# AN OPTIMAL INEQUALITY FOR THE TANGENT FUNCTION

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Abstract. In this note we deal with some inequalities for the tangent function that are valid for $x$ in $(-\pi/2, \pi/2)$. These inequalities are optimal in the sense that the best values of the exponents involved are obtained.

## 1. Introduction

The story started when I wanted to provide my students of "Basic Calculus" class, with a way to prove that

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} = \frac{1}{3} \tag{1}$$

without recourse to any advanced topics or to the L'Hôpital's rule. So, I came up with the following proposition :

**Proposition.** *For every $x \in (0, \pi/2)$ the following inequality holds:*

$$x + \frac{x^3}{3} < \tan x < x + \frac{\tan^3 x}{3} \tag{2}$$

Clearly, the limit in (1) follows easily from this Proposition. But this was not the end of the story, it was just the beginning of my investigation. In fact, the inequality (2) means that $3(\tan x - x)$ is somewhere between $x^3$ and $\tan^3 x$, but where exactly ?

In order to describe our results, an important role is played by the family of functions $(f_\gamma)_{\gamma\in[0,3]}$ defined on $[0, \pi/2)$ by

$$f_\gamma(x) = x^{3-\gamma}\tan^\gamma x. \tag{3}$$

Because of the well-known inequality $\tan x \geq x$ for $0 \leq x < \pi/2$, we see that the family $(f_\gamma)_{\gamma\in[0,3]}$ is increasing in the sense that $f_\alpha \leq f_\beta$ for $\alpha < \beta$. Using this family, we can reformulate (2) by saying that

$$f_0(x) < 3(\tan x - x) < f_3(x), \qquad \text{for } 0 < x < \tfrac{\pi}{2},$$

So, it is natural to be interested in identifying the best $\alpha$ and $\beta$ such that $f_\alpha(x) < 3(\tan x - x) < f_\beta(x)$ for $0 < x < \pi/2$. We were able to completely answer this question, our results are summarized in the following two statements :

2010 *Mathematics Subject Classification.* 26D05.
*Key words and phrases.* inequality, tangent function, power series expansion.
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**Proposition.** *If for every $x \in (0, \pi/2)$ we have $f_\alpha(x) \le 3(\tan x - x) \le f_\beta(x)$, where $f_\gamma$ is defined in* (3)*, then $\alpha \le 1$ and $\beta \ge 6/5$.*

**Main Theorem.** *The following two inequalities hold :*
(a) *For every $x \in (0, \pi/2)$ we have $f_1(x) < 3(\tan x - x)$,*
(b) *For every $x \in (0, \pi/2)$ we have $3(\tan x - x) < f_{6/5}(x)$.*
*where $f_\gamma$ is defined in* (3)*. Equivalently,*

$$\forall\, x \in \left(0, \frac{\pi}{2}\right), \qquad x + \frac{1}{3}x^2 \tan x < \tan x < x + \frac{1}{3}x^{9/5}\tan^{6/5} x. \tag{4}$$

Before we embark in the proof of our results, it is worth mentioning that there is a lot of similar inequalities involving trigonometric functions in the literature [1, 2, 3, 4]. For instance, the Becker-Stark's inequality [1] states that

$$\frac{8x}{\pi^2 - 4x^2} < \tan x < \frac{\pi^2 x}{\pi^2 - 4x^2}, \qquad \text{for } 0 < x < \tfrac{\pi}{2}.$$

Also, in [4] the authors prove, among other things, that for $0 < x < \frac{\pi}{2}$, one has

$$x + \frac{x^3}{3} + \frac{2}{15}x^4 \tan x < \tan x < x + \frac{x^3}{3} + \left(\frac{2}{\pi}\right)^4 x^4 \tan x. \tag{5}$$

Numerical evidence shows that the upper inequality in (4) is sharper than the upper inequality in (5) for $x \in (0, x_0)$ where $x_0 \approx 1.2332$, and that the lower inequality in (4) is sharper than the lower inequality in (5) for $x \in (x_1, \pi/2)$ where $x_1 \approx 1.5255$. So the two results are complementary but not comparable.

