$, we can choose n_0 large enough to ensure that $(\theta_n)_i \leq 2\theta_i$ for all $i \in \mathcal{I}$ and $n \geq n_0$. Now if we choose G in (38) large enough so that $-\kappa_1 + G > \kappa$ with κ as above we have for $t > \tau_n, n \geq n_0$,

$$2\theta \cdot \tilde{X}_n(t) \geq \theta_n \cdot \tilde{X}_n(t) > \kappa,$$

and hence by our choice of κ we have on the indicated event, for all $t > \tau_n$,

$$h(\tilde{X}_n(t)) \geq |h(\varphi)|^* \text{ and } g(\tilde{X}_n(t)) \geq g(\varphi(T)) \text{ for all sufficiently large } n. \tag{40}$$

Now we fix G as above and consider $t \le \tau_n$, on the same event $(\tilde{A}_n, \tilde{S}_n) \in \mathcal{A}_r$. The nonnegativity of $\tilde{X}^i_n$ and (36) imply a lower bound on each of the terms Z^i_n, namely

$$Z^i_n(t) \ge -\tilde{X}_n(0) - y^i_n t - \tilde{A}^i_n(t) + \tilde{S}^i_n(T^i_n(t)).$$

Therefore using (34) there exists a constant κ_2 such that for all sufficiently large n, $Z^i_n(t) \ge -\kappa_2$. Combining this with the definition of τ_n in terms of G, we have for $t \le \tau_n$ and all large n,

$$\|Z_n(t)\| \le \kappa_3. \tag{41}$$

Consider the stochastic processes $Y_n, \tilde{Y}_n, \tilde{Z}_n$, with values in $\mathbb{R}^I$,

$$\begin{aligned} Y^i_n(t) &= \tilde{A}^i_n(t \wedge \tau_n), \\ \tilde{Y}^i_n(t) &= x_i - \tilde{X}^i_n(0) + (y_i - y^i_n)t + \tilde{S}^i_n(T^i_n(t \wedge \tau_n)) - (1 - \mu^i \theta^i_n) Z^i_n(t \wedge \tau_n), \\ \tilde{Z}^i_n(t) &= \mu^i \theta^i_n Z^i_n(t). \end{aligned}$$

Then by (36),

$$\tilde{X}^i_n(t) = x_i + y_i t + Y^n_i(t) - \tilde{Y}^i_n(t) + \tilde{Z}^i_n(t), \qquad t \in [0, \tau_n]. \tag{42}$$

Note that $Y_n, \tilde{Y}_n$ have RCLL sample paths, and consider $\hat{\alpha}_\theta[Y_n, \tilde{Y}_n]$. Denote by W_n the corresponding dynamics, namely

$$W_n(t) = x + yt + Y_n(t) - \tilde{Y}_n(t) + \hat{\alpha}_\theta[Y_n, \tilde{Y}_n](t). \tag{43}$$

Use (26) with $\psi(t) = x + yt + Y_n(t) - \tilde{Y}_n(t)$, $\zeta = \tilde{Z}_n$. Note that $\tilde{X}_n = \psi + \zeta$ takes values in $\mathbb{R}^I_+$, by definition, and that $\theta \cdot \tilde{Z}_n$ is nonnegative and nondecreasing, by (10). Moreover, by definition of $\hat{\alpha}_\theta$, $W_n = f(\Gamma[\theta \cdot \psi])$. Hence (26) gives

$$h(\tilde{X}_n(t)) \ge h(W_n(t)) \ \text{ and } \ g(\tilde{X}_n(t)) \ge h(W_n(t)), \qquad t \in [0, \tau_n]. \tag{44}$$

Let $\kappa_4 = \|\tilde{\psi}\|^*$. By (34), on the indicated event, $(\tilde{A}_n, \tilde{S}_n) \in D(2(1 + \kappa_4))$ where $D(\kappa)$ is defined in Section 3. Note that $x + Y_n(0) - \tilde{Y}_n(0) = \tilde{X}_n(0) \in \mathbb{R}^I_+$ and, from (41), that $(Y_n, \tilde{Y}_n) \in D(2(2 + \kappa_4))$ for all large n. Since $0 \le B^i_n(s) \le 1$, $T^i_n(s) \in [0, \tau_n]$ for all $s \in [0, \tau_n]$. Hence from (34) we have for $(\tilde{A}_n, \tilde{S}_n) \in \mathcal{A}_r$

$$\sup_{[0,\tau_n]} |\tilde{\psi}^2_i(\rho_i t) - \tilde{S}^i_n(T^i_n(t))| \le r_1 + \sup_{[0,\tau_n]} |\tilde{\psi}^2_i(\rho_i t) - \tilde{\psi}^2(T^i_n(t))|.$$

Again using the continuity of $\tilde{\psi}^2$, we can choose $r_2 > 0$ small enough such that

$$\mathrm{osc}_{r_2}[\tilde{\psi}^2] < r_1.$$

Since $\frac{b_n}{\sqrt{n}} \to 0$, we note from (41) that for all large n, and all i, $\sup_{[0,\tau_n]} |\rho^i t - T^i_n(t)| < r_2$. Since $\tilde{X}_n(0) \to x$, $y_n \to y$ and $\theta_n \to \theta = (\frac{1}{\mu^1}, \dots, \frac{1}{\mu^I})$, it follows that,

$$\sup_{[0,\tau_n]} |\tilde{Y}^i_n(t) - \tilde{\psi}^2_i(\rho_i t)| < 3r_1,$$

for all large n. Now taking $\kappa = 2(2+\kappa_4)$, we choose r_1 sufficiently small (see (32)) so that for all n large we have

$$\sup_{[0,\tau_n]} \|\alpha_\theta[\tilde{\psi}](t) - \hat{\alpha}_\theta[Y_n, \tilde{Y}_n](t)\| \le \varepsilon.$$

Now choosing $r < \varepsilon/(3\sqrt{I})$ and using (39) and (43), for $(\tilde{A}_n, \tilde{S}_n) \in \mathcal{A}_r$ and all large n, we have

$$\|\varphi - W_n\|^*_{\tau_n} \le 4\varepsilon. \tag{45}$$

Let $\kappa_5 = (\|\varphi\|^* + 4)$. Denote by ω_h $[\omega_g]$ the modulus of continuity of h [resp., g] over $\{q : \|q\| \le \kappa_5\}$. Then by (44), on the indicated event, for all large n,

$$\int_0^{\tau_n} h(\tilde{X}_n(s))ds \ge \int_0^{\tau_n} h(W_n(s))ds \ge \int_0^{\tau_n} h(\varphi(s))ds - T\omega_h(4\varepsilon).$$

Combined with (40) this gives

$$\int_0^T h(\tilde{X}_n(s))ds \ge \int_0^T h(\varphi(s))ds - T\omega_h(4\varepsilon).$$

A similar argument gives

$$g(\tilde{X}_n(T)) = g(\varphi(T))\chi_{\{T\le\tau_n\}} + g(\varphi(T))\chi_{\{T>\tau_n\}} \ge g(\varphi(T)) - \omega_g(4\varepsilon).$$

Hence for all large n,

$$\begin{aligned}
\mathbb{E}[e^{b_n^2[\int_0^T h(\tilde{X}_n(s))ds + g(\tilde{X}_n(T))]}] &\ge \mathbb{E}\Big[e^{b_n^2[\int_0^T h(\tilde{X}_n(s))ds + g(\tilde{X}_n(T))]}\chi_{\{(\tilde{A}_n,\tilde{S}_n)\in\mathcal{A}_r\}}\Big] \\
&\ge \mathbb{E}\Big[e^{b_n^2[\int_0^T h(\varphi(s))ds + g(\varphi(T)) - a(\varepsilon)]}\chi_{\{(\tilde{A}_n,\tilde{S}_n)\in\mathcal{A}_r\}}\Big],
\end{aligned}$$

where $a(\varepsilon) = [T\omega_h(4\varepsilon) + \omega_g(4\varepsilon)] \to 0$ as $\varepsilon \to 0$. We now use condition 2.1. Since $\mathcal{A}_\eta$ is open,

$$\mathbb{P}((\tilde{A}_n, \tilde{S}_n) \in \mathcal{A}_r) \ge e^{-b_n^2[\inf_{\psi\in\mathcal{A}_r}\mathbb{I}(\psi)+\varepsilon]}$$

holds for all sufficiently large n. Hence we have from (35) that for all large n,

$$\begin{aligned}
V^n(\tilde{X}_n(0)) + \varepsilon &\ge J(\tilde{X}_n(0), B_n) \\
&\ge \int_0^T h(\varphi(s))ds + g(\varphi(T)) - \mathbb{I}(\tilde{\psi}) - a(\varepsilon) - \varepsilon.
\end{aligned}$$