## 2. Results and Proofs

Clearly, the next Proposition 1 follows from our main Theorem 4, but it can be elementarily proved directly, our aim from presenting the proof is just to compare the degree of difficulty.

**Proposition 1.** *For every $x \in (0, \pi/2)$ the following inequality holds:*

$$x + \frac{x^3}{3} < \tan x < x + \frac{\tan^3 x}{3}$$

*Proof.* Indeed, let $g$ and $h$ be the functions defined on $[0, \pi/2)$ by

$$g(x) = \tan x - x - \frac{x^3}{3},$$

$$h(x) = \frac{\tan^3 x}{3} + x - \tan x.$$

Clearly, for $x \in (0, \pi/2)$, we have $g'(x) = \tan^2 x - x^2 > 0$ and $h'(x) = \tan^4 x > 0$. Thus, both $g$ and $h$ are monotonous increasing on the interval $(0, \pi/2)$, and the desired inequality follows since $g(0) = h(0) = 0$. ☐

**Proposition 2.** *If for some* $0 \leq \alpha, \beta \leq 3$*, we have*

$$\left(\frac{\tan x}{x}\right)^{\alpha} \leq \frac{3(\tan x - x)}{x^3} \leq \left(\frac{\tan x}{x}\right)^{\beta}$$

*for every* $x \in (0, \pi/2)$*, then* $\alpha \leq 1$ *and* $\beta \geq 6/5$*.*

*Proof.* Suppose that for $x \in (0, \pi/2)$ we have

$$\left(\frac{\tan x}{x}\right)^{\alpha} \leq \frac{3(\tan x - x)}{x^3} \leq \left(\frac{\tan x}{x}\right)^{\beta},$$

that is $\alpha \leq \varphi(x) \leq \beta$ where $\varphi$ is defined on $(0, \pi/2)$ by

$$\varphi(x) = \log\left(\frac{3(\tan x - x)}{x^3}\right) \Big/ \log\left(\frac{\tan x}{x}\right).$$

Now, since

$$\varphi(x) = \frac{\log(\tan x) + \log(1 - x/\tan x) + \log 3 - 3\log(x)}{\log(\tan x) - \log x}$$

we conclude that

$$\lim_{x \to \left(\frac{\pi}{2}\right)^-} \varphi(x) = 1. \tag{6}$$

On the other hand, since in the neighborhood of 0 we have

$$\frac{\tan x}{x} = 1 + \frac{x^2}{3} + \frac{2}{15}x^4 + O(x^6),$$

we deduce that

$$\begin{aligned} \log\left(\frac{\tan x}{x}\right) &= \frac{x^2}{3} + O(x^4) \\ \log\left(\frac{3(\tan x - x)}{x^3}\right) &= \log\left(1 + \frac{2}{5}x^2 + O(x^4)\right) \\ &= \frac{2}{5}x^2 + O(x^4). \end{aligned}$$

Thus, $\varphi(x) = \frac{6}{5} + O(x^2)$, and consequently

$$\lim_{x \to 0^+} \varphi(x) = \frac{6}{5}. \tag{7}$$

Therefore, (6) and (7), together with the fact that $\alpha \leq \varphi(x) \leq \beta$ for every $x \in (0, \pi/2)$, imply that $\alpha \leq 1$ and $\beta \geq \frac{6}{5}$ as desired. □

Before we come to the proof of our main theorem, we will need the following technical lemma.