Therefore

$$\liminf_{n\to\infty} V^n(\tilde{X}_n(0)) \ge c(\tilde{\psi}, \alpha_\theta[\tilde{\psi}]) - a(\varepsilon) - 2\varepsilon,$$

and letting $\varepsilon \to 0$, we obtain

$$\liminf_{n\to\infty} V^n(\tilde{X}_n(0)) \ge c(\tilde{\psi}, \alpha_\theta[\tilde{\psi}]).$$

Finally, since $\tilde{\psi}$ is arbitrary we have from (25)

$$\liminf_{n\to\infty} V^n(\tilde{X}_n(0)) \ge V(x).$$

□

4.2 Upper bound

Theorem 4.2 *Assume Conditions 2.1, 2.2 and 2.3 to hold. Then* $\limsup V^n(\tilde{X}_n(0)) \le V(x)$.

Remark 4.1 *If the functions h, g are bounded then Condition 2.3 is not required in the above statement.*

The proof is based on the construction and analysis of a particular policy, described below in equations (50)–(54). To see the main idea behind the structure of the policy, refer to equations (8) and (9), which describe the dependence of the scaled process $\tilde{X}_n$ on the stochastic primitives $\tilde{A}_n$, $\tilde{S}_n$, and the control process B_n (recall from (4) that T_n is an integral form of B_n). Because of the amplifying factor $\sqrt{n}/b_n$ which appears in the expression (9) in front of

$$\rho^i t - T_n^i(t) = \int_0^t (\rho^i - B_n^i(s))ds,$$

it is seen that fluctuations at scale as small as $b_n/\sqrt{n}$ of B_n, about its center ρ, cause order-one displacements in $\tilde{X}_n$. Initially, the policy drives the process $\tilde{X}_n$ from from the initial position $\tilde{X}^n(0) \approx x$ to the corresponding point on the minimizing curve, $f(\theta \cdot x)$, in a short time. This is reflected in the choice of the constant ℓ applied during the first time interval $[0, v)$ (see first line of (53)). Afterwards, the policy mimics the behavior of the optimal strategy for the game, namely $\hat{\alpha}_\theta$. This is performed by applying F_n (see third line of (53)), which consists of the response of $\hat{\alpha}_\theta$, in differential form, to the stochastic data P_n (see (51)).

Proof: Given a constant $\varDelta$, define

$$\mathcal{Q} = \{\psi \in D([0,T], \mathbb{R}^{2I}) \ : \ \mathbb{I}(\psi) \le \varDelta\}. \tag{46}$$

By the definition of $\mathbb{I}$ (from Section 2), $\mathcal{Q}$ is a compact set containing absolutely continuous paths starting from zero (particularly, $\mathcal{Q} \subset P$), with derivative having L^2-norm uniformly bounded. Consequently, for a constant $M = M_\varDelta$ and all $\psi \in \mathcal{Q}$, one has $\|\psi^1\|^* + \|\psi^2\|^* \le M$. Consider the set $D(M+1)$ (28), let $\varepsilon \in (0,1)$ be given, and choose $\delta_1, \delta, \eta > 0$ as in (32) and (33), corresponding to ε and $\kappa = M + 1$. Assume, without loss of generality, that $\delta_1 \vee \delta < \varepsilon$. It follows from the L^2 bound alluded to above, that for each fixed $\varDelta$, the members of $\mathcal{Q}$ are equicontinuous. Hence one can choose $v_0 \in (0, \eta)$ (depending on $\varDelta$), such that

$$\mathrm{osc}_{v_0}(\psi_i^l) < \frac{\delta_1 \wedge \delta}{4\sqrt{2I}}, \text{ for all } \psi = (\psi^1, \psi^2) \in \mathcal{Q},\, l = 1, 2,\, i \in \mathcal{I}. \tag{47}$$

Recall

$$\mathcal{A}_r(\tilde{\psi}) = \{\psi \in D([0,T], \mathbb{R}^{2I}) \ : \ d(\psi, \tilde{\psi}) < r\}.$$

Noting that, for any $f \in \varUpsilon$ (see Notations),

$$\begin{aligned} \|\psi(t) - \tilde{\psi}(t)\| &\le \|\psi(t) - \tilde{\psi}(f(t))\| + \|\tilde{\psi}(f(t)) - \tilde{\psi}(t)\|, \\ |f(t) - t|_T^* &\le T(e^{\|f\|^\circ} - 1), \end{aligned}$$

it follows, by the equicontinuity of the members of $\mathcal{Q}$, that it is possible to choose $v_1 > 0$ such that, for any $\tilde{\psi} \in \mathcal{Q}$,

$$\psi \in \mathcal{A}_{v_1}(\tilde{\psi}) \quad \text{implies} \quad \|\psi - \tilde{\psi}\|^* < \frac{\delta_1}{4}. \tag{48}$$

Let $v_2 = \min\{v_0, v_1, \frac{\varepsilon}{2}\}$. Since $\mathcal{Q}$ is compact and $\mathbb{I}$ is lower semicontinuous, one can find a finite number of members $\bar{\psi}^1, \bar{\psi}^2, \dots, \bar{\psi}^N$ of $\mathcal{Q}$, and positive constants $v^1, \dots, v^N$ with $v^k < v_2$, satisfying $\mathcal{Q} \subset \cup_k \mathcal{A}^k$, and

$$\inf\{\mathbb{I}(\psi) : \psi \in \overline{\mathcal{A}^k}\} \geq \mathbb{I}(\bar{\psi}^k) - \frac{\varepsilon}{2}, \qquad k = 1, 2, \dots, N, \tag{49}$$

where, throughout, $\mathcal{A}^k := \mathcal{A}_{v^k}(\bar{\psi}^k)$.

We next define a policy for which we shall prove that the lower bound is asymptotically attained. Denote

$$\Theta(a, b) = a\chi_{[0,1]}(a)\chi_{[0,1]}(b), \qquad a, b \in \mathbb{R}.$$

Fix $n \in \mathbb{N}$. Recall (3), (4) and (5) by which

$$\begin{cases} D_n^i = S_n^i \circ T_n^i, \\ T_n^i = \int_0^{\cdot} B_n^i(s)ds, \\ X_n^i = X_n^i(0) + A_n^i - D_n^i. \end{cases} \tag{50}$$

Recall the scaled processes (7) and let also

$$\begin{cases} \tilde{D}_n^i = \tilde{S}_n^i \circ T_n^i, \\ P_n = (\tilde{A}_n, \tilde{D}_n). \end{cases} \tag{51}$$

Let $\ell = f(x \cdot \theta) - x$ and $v = \frac{v_2}{2} \wedge \frac{T}{4}$. For $i \in \mathcal{I}$, assume that B_n^i is given by

$$B_n^i(t) = C_n^i(t)\chi_{\{X_n^i(t) > 0\}}, \qquad t \in [0, T], \tag{52}$$

where, for $t \in [0, T]$,

$$C_n^i(t) = \begin{cases} \Theta\Big(\rho^i - \frac{b_n}{\mu^i\sqrt{n}}\frac{\ell_i}{v}, \sum_{k=1}^{I}\Big(\rho^k - \frac{b_n}{\mu^k\sqrt{n}}\frac{\ell_k}{v}\Big)^+\Big), & \text{if} \quad t \in [0, v), \\ \rho^i, & \text{if} \quad t \in [v, 2v), \\ \Theta\Big(\rho^i - F_n^i(t - v), \sum_{k=1}^{I}(\rho^k - F_n^k(t - v))^+\Big), & \text{if} \quad \begin{array}{l} \|P_n\|_{t-v}^* < M + 2, \\ t \in [jv, (j+1)v), j = 2, 3, \dots, \end{array} \\ \rho^i, & \text{if} \quad \begin{array}{l} \|P_n\|_{t-v}^* \geq M + 2, \\ t \in [jv, (j+1)v),\ j = 2, 3, \dots, \end{array} \end{cases} \tag{53}$$

and

$$F_n^i(u) = \frac{b_n}{\mu^i\sqrt{n}}\frac{\hat{\alpha}_\theta^i[P_n](jv) - \hat{\alpha}_\theta^i[P_n]((j-1)v)}{v}, \quad u \in [jv, (j+1)v),\ j = 1, 2, \dots. \tag{54}$$

Let us argue that these equations uniquely define a policy. To this end, consider equations (50), (51), (52), (53), (54), along with the obvious relations between scaled and unscaled processes, as a set of equations for $X_n, D_n, T_n, P_n, B_n, C_n, F_n$ (and the scaled versions $\tilde{X}_n, \tilde{D}_n$), driven by the data (A_n, S_n) (equivalently, $(\tilde{A}_n, \tilde{S}_n)$), and satisfying the initial condition $X_n(0)$. Arguing by induction on the jump times of the processes A_n and S_n, and using the causality of the map $\hat{\alpha}_\theta$, it is easy to see that this set of equations has a unique solution. Moreover, this solution is consistent with the model equations (3)–(5). The processes alluded to above are therefore well-defined.

We now show that $B_n \in \mathfrak{U}_n$. To see that B_n has RCLL sample paths, note first that, by construction, F_n, X_n are piecewise constant with finitely many jumps, locally, hence so is B_n. Therefore the existence of left limits follows. Right continuity follows from the fact that X_n, F_n and consequently C_n have this property. The other elements in the definition of an admissible control hold by construction. Thus $B_n \in \mathfrak{U}_n$ for $n \in \mathbb{N}$. As a result,

$$V^n(\tilde{X}_n(0)) \le J^n(\tilde{X}_n(0), B_n). \tag{55}$$

Our convention in this proof will be that $c_1, c_2, \dots$ denote positive constants that do not depend on $n, \varepsilon, v, \Delta$.

Define $\varphi^k(t) = f(\varphi_\theta[\bar{\psi}^k](t))$. Note that φ^k is the dynamics corresponding to $\bar{\psi}^k$ and $\alpha_\theta[\bar{\psi}^k]$. Let $\tilde{\Lambda}_n = \Lambda_T(\tilde{A}_n, \tilde{S}_n)$ and denote by Ω_n^k the event $\{(\tilde{A}_n, \tilde{S}_n) \in \mathcal{A}^k\}$. We prove the result in number of steps. In Steps 1–4 we shall show that for a constant c_1, for all $n \ge n_0(\varepsilon, v)$,

$$\|\tilde{X}_n\|_T^* \le c_1(1 + \tilde{\Lambda}_n), \tag{56}$$

and

$$\sup_{[v,T]} \|\tilde{X}_n - \varphi^k\| \le c_1\varepsilon, \qquad \text{on } \Omega_n^k,\ k = 1, 2, \dots, N. \tag{57}$$

The final step will then use these estimates to conclude the result.

Step 1: The goal of this step is to show (63) below. By (27),

$$\|\hat{\alpha}_\theta[P_n]\|_t^* \le c_2(1 + \|P_n\|_t^*). \tag{58}$$

Therefore

$$\|F_n\|_t^* \le \frac{b_n}{\sqrt{n}} \frac{c_3}{v}(1 + \|P_n\|_t^*). \tag{59}$$

Since $\rho^i \in (0,1)$ for all $i \in \mathcal{I}$, we note from (59) that for all sufficiently large n, for any $t \in [2v, T]$,

$$\|P_n\|_{t-v}^* < M + 2 \quad \text{implies} \quad \sum_i (\rho^i - F_n^i(t-v))^+ = \sum_i (\rho^i - F_n^i(t-v)) \le 1,$$

as $\sum_i F_n^i(u) \ge 0$ for all $u \in [v, T]$. Define

$$\hat{\tau}_n = \inf\{t \ge 0 \ : \ \|P_n(t)\| \ge M + 2\}.$$

It is easy to check by definition of C_n^i, and using the fact $\rho_i \in (0,1)$ and the convergence $b_n/\sqrt{n} \to 0$, that for all large n, on the event $\{\hat{\tau}_n \le v\}$,

$$\sup_{t\in[0,T]} \frac{\sqrt{n}}{b_n}\Big|\rho^i t - \int_0^t C_n^i(s)ds\Big| \le c_4.$$

Next consider the event $\{\hat{\tau}_n > v\}$. Using (59), the one has for all sufficiently large n

$$C_n^i(t) = \begin{cases} \rho^i - \dfrac{b_n}{\mu^i\sqrt{n}}\dfrac{\ell_i}{v}, & \text{if} \quad t\in[0,v), \\ \rho^i, & \text{if} \quad t\in[v,2v), \\ \rho^i - F_n^i(t-v), & \text{if} \quad t\in[2v,\hat{\tau}_n+v) \\ \rho^i, & \text{if} \quad t\in[\hat{\tau}_n+v,T]. \end{cases} \tag{60}$$

Thus, on $\{\hat{\tau}_n > v\}$,

$$\sup_{t\in[0,2v]} \Big|\rho^i t - \int_0^t C_n^i(s)ds\Big| \le c_5\frac{b_n}{\sqrt{n}},$$

while

$$\sup_{t\in[2v,T]} \Big|\rho^i t - \int_0^t C_n^i(s)ds\Big| \le c_5\frac{b_n}{\sqrt{n}} + \sup_{t\in[2v,\hat{\tau}_n+v]} \Big|\int_{2v}^t F_n^i(s-v)ds\Big|. \tag{61}$$

Consider $j \ge 2$ and $jv \le t < (j+1)v$. Then by the definition of F_n,

$$\begin{aligned}\int_{2v}^t F_n^i(s-v)ds &= \int_{2v}^{jv} F_n^i(s-v)ds + \int_{jv}^t F_n^i(s-v)ds \\ &= \frac{b_n}{\mu^i\sqrt{n}}[\hat{\alpha}_\theta^i[P_n]((j-2)v) - \hat{\alpha}_\theta^i[P_n](0)] \\ &\quad + \frac{b_n}{\mu^i\sqrt{n}}\frac{t-jv}{v}[\hat{\alpha}_\theta^i[P_n]((j-1)v) - \hat{\alpha}_\theta^i[P_n]((j-2)v)]. \end{aligned} \tag{62}$$

Combining this identity with (58) shows that the last term on (61) is bounded by

$$\sup_{t\in[2v,\hat{\tau}_n+v]} \frac{b_n}{\mu^i\sqrt{n}}4c_2(1+\|P_n\|_{t-v}^*) \le \frac{b_n}{\mu^i\sqrt{n}}4c_2(1+\tilde{\Lambda}_n),$$

where in the last inequality we also used the fact that $T_n^i(t) \le t$, by which $|\tilde{D}_n^i|_t^* = |\tilde{S}_n^i \circ T_n^i|_t^* \le |\tilde{S}_n^i|_t^*$. We conclude that, for all sufficiently large n,

$$\sup_{t\in[0,T]} \frac{\sqrt{n}}{b_n}\Big|\rho^i t - \int_0^t C_n^i(s)ds\Big| \le c_6(1+\tilde{\Lambda}_n). \tag{63}$$