**Lemma 3.** *Let $\varphi$ be the function defined on $\mathbb{R}$ by*
$$\varphi(x) = (9 - 24x^2)\cos(x) - 9\cos(3x) - 4x\sin(3x). \tag{8}$$
*Then $\varphi(x) > 0$ for $0 < x \leq \pi/2$.*

*Proof.* In order to determine the sign of $\varphi(x)$ for $x \in (0, \pi/2]$, we will use power series expansion. Clearly, for every real $x$ we have
$$\begin{aligned}\varphi(x) &= (9-24x^2)\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} - 9\sum_{n=0}^{\infty}\frac{(-1)^n 3^{2n}x^{2n}}{(2n)!} - 4x\sum_{n=1}^{\infty}\frac{(-1)^{n-1}3^{2n-1}x^{2n-1}}{(2n-1)!}\\ &= \sum_{n=0}^{\infty}\big(9 + 24(2n)(2n-1) - 9\cdot 3^{2n} + 4(2n)\cdot 3^{2n-1}\big)\frac{(-1)^n x^{2n}}{(2n)!}\\ &= 3\sum_{n=0}^{\infty}\big(32n^2 - 16n + 3) + (8n-27)\cdot 3^{2n-2}\big)\frac{(-1)^n x^{2n}}{(2n)!}\end{aligned}$$
Thus, for a real $x$ we have
$$\varphi(x) = 3\sum_{n=0}^{\infty}(-1)^n\frac{T_n}{(2n)!}x^{2n}, \tag{9}$$
where,
$$T_n = 2(4n-1)^2 + 1 + (8n-27)9^{n-1}. \tag{10}$$
Noting that $T_0 = T_1 = T_2 = T_3 = 0$ we conclude that (9) can be written as follows
$$\varphi(x) = 3\sum_{n=4}^{\infty}(-1)^n\frac{T_n}{(2n)!}x^{2n}. \tag{11}$$

We recognize an alternating series since it is clear from (10) that $T_n > 0$ for $n \geq 4$. Now, if we show that the sequence $\left(\frac{T_n}{(2n)!}x^{2n}\right)_{n\geq 4}$ is decreasing for any $x \in (0, \pi/2]$ then this would imply that $\varphi(x) > 0$ for $x \in (0, \pi/2]$, because the first term in the series (11) is positive.

Let $U_n$ be defined by,
$$U_n = (2n+2)(2n+1)T_n - 3T_{n+1}. \tag{12}$$
a simple calculation shows that
$$\begin{aligned}U_n &= (4n^2 + 6n + 2)\big(32n^2 - 16n + 3 + (8n-27)9^{n-1}\big)\\ &\qquad - 3(32n^2 + 48n + 19 + (8n-19)9^n)\\ &= 128n^4 + 128n^3 - 116n^2 - 158n - 51 + (32n^3 - 60n^2 - 362n + 459)9^{n-1}\\ &= B_n + A_n\cdot 9^{n-1} \end{aligned}\tag{13}$$

where

$$\begin{aligned} B_n &= 128n^4 + 128n^3 - 116n^2 - 158n - 51 \\ A_n &= 32n^3 - 60n^2 - 362n + 459 \end{aligned}$$

Now, it is straightforward to check that

$$\begin{aligned} B_{n+1} &= 128n^4 + 640n^3 + 1036n^2 + 437n + 69(n-1), \\ A_{n+4} &= 32n^3 + 324n^2 + 694n + 99. \end{aligned}$$

Thus, $A_n$ and $B_n$ are positive for $n \geq 4$, and according to (13) we have $U_n > 0$ for $n \geq 4$. Using (12) we conclude that for $n \geq 4$ and $x \in (0, \sqrt{3}]$ we have

$$(2n+2)(2n+1)T_n > x^2 T_{n+1}$$

or, equivalently,

$$\forall\, n \geq 4, \quad \forall\, x \in \left(0, \sqrt{3}\right], \quad \frac{T_n}{(2n)!}x^{2n} > \frac{T_{n+1}}{(2n+2)!}x^{2n+2}.$$

It follows that the sequence $\left(\frac{T_n}{(2n)!}x^{2n}\right)_{n\geq 4}$ is decreasing for any $x \in (0, \sqrt{3}]$, and, as we have already explained, this implies using (11) that $\varphi(x) > 0$ for $x \in \left(0, \sqrt{3}\right]$, and the Lemma follows since $\frac{\pi}{2} < \sqrt{3}$. □

With this technical lemma at hand, we can prove our Main Theorem.