Step 2: We prove (56). To this end, rewrite (8) as $\tilde{X}_n^i = \hat{Y}_n^i + \hat{Z}_n^i$, where

$$\begin{aligned}\hat{Y}_n^i(t) &= \tilde{X}_n^i(0) + y_n^i t + \tilde{A}_n^i(t) - \tilde{S}_n^i(T_n^i(t)) + \frac{\mu_n^i}{n}\frac{\sqrt{n}}{b_n}\Big(\rho^i t - \int_0^t C_n^i(s)ds\Big), \\ \hat{Z}_n^i(t) &= \frac{\mu_n^i}{n}\frac{\sqrt{n}}{b_n}\int_0^t C_n^i(s)\chi_{\{\tilde{X}_n^i(s)=0\}}ds.\end{aligned}$$

Since for each i, $\tilde{X}_n^i$ is nonnegative and $\hat{Z}_n^i$ is nonnegative, nondecreasing, and increases only when $\tilde{X}_n^i$ is equal to zero, it follows that $(\tilde{X}_i^n, \hat{Z}_n^i)$ is the solution to the Skorohod problem for data $\hat{Y}_n^i$ (see [7] for this well-known characterization of the Skorohod map (16)). As a result, for all large n,

$$|\hat{Z}_n^i|_T^* + |\tilde{X}_n^i|_T^* \leq 4|\hat{Y}_n^i|_T^* \leq c_7(1 + \tilde{\Lambda}_n), \tag{64}$$

where we used (63) and the convergence of μ_n^i/n, $\tilde{X}_n^i(0)$ and y_n^i. This shows (56).

Step 3: Here we analyze the events Ω_n^k. First, using

$$\rho^i t - T_n^i(t) = \rho^i t - \int_0^t C_n^i(s)ds + \int_0^t C_n^i(s)\chi_{\{\tilde{C}_n^i(s)=0\}}ds,$$

we obtain from (63) and (64), for all large n,

$$\sup_{t\in[0,T]} \frac{\mu_n^i}{n}\frac{\sqrt{n}}{b_n}|\rho^i t - T_n^i(t)| \leq c_8(1 + \tilde{\Lambda}_n).$$

In particular, for all large n,

$$\sup_{t\in[0,T]} |\rho^i t - T_n^i(t)| \leq \frac{v}{2} \tag{65}$$

holds on the event $\cup_k \Omega_n^k$.

Next, we estimate $\tilde{S}_n(T_n(t)) - R[\bar{\psi}^{k,2}](t-v)$ on the set Ω_n^k. If we define $\bar{\psi}^{k,2}(T_n(\cdot)) = (\bar{\psi}_1^{k,2}(T_n^1(\cdot)), \dots, \bar{\psi}_I^{k,2}(T_n^I(\cdot)))$ then, for all large n,

$$\begin{aligned}
\sup_{t\in[v,T]} &\|\tilde{S}_n(T_n(t)) - R[\bar{\psi}^{k,2}](t-v)\| \\
&\leq \|\tilde{S}_n \circ T_n - \bar{\psi}^{k,2} \circ T_n\|^* + \sup_{t\in[v,T]} \|\bar{\psi}^{k,2}(T_n(t)) - R[\bar{\psi}^{k,2}](t-v)\| \\
&\leq \frac{\delta_1}{4} + \frac{\delta_1}{4} = \frac{\delta_1}{2},
\end{aligned} \tag{66}$$

where for the first estimate we have used (48) and for second we have used (47) and (65).

Finally, we show the two estimates (67) and (69) below. Note that on Ω_n^k one has $\hat{\tau}_n \geq T$ for all large n (as follows by $\|P_n\|_T^* = \|\tilde{A}_n\|_T^* + \|\tilde{D}_n\|_T^* \leq \|\tilde{A}_n\| + \|\tilde{S}_n\| < M+2$ by the discussion in the beginning of the proof (48)). As a result, (60) is applicable. In particular, for all large n,

$$\mu^i \frac{\sqrt{n}}{b_n}\Big(\rho^i t - \int_0^t C_n^i(s)ds\Big) - \frac{t}{v}\ell_i = 0, \qquad t \in [0, v). \tag{67}$$

Now for $k = 1, 2, \dots, N$, consider

$$\hat{W}_n^{i,k}(t) := \mu^i \frac{\sqrt{n}}{b_n}\Big(\rho^i t - \int_0^t C_n^i(s)ds\Big) - \alpha_\theta[\bar{\psi}^k](t-v), \qquad t \in [v, T],$$

on the event Ω_n^k. We note from (20) that $\alpha_\theta[\bar{\psi}^k](0) = \ell$. Hence for $t \in [v, 2v)$ and all large n, we have from (47) and (33) that

$$|\hat{W}_n^{i,k}(t)| = |\ell - \alpha_\theta[\bar{\psi}^k](t-v)| \leq \varepsilon.$$

Next consider $t \in [2v, T]$ and integer j for which $jv \le t < (j+1)v$. From the calculation (62), for large n,

$$\begin{aligned}
\mu^i \frac{\sqrt{n}}{b_n}\Big(\rho^i t - \int_0^t C_n^i(s)ds\Big) &= \ell_i + \mu^i \frac{\sqrt{n}}{b_n}\int_{2v}^t F_n^i(s-v)ds \\
&= \hat{\alpha}_\theta^i[P_n]((j-2)v) \\
&\quad + \frac{t-jv}{v}[\hat{\alpha}_\theta^i[P_n]((j-1)v) - \hat{\alpha}_\theta^i[P_n]((j-2)v)].
\end{aligned}$$

Hence

$$|\hat{W}_n^{i,k}(t)| \le |\hat{\alpha}_\theta^i[P_n]((j-2)v) - \alpha_\theta[\bar{\psi}^k](t-v)| + |\hat{\alpha}_\theta^i[P_n]((j-1)v) - \hat{\alpha}_\theta^i[P_n]((j-2)v)|.$$

For large n,

$$\begin{aligned}
&|\hat{\alpha}_\theta^i[P_n]((j-2)v) - \alpha_\theta[\bar{\psi}^k](t-v)| \\
&\qquad \le |\hat{\alpha}_\theta^i[P_n]((j-2)v) - \hat{\alpha}_\theta[\bar{\psi}^{k,1}, \bar{\psi}^{k,2}\circ T_n]((j-2)v)| \\
&\qquad\quad + |\hat{\alpha}_\theta[\bar{\psi}^{k,1}, \bar{\psi}^{k,2}\circ T_n]((j-2)v) - \hat{\alpha}_\theta[\bar{\psi}^{k,1}, R[\bar{\psi}^{k,2}]]((j-2)v)| \\
&\qquad\quad + |\alpha_\theta[\bar{\psi}^k]((j-2)v) - \alpha_\theta[\bar{\psi}^k](t-v)| \\
&\qquad \le 3\varepsilon,
\end{aligned}$$

where the first quantity is estimated using (48) and (32), the second using (65) and (32), and the third using (47) and (33). A similar estimate gives, for all large n,

$$|\hat{\alpha}_\theta^i[P_n]((j-1)v) - \hat{\alpha}_\theta^i[P_n]((j-2)v)| \le 3\varepsilon.$$

Hence for all large n, for each k,

$$\sup_{t\in[v,T]} |\hat{W}_n^{i,k}(t)| \le 6\varepsilon, \tag{68}$$

on Ω_n^k. Using (68) and (63), for all large n, for each k,

$$\sup_{t\in[v,T]}\Big|\frac{\mu_n^i}{n}\frac{\sqrt{n}}{b_n}\Big(\rho^i t - \int_0^t C_n^i(s)ds\Big) - \alpha_\theta[\bar{\psi}^k](t-v)\Big| \le 7\varepsilon, \tag{69}$$

on Ω_n^k.

Step 4: Recall $\varphi^k(t) = f(\varphi_\theta[\bar{\psi}^k](t))$. The goal of this step is to estimate the difference between $\tilde{X}_n$ and φ^k on Ω_n^k. To this end, let first

$$\tilde{\varphi}^k(t) = \begin{cases} x + \frac{t}{v}\ell & \text{for } t \in [0, v) \\ f(\varphi_\theta[\bar{\psi}^k](t-v)) & \text{for } t \in [v, T]. \end{cases}$$

Recall from Step 2 that $\tilde{X}_n^i$ solves the Skorohod problem for $\hat{Y}_n^i$. Note also that $\tilde{\varphi}_i^k \ge 0$. Thus using the Lipschitz property of the Skorohod map we have on Ω_n^k

$$|\tilde{X}_n^i - \tilde{\varphi}_i^k|_T^* \le 2|\hat{Y}_n^i - \tilde{\varphi}_i^k|_T^*. \tag{70}$$

Now for $t \in [0, v]$ and for all n large we have, using the definition of $\hat{Y}_n$ and (67), for all large n,

$$
\begin{aligned}
&|\hat{Y}_n^i(t) - \tilde{\varphi}_i^k(t)| \\
&\le |\tilde{X}_n^i(0) - x_i| + v|y_n^i| + |\tilde{A}_n^i(t) - \tilde{S}_n^i(T_n^i(t))| + \Big|\frac{\mu_n^i}{n} - \mu^i\Big|\Big|\frac{\sqrt{n}}{b_n}\Big(\rho^i t - \int_0^t C_n^i(s)ds\Big)\Big| \\
&\le c_9\varepsilon
\end{aligned} \tag{71}
$$

on Ω_n^k, where we use (47), (48) and (63). Moreover, for $t \in [v, T]$, by the definition of $\hat{Y}_n$ and $\tilde{\varphi}^k$,

$$
\begin{aligned}
\hat{Y}_n^i(t) - \tilde{\varphi}_i^k(t) = \tilde{X}_n^i(0) + y_n^i t + \tilde{A}_n^i(t) - \tilde{S}_n^i(T_n^i(t)) + \frac{\mu_n^i}{n}\frac{\sqrt{n}}{b_n}\Big(\rho^i t - \int_0^t C_n^i(s)ds\Big) \\
- \alpha_\theta[\bar{\psi}^k]_i(t-v) - x_i - y_i(t-v) - \bar{\psi}_i^{k,1}(t-v) + R[\bar{\psi}^{k,2}]_i(t-v).
\end{aligned}
$$

Hence, using (47), (48),(66) and (69), the estimate (71) is valid for $t \in [v, T]$ as well. Namely, $|\hat{Y}_n^i - \tilde{\varphi}_i^k|_T^* \le c_9\varepsilon$ on Ω_n^k for large n. Thus using (70), $\|\tilde{X}_n - \tilde{\varphi}^k\|^* \le c_{10}\varepsilon$ on Ω_n^k for large n. Comparing the definition of $\tilde{\varphi}^k$ and φ^k we obtain that, for all sufficiently large n, (57) holds.

Step 5: Since φ^k is bounded, and so is $\tilde{X}_n$ on Ω_n^k, it follows from (57) by continuity of h and g that, for all large n, on Ω_n^k,

$$
\Big|\int_0^T h(\varphi^k(s)ds + g(\varphi^k(T)) - H_n\Big| \le \omega(\varepsilon), \tag{72}
$$

where

$$
H_n = \int_0^T h(\tilde{X}_n(s))ds + g(\tilde{X}(T)),
$$

and $\omega = \omega_\Delta$ satisfies $\omega(a) \to 0$ as $a \to 0$ (for any Δ).

By (56) and the growth condition on h and g, $H_n \le c_{11}(1 + \tilde{\Lambda}_n)$. Hence given any $\Delta_1 > 0$,

$$
H_n > \Delta_1 \quad \text{implies} \quad \tilde{\Lambda}_n > c_{11}^{-1}\Delta_1 - 1 =: G(\Delta_1).
$$

Therefore

$$
\begin{aligned}
\mathbb{E}[e^{b_n^2 H_n}] &\le \mathbb{E}[e^{b_n^2[H_n\wedge\Delta_1]}] + \mathbb{E}[e^{b_n^2 H_n}\chi_{\{H_n>\Delta_1\}}] \\
&\le \mathbb{E}[e^{b_n^2[H_n\wedge\Delta_1]}] + \mathbb{E}[e^{b_n^2 c_{11}(1+\tilde{\Lambda}_n)}\chi_{\{\tilde{\Lambda}_n>G(\Delta_1)\}}].
\end{aligned} \tag{73}
$$

Now we estimate both terms on the right hand side of (73). Denote $\mathcal{B} = (\cup_{k=1}^N \mathcal{A}^k)^c$. Using (72), for all large n,

$$
\begin{aligned}
\mathbb{E}[e^{b_n^2[H_n\wedge\Delta_1]}] &\le \sum_{k=1}^N \mathbb{E}[e^{b_n^2[H_n\wedge\Delta_1]}\chi_{\{(\tilde{A}_n,\tilde{S}_n)\in\mathcal{A}^k\}}] + \mathbb{E}[e^{b_n^2[H_n\wedge\Delta_1]}\chi_{\{(\tilde{A}_n,\tilde{S}_n)\in\mathcal{B}\}}] \\
&\le \sum_{k=1}^N \mathbb{E}[e^{b_n^2[\int_0^T h(\varphi^k(s))ds + g(\varphi^k(T)) + \omega(\varepsilon)]}\chi_{\{(\tilde{A}_n,\tilde{S}_n)\in\mathcal{A}^k\}}] + \mathbb{E}[e^{b_n^2\Delta_1}\chi_{\{(\tilde{A}_n,\tilde{S}_n)\in\mathcal{B}\}}].
\end{aligned}
$$

Now by Condition 2.1, for all large n,

$$\frac{1}{b_n^2}\log \mathbb{P}((\tilde{A}_n,\tilde{S}_n)\in \overline{\mathcal{A}^k}) \le -\inf_{\psi\in\overline{\mathcal{A}^k}} \mathbb{I}(\psi)+\frac{\varepsilon}{2}, \qquad \frac{1}{b_n^2}\log \mathbb{P}((\tilde{A}_n,\tilde{S}_n)\in \mathcal{B}) \le -\inf_{\psi\in\mathcal{B}} \mathbb{I}(\psi)+\varepsilon.$$

Hence for large n,

$$\begin{aligned}\mathbb{E}[e^{b_n^2[H_n\wedge\Delta_1]}] &\le \sum_{k=1}^N e^{b_n^2[\int_0^T h(\varphi^k(s))ds+g(\varphi^k(T))+\omega(\varepsilon)-\inf_{\psi\in\bar{A}_{vk}}\mathbb{I}(\psi)+\frac{\varepsilon}{2}]}+e^{b_n^2[\Delta_1-\inf_{\psi\in\mathcal{B}}\mathbb{I}(\psi)+\varepsilon]}\\ &\le \sum_{k=1}^N e^{b_n^2[\int_0^T h(\varphi^k(s))ds+g(\varphi^k(T))-\mathbb{I}(\bar{\psi}^k)+\omega(\varepsilon)+\varepsilon]}+e^{b_n^2[\Delta_1-\Delta+\varepsilon]},\end{aligned}$$

where for the first term on the r.h.s. we used (49) and for the second term we used the fact $\mathcal{B}\subset\mathcal{Q}^c$ and the definition of $\mathcal{Q}$.