**Theorem 4.** *The following two inequalities hold :*
(a) *For every* $x \in (0, \pi/2)$ *we have* $x^2 \tan x < 3(\tan x - x)$*,*
(b) *For every* $x \in (0, \pi/2)$ *we have* $3(\tan x - x) < x^{9/5}(\tan x)^{6/5}$*.*

*Proof.* **(a)** Consider the function $g$ defined on the interval $(0, \pi/2)$ by

$$g(x) = 3 - x^2 - 3x \cot x \tag{14}$$

Clearly we have

$$g'(x) = x - 3\cot x + 3x\cot^2 x = (1 + 3\cot^2 x)h(x) \tag{15}$$

where $h(x) = x - \dfrac{3\tan x}{3 + \tan^2 x}$. Similarly $h$ has a derivative on $[0, \pi/2)$ that is given by

$$\begin{aligned} h'(x) &= 1 - 3\frac{(3 - \tan^2 x)(1 + \tan^2 x)}{(3 + \tan^2 x)^2} \\ &= \frac{4\tan^2 x}{(3 + \tan^2 x)^2} \end{aligned}$$

So, $h$ is monotonous increasing, with $h(0) = 0$. This implies that $h$ is positive on the interval $(0, \pi/2)$. Going back to (15) we conclude that $g$ is also monotonous increasing on $(0, \pi/2)$. Finally, since $\lim_{x\to 0^+} g(x) = 0$, we conclude that $g$ is positive on $(0, \pi/2)$, but it is straightforward to check that this is equivalent to the fact that $3(\tan x - x) > x^2 \tan x$ for $x \in (0, \pi/2)$ which is the desired inequality.

**(b)** This inequality is more delicate to prove. Again, we will consider an auxiliary function. Let $g$ be the function defined on $(0, \pi/2)$ by

$$g(x) = 6 \log\left(\frac{\tan x}{x}\right) - 5 \log\left(\frac{3(\tan x - x)}{x^3}\right). \tag{16}$$

Clearly we have

$$\begin{aligned} g'(x) &= \frac{6}{\cos x\, \sin x} + \frac{9}{x} - \frac{5 \sin^2 x}{\cos x(\sin x - x \cos x)} \\ &= \frac{(9 - 6x^2)\cos x + x(4\sin^3 x - 3 \sin x) - 9 \cos^3 x}{x \cos x\, \sin x\,(\sin x - x \cos x)} \end{aligned}$$

So, recalling the expression of $\cos(3x)$ and $\sin(3x)$ in terms of $\cos x$ and $\sin x$ we see that

$$\begin{aligned} g'(x) &= \frac{(9 - 24x^2)\cos(x) - 9\cos(3x) - 4x \sin(3x)}{4x \cos^2 x\, \sin x\,(\tan x - x)}, \\ &= \frac{\varphi(x)}{4x \cos^2 x\, \sin x\,(\tan x - x)}, \end{aligned} \tag{17}$$

where $\varphi$ is the function considered in Lemma 3. Using the conclusion of that Lemma we see that $g$ is monotonous increasing on $(0, \pi/2)$. But $\lim_{x \to 0^+} g(x) = 0$, so $g$ is positive on $(0, \pi/2)$, and this is equivalent to $3(\tan x - x) < x^{9/5} \tan^{6/5} x$ which is the desired inequality. □

**Corollary 5.** *The necessary and sufficient condition, on the real numbers $\alpha$ and $\beta$, for the following inequality*

$$1 + \frac{x^2}{3}\left(\frac{\tan x}{x}\right)^{\alpha} < \frac{\tan x}{x} < 1 + \frac{x^2}{3}\left(\frac{\tan x}{x}\right)^{\beta}$$

*to hold for every nonzero real $x$ from $(-\pi/2, \pi/2)$, is that $\alpha \le 1$ and $\beta \ge 6/5$.*

*Proof.* This follows from Proposition 2, Theorem 4, and from the fact that the considered functions are even. □

## References

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