The last term on (73) is bounded by

$$\mathbb{E}[e^{b_n^2(c_{11}\tilde{A}_n+c_{11}+\tilde{A}_n-G(\Delta_1))}].$$

From Condition 2.3, there exists a constant c_{12} such that for all large n,

$$\frac{1}{b_n^2}\log\mathbb{E}[e^{b_n^2(c_{11}+1)\tilde{A}_n}]<c_{12}.$$

Therefore from (73) we obtain

$$\begin{aligned}&\limsup \frac{1}{b_n^2}\log\mathbb{E}[e^{b_n^2 H_n}]\\ &\le \max_{1\le k\le N}\Big[\int_0^T h(\varphi^k(s))ds+g(\varphi^k(T))-\mathbb{I}(\bar{\psi}^k)+\omega(\varepsilon)+\varepsilon\Big]\\ &\qquad\vee[\Delta_1-\Delta+\varepsilon]\vee[c_{11}+c_{12}-G(\Delta_1)]\\ &\le \sup_{\psi\in P}[c(\psi,\alpha_\theta[\psi])+\omega(\varepsilon)+\varepsilon]\vee[\Delta_1-\Delta+\varepsilon]\vee[c_{11}+c_{12}-G(\Delta_1)].\end{aligned}$$

Now let $\varepsilon\to 0$ first, then $\Delta\to\infty$, recalling that c_{11}, c_{12} and G do not depend on Δ. Finally let $\Delta_1\to\infty$, so $G(\Delta_1)\to\infty$, to obtain

$$\limsup V_n(\tilde{X}_n(0))\le \limsup\frac{1}{b_n^2}\log\mathbb{E}[e^{b_n^2 H_n}]\le \sup_{\psi\in P} c(\psi,\alpha_\theta[\psi])=V(x),$$

where for the first inequality we used (55) and for the equality we used (25). This completes the proof. $\square$

5 The linear case and asymptotic optimality

Section 4.2 describes a policy for the queueing control problem, that is asymptotically optimal. While the construction of this policy and its analysis facilitate the proof of the main result, they fail to provide a simple, closed-form asymptotically optimal policy. In this section we focus on cost with either h linear and $g = 0$ or g linear and $h = 0$, aiming at a simple control policy.

More precisely, the assumption on the functions h and g is that

$$h(x) = \sum_{i=1}^{I} c^i x_i, \qquad g(x) = \sum_{i=1}^{I} d^i x_i,$$

where c^i and d^i are nonnegative constants, and, in addition,

$$c^1\mu^1 \geq c^2\mu^2 \geq \cdots \geq c^I\mu^I \quad \text{and} \quad d^1\mu^1 \geq d^2\mu^2 \geq \cdots \geq d^I\mu^I.$$

We consider the so-called $c\mu$ rule, namely the policy that prioritizes according to the ordering of the class labels, with highest priority to class 1. Let us construct this policy rigorously. Consider the set of equations

$$B_n^1(t) = \chi_{\{X_n^i(t)>0\}}, B_n^2(t) = \chi_{\{X_n^1(t)=0, X_n^2(t)>0\}}, \ldots, B_n^I(t) = \chi_{\{X_n^1(t)=0,\ldots,X_n^{I-1}(t)=0, X_n^I(t)>0\}}. \tag{74}$$

Arguing as in Section 4.2, considering (74) along with the model equations (3)–(5), it is easy to see that there exists a unique solution, this solution is used to define the processes X_n, D_n, T_n, B_n, and moreover B_n is an admissible policy.

The result below states that the policy is asymptotically optimal.

Theorem 5.1 *Assume Conditions 2.1, 2.3 hold. Then, under the priority policy* $\{B_n\}$ *of* (74)*,*

$$\lim_{n\to\infty} J^n(\tilde{X}_n(0), B_n) = V(x).$$

Proof: By Example 2.1, Condition 2.2 holds. As a result, the lower bound, namely Theorem 4.1, is valid. It therefore suffices to prove that $\limsup_{n\to\infty} J^n(\tilde{X}_n(0), B_n) \leq V(x)$. The general strategy of the proof of Theorem 4.2 is repeated here; the details of proving the main estimates are, of course, different.

Thus, given constants $\varDelta$ and ε we consider $\mathcal{Q}$ (46), M, the constants $\delta_1, \delta, \eta, v_0, v_2$, the members $\bar{\psi}^k$ of $\mathcal{Q}$, the sets $\mathcal{A}^k = \mathcal{A}_{v^k}(\bar{\psi}^k)$ and the events $\varOmega_n^k$ precisely as in the proof of Theorem 4.2. We also set $\varphi^k = f(\varphi_\theta[\bar{\psi}^k])$ as in that proof.

In what follows, c_1, $c_2, \ldots$ denote constants independent of $\varDelta$, ε, δ_1, δ, η, v_0, v_2 and n. Analogously to (56) and (57), we aim at proving that there exists a constant c_1, such that for all sufficiently large n,

$$\|\tilde{X}_n\|_T^* \leq c_1(1 + \tilde{\varLambda}_n), \tag{75}$$

(where, as before, $\tilde{\varLambda}_n = \varLambda_T(\tilde{A}_n, \tilde{S}_n)$), and

$$\sup_{[v_2,T]} \|\tilde{X}_n - \varphi^k\| \leq c_1\varepsilon, \qquad \text{on } \varOmega_n^k,\ k = 1, 2, \ldots, N. \tag{76}$$

Once these estimates are established, the proof can be completed exactly as in Step 5 of the proof of Theorem 4.2. We therefore turn to proving (75) and (76).

Recall that $\theta_n = (\frac{n}{\mu_n^1}, \frac{n}{\mu_n^2}, \ldots, \frac{n}{\mu_n^I})$. Therefore by (37),

$$\theta_n \cdot Z_n(t) = \frac{\sqrt{n}}{b_n}\Big(t - \int_0^t \sum_{i=1}^{I} B_n^i(s)ds\Big) = \frac{\sqrt{n}}{b_n}\int_0^t \chi_{\{\theta_n \cdot \tilde{X}_n(s)=0\}} ds,$$

where we used (74), by which $\sum B_n^i = 0 \iff$ for all i, $X_n^i = 0 \iff \theta_n \cdot \tilde{X}_n = 0$. Hence from (8), with

$$Y_n^{\#,i}(t) = \tilde{X}_n^i(0) + y_n^i t + \tilde{A}_n^i(t) - \tilde{S}_n^i(T_n^i(t)), \tag{77}$$

we have

$$\theta_n \cdot \tilde{X}_n(t) = \theta_n \cdot Y_n^{\#} + \frac{\sqrt{n}}{b_n} \int_0^t \chi_{\{\theta_n \cdot \tilde{X}_n(s)=0\}} ds. \tag{78}$$

Since $\theta_n \cdot \tilde{X}_n$ is nonnegative, and $\theta_n \cdot Z_n$ increases only when $\theta_n \cdot \tilde{X}_n$ vanishes, it follows that $(\theta_n \cdot \tilde{X}_n, \theta_n \cdot Z_n)$ solve the Skorohod problem for $\theta_n \cdot Y_n^{\#}$. As a result,

$$|\theta_n \cdot \tilde{X}_n|_T^* + |\theta_n \cdot Z_n|_T^* \le 4|\theta_n \cdot Y_n^{\#}|_T^*.$$

Also, using (8), the non-negativity of $\tilde{X}_n^i$ implies

$$Z_n^i(t) \ge -Y_n^{\#,i}(t).$$

Since $\theta_n \to \theta, y_n^i \to y_i, \tilde{X}_n(0) \to x$, it follows that there exists a constant c_1 such that for all n, (75) holds, as well as

$$\|Z_n\|_T^* \le c_1(1 + \tilde{\Lambda}_n). \tag{79}$$

Toward proving (76), let us compute the paths φ^k. By Example 2.1, the corresponding minimizing curve is given by $f(w) = (0, \dots, 0, w\mu^I)$, $w \ge 0$. Recall the notation $\bar{R}$ (18), and let $\hat{\psi}^k = \bar{R}[\bar{\psi}^k]$, that is,

$$\hat{\psi}^k(t) = x + yt + \bar{\psi}^{k,1}(t) - R[\bar{\psi}^{k,2}](t).$$

Then $\varphi^k = f(\varphi_\theta[\bar{\psi}^k]) = f(\varGamma[\theta \cdot \hat{\psi}^k])$. Thus

$$\varphi_i^k = \begin{cases} 0, & \text{if } i = 1, 2, \dots, I-1, \\ \\ \mu^I \varGamma[\theta \cdot \hat{\psi}^k], & \text{if } i = I. \end{cases} \tag{80}$$

Define $\mathcal{I}' = \{1, 2, \dots, I-1\}$ and $\rho' = \sum_{i=1}^{I-1} \rho^i$. Then by (8) and (9),

$$\begin{aligned} \tilde{X}_n'(t) := \sum_{i \in \mathcal{I}'} \theta_n^i \tilde{X}_n^i(t) &= \sum_{i \in \mathcal{I}'} \theta_n^i Y_n^{\#,i}(t) + \frac{\sqrt{n}}{b_n} \sum_{i \in \mathcal{I}'} (\rho^i t - T_n^i(t)) \\ &= U_n(t) + \frac{\sqrt{n}}{b_n} \int_0^t \chi_{\{\tilde{X}_n'(s)=0\}} ds, \end{aligned}$$

where

$$U_n(t) = \sum_{i \in \mathcal{I}'} \theta_n^i Y_n^{\#,i}(t) + \frac{\sqrt{n}}{b_n}(\rho' - 1)t,$$

and we used (74) by which $\sum_{\mathcal{I}'} B_n^i = 0 \iff X_n^i = 0$ for all $i \in \mathcal{I}'$. Hence, invoking again the Skorohod map,

$$\tilde{X}_n'(t) = U_n(t) + \sup_{[0,t]} \{-U_n \vee 0\}. \tag{81}$$

We will argue that, on $\Omega_n := \cup_k \Omega_n^k$, for all sufficiently large n,

$$\sup_{[v_2,T]} |\tilde{X}_n'| \le c_2\varepsilon. \tag{82}$$

To this end, let us fisrt show that, for all sufficiently large n, the following holds: On Ω_n, $U_n(t_2) \le U_n(t_1)$ whenever $t_1, t_2 \in [0,T]$ are such that $t_2 - t_1 \ge v_2$. Suppose this claim is false. Then there are infinitely many n for which there exist (n-dependent) $t_1, t_2 \in [0,T]$ with $t_2 - t_1 \ge v_2$ but $U_n(t_2) > U_n(t_1)$ on Ω_n. Thus

$$\begin{aligned}
&\sum_{i\in\mathcal{I}'} \theta_n^i[\tilde{X}_n^i(0) + y_n^i t_1 + \tilde{A}_n^i(t_1) - \tilde{S}_n^i(T_n^i(t_1))] \\
&- \sum_{i\in\mathcal{I}'} \theta_n^i[\tilde{X}_n^i(0) + y_n^i t_2 + \tilde{A}_n^i(t_2) - \tilde{S}_n^i(T_n^i(t_2))] \\
&< \frac{\sqrt{n}}{b_n}(\rho' - 1)(t_2 - t_1) \le \frac{\sqrt{n}}{b_n}(\rho' - 1)v_2.
\end{aligned}$$

However, this is a contradiction because the r.h.s. tends to $-\infty$ as $n \to \infty$ whereas the l.h.s. remains bounded. This proves the claim.

Next, note that, for a similar reason, for all sufficiently large n, $U_n(t) < 0$ on Ω_n, for $t \ge v_2$. Hence for $t \ge v_2$ and n large, we have on Ω_n,

$$\sup_{[0,t]}\{-U_n \vee 0\} = \sup_{[0,t]}\{-U_n\} = \sup_{[t-v_2,t]}\{-U_n\}.$$

Thus using (81), on Ω_n, we have for all n large and $t \ge v_0$,

$$\begin{aligned}
\tilde{X}_n'(t) &= U_n(t) + \sup_{[t-v_2,t]}\{-U_n\} \\
&\le \sum_{i\in\mathcal{I}'} \theta_n^i Y_n^{\#,i}(t) + \frac{\sqrt{n}}{b_n}(\rho' - 1)t + \sup_{[t-v_2,t]}\Big[-\sum_{i\in\mathcal{I}'} \theta_n^i Y_n^{\#,i}(s) - \frac{\sqrt{n}}{b_n}(\rho' - 1)s\Big] \\
&\le \sum_{i\in\mathcal{I}'} \theta_n^i Y_n^{\#,i}(t) + \sup_{[t-v_2,t]}\Big[-\sum_{i\in\mathcal{I}'} \theta_n^i Y_n^{\#,i}(s)\Big] \\
&\le c_3\varepsilon + c_3[\mathrm{osc}_{v_2}(\tilde{A}_n) + \mathrm{osc}_{v_2}(\tilde{S}_n)],
\end{aligned} \tag{83}$$

where we used (77) and the fact that T_n^i are Lipschitz with constant 1. On Ω_n^k,

$$\mathrm{osc}_{v_2}(\tilde{A}_n) \le 2\|\tilde{A}_n - \bar{\psi}^{k,1}\|^* + \mathrm{osc}_{v_2}(\bar{\psi}^{k,1}) \le 3\varepsilon, \tag{84}$$

where we used (48) and (47). Similarly, $\mathrm{osc}_{v_2}(\tilde{S}_n) \le 3\varepsilon$. Using this in (83) gives (82).

Next, recall that $\theta_n \cdot \tilde{X}_n = \Gamma[\theta_n \cdot Y_n^{\#}]$. Note by (80) that $\theta \cdot \varphi^k = \Gamma[\theta \cdot \hat{\psi}^k]$. Therefore using the Lipschitz property of Γ we have, for all sufficiently large n,

$$\begin{aligned}
|\theta_n \cdot \tilde{X}_n - \theta \cdot \varphi^k|_T^* &\le 2|\theta_n \cdot Y_n^{\#} - \theta_n \cdot \hat{\psi}^k|_T^* + 2\|\theta_n - \theta\|\|\hat{\psi}^k\|_T^* \\
&\le c_4\|Y_n^{\#} - \hat{\psi}^k\|_T^* + \varepsilon \\
&\le c_4 \sum_i \{|\tilde{A}_n^i - \bar{\psi}^{k,1,i}|_T^* + |\tilde{S}_n^i \circ T_n^i - R[\bar{\psi}^{k,2,i}]|_T^*\} + 2\varepsilon.
\end{aligned} \tag{85}$$

Now, on Ω_n^k, $\|\tilde{A}_n - \bar{\psi}^{k,1}\| \le \varepsilon$ and $\|\tilde{S}_n - R[\bar{\psi}^{k,2}]\| \le \varepsilon$. Moreover, from (79),

$$\sup_{[0,T]} |(\rho^i t - T_n^i(t))| \le v_2,$$

on Ω_n, for all sufficiently large n. It follows that, on Ω_n^k, for all sufficiently large n,

$$|\theta_n \cdot \tilde{X}_n - \theta \cdot \varphi^k|_T^* \le c_5\varepsilon + \mathrm{osc}_{v_0}(\bar{\psi}^{k,2}) \le c_6\varepsilon, \tag{86}$$

where the last inequality follows from (47).

Now, by (82) and the fact that $\varphi_i^k = 0$ for $i < I$ (80), we have $\sup_{[v_2,T]} |\tilde{X}_n^i - \varphi_i^k| \le c_7\varepsilon$ for $i < I$, on Ω_n^k for large n. Combining this with (86), the convergence $\theta_n \to \theta$ and the fact that the I vectors θ and e_i, $i < I$ are linearly independent, gives $\sup_{[v_2,T]} \|\tilde{X}_n - \varphi^k\| \le c_8\varepsilon$, on Ω_n^k, for all sufficiently large n. This proves (76) and completes the proof of the result. □

A Appendix

Proof of Proposition 2.1: We borrow some ideas from the proof of Lemma A.1 in [21]. Clearly, the statements regarding $\tilde{A}_n$ and $\tilde{S}_n$ are identical, hence it suffices to consider only the former. Define $M_A^i(u) = \mathbb{E}[e^{u(IA^i)}]$ for $u \in \mathbb{R}$. It suffices to prove that for any positive $K > 0$ and $i \in \mathcal{I}$,

$$\limsup \frac{1}{b_n^2} \log \mathbb{E}[e^{b_n^2(K|\tilde{A}_n^i|^*)}] \quad < \quad \infty.$$

Assume $i = 1$. Since $M_A^1(u) = \mathbb{E}[e^{uIA^1}]$ is finite around 0, it is C^2 there, and so is $H_A^1(u) := \log M_A^1(u)$. Therefore by Taylor expansion there exist $\gamma, \delta > 0$ such that

$$|H_A^1(u) - u| \le \gamma u^2, \text{ for all } u \text{ with } |u| \le \delta. \tag{87}$$

Here we have used the fact that $\frac{dM_A^1}{du}(0) = \mathbb{E}[IA^1] = 1$. Note that

$$\begin{aligned}&\mathbb{E}[e^{b_n^2(K|\tilde{A}_n^1|^*)}] \\ &\quad = 1 + b_n^2 K \int_0^\infty e^{b_n^2 Kt} \mathbb{P}(|\tilde{A}_n^1|^* > t)dt \le 1 + b_n^2 K e^{Kb_n^2} + b_n^2 K \int_1^\infty e^{b_n^2 Kt} \mathbb{P}(|\tilde{A}_n^1|^* > t)dt.\end{aligned}$$

For $t \ge 1$,

$$\begin{aligned}\mathbb{P}(|\tilde{A}_n^i|^* > t) &= \mathbb{P}(\exists v \in [0,T] \text{ such that } |\tilde{A}_n^1(v)| > t) \\ &\le \mathbb{P}(\exists v \in [0,T] \text{ such that } \tilde{A}_n^1(v) < -t) + \mathbb{P}(\exists v \in [0,T] \text{ such that } \tilde{A}_n^1(v) > t).\end{aligned}$$

Now

$$\begin{aligned}\tilde{A}_n^1(v) > t &\Leftrightarrow A_n^1(v) > b_n\sqrt{n}t + \lambda_n^1 v, \\ \tilde{A}_n^1(v) < -t &\Leftrightarrow A_n^1(v) < -b_n\sqrt{n}t + \lambda_n^1 v.\end{aligned}$$

Let $\lfloor x \rfloor$ denote the largest integer less than or equal to x. Also assume $-b_n\sqrt{n}t + \lambda_n^1 T > 0$. Then

$$\begin{aligned}
& \mathbb{P}(\exists v \in [0,T] \text{ such that } \tilde{A}_n^1(v) < -t) \\
= \;& \mathbb{P}(\exists v \in [0,T] \text{ such that } A_n^1(v) < -b_n\sqrt{n}t + \lambda_n^1 v) \\
\leq \;& \mathbb{P}(\exists v \in [0,T] \text{ such that } \sum_{l=1}^{\lfloor -b_n\sqrt{n}t+\lambda_n^1 v+1 \rfloor} IA^1(l) > \lambda_n^1 v) \\
\leq \;& \mathbb{P}(\exists v \in [0,T] \text{ such that } \sum_{l=1}^{\lfloor -b_n\sqrt{n}t+\lambda_n^1 v+1 \rfloor} (IA^1(l)-1) > \lambda_n^1 v - \lfloor -b_n\sqrt{n}t + \lambda_n^1 v + 1 \rfloor) \\
\leq \;& \mathbb{P}(\exists v \in [0,T] \text{ such that } \sum_{l=1}^{\lfloor -b_n\sqrt{n}t+\lambda_n^1 v+1 \rfloor} (IA^1(l)-1) > b_n\sqrt{n}t - 1).
\end{aligned}$$

We define $V_k = \sum_{l=1}^{k}(IA^1(l)-1)$. Then $\{V_k\}$ is a martingale w.r.t. the filtration generated by $\{IA^1(l)\}$. For all large n, $b_n\sqrt{n}t - 1 > 0$ for all $t \geq 1$. Denote $L_n = \lfloor -b_n\sqrt{n}t + \lambda_n^1 T + 1 \rfloor$. Hence

$$\begin{aligned}
\mathbb{P}(\exists v \in [0,T] \text{ such that } \tilde{A}_n^1(v) < -t) \;&\leq\; \mathbb{P}(\sup_{1\leq k\leq L_n} |V_k| > b_n\sqrt{n}t - 1) \\
&\leq\; e^{-\beta_n(b_n\sqrt{n}t-1)}\mathbb{E}[\sup_{1\leq k\leq L_n} e^{\beta_n|V_k|}],
\end{aligned}$$

where $\beta_n > 0$ is a constant. We note that $\{e^{\beta_n|V_k|}\}_k$ is a sub-martingale. Hence by Doob's martingale inequality

$$\mathbb{E}[\sup_{1\leq k\leq L_n} e^{\beta_n|V_k|}] \leq \mathbb{E}[\sup_{1\leq k\leq L_n} e^{2\beta_n|V_k|}]^{\frac{1}{2}} \leq 2\mathbb{E}[e^{2\beta_n|V_{L_n}|}]^{\frac{1}{2}}.$$

Thus

$$\begin{aligned}
&\mathbb{P}(\exists v \in [0,T] \text{ such that } \tilde{A}_n^1(v) < -t) \\
&\qquad \leq 2e^{-\beta_n(b_n\sqrt{n}t-1)}\mathbb{E}[e^{2\beta_n|V_{L_n}|}]^{\frac{1}{2}} \\
&\qquad \leq 2e^{-\beta_n(b_n\sqrt{n}t-1)}[\mathbb{E}[e^{2\beta_n V_{L_n}}] + \mathbb{E}[e^{-2\beta_n V_{L_n}}]]^{\frac{1}{2}} \\
&\qquad \leq 2e^{-\beta_n(b_n\sqrt{n}t-1)}[e^{L_n(H_A^1(2\beta_n)-2\beta_n)} + e^{L_n(H_A^1(-2\beta_n)+2\beta_n)}]^{\frac{1}{2}}.
\end{aligned}$$

If $2\beta_n \leq \delta$ and n is large enough so that $\frac{b_n\sqrt{n}t}{2} - 1 > 0$ holds then using (87) we have

$$\begin{aligned}
\mathbb{P}(\exists v \in [0,T] \text{ such that } \tilde{A}_n^1(v) < -t) \;&\leq\; 2\sqrt{2}e^{-\beta_n\frac{b_n\sqrt{n}t}{2}}e^{2L_n\gamma\beta_n^2} \\
&\leq\; 2\sqrt{2}e^{-\beta_n\frac{b_n\sqrt{n}t}{2}}e^{2(-b_n\sqrt{n}t+\lambda_n^1 T+1)\gamma\beta_n^2}.
\end{aligned}$$

Now we choose $\beta_n = \frac{b_n}{\sqrt{n}}(2K+2)$ and we choose n_1 such that for $n \geq n_1$, $2\beta_n \leq \delta$. Hence

$$\mathbb{P}(\exists v \in [0,T] \text{ such that } \tilde{A}_n^1(v) < -t) \leq 2\sqrt{2}e^{b_n^2 8\frac{\lambda_n^1 T+1}{n}\gamma(K+1)^2}e^{-b_n^2(K+1)t}. \tag{88}$$

In a similar way we obtain n_2 such that for all $n \geq n_2$

$$\mathbb{P}(\exists v \in [0,T] \text{ such that } \tilde{A}_n^1(v) > t) \leq 2\sqrt{2} e^{b_n^2 8 \frac{\lambda_n^1 T}{n} \gamma (K+2)^2} e^{-b_n^2 (K+1)t}. \tag{89}$$

Thus from (88) and (89) we have constants n_3, γ_1, γ_2 such that for all $n \geq n_3$

$$\mathbb{P}(|\tilde{A}_n^i|^* > t) \leq \gamma_1 e^{b_n^2 \gamma_2} e^{-b_n^2 (K+1)t}.$$

Hence for $n \geq n_3$,

$$\int_1^\infty e^{b_n^2 K t} \mathbb{P}(|\tilde{A}_n^1|^* > t) dt \leq \gamma_1 e^{b_n^2 \gamma_2} \int_1^\infty e^{-b_n^2 t} dt = \frac{1}{b_n^2} \gamma_1 e^{b_n^2 (\gamma_2 - 1)},$$

and

$$\mathbb{E}[e^{b_n^2 (K |\tilde{A}_n^1|^*)}] \leq 1 + b_n^2 K e^{K b_n^2} + K \gamma_1 e^{b_n^2 (\gamma_2 - 1)} \leq 3 \max(1, b_n^2 K e^{K b_n^2}, K \gamma_1 e^{b_n^2 (\gamma_2 - 1)}).$$

This gives the required estimate and completes the proof. □

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