Solving Linear Differential Equations: A Novel Approach

[a]N. Gurappa [1], [b]Abhijit Sen [2],
[c]Rajneesh Atre [3] and [b]Prasanta K. Panigrahi[4]
[a]Priyadarshini College of Engineering, Sullurpet-524121, Andhra Pradesh,
[b]Indian Institute of Science Education and Research-Kolkata, Mohanpur,
Nadia-741252, West Bengal, India
[c]Department of Physics, Jaypee Institute of Engineering and Technology,
Guna-473226, India

Abstract

We explicate a procedure to solve general linear differential equations, which connects the desired solutions to monomials x^λ of an appropriate degree λ. In the process the underlying symmetry of the equations under study, as well as that of the solutions are made transparent. We demonstrate the efficacy of the method by showing the common structure of the solution space of a wide variety of differential equations *viz.* Hermite, Laguerre, Jocobi, Bessel and hypergeometric etc. We also illustrate the use of the procedure to develop approximate solutions, as well as in finding solutions of many particle interacting systems.

1 Introduction

As is evident, the basic building blocks of polynomials and functions are the monomials. Superposition of finite and infinite number of these constituents, with appropriate coefficients, respectively leads to various polynomials and functions. Well known classical orthogonal polynomials and functions like Bessel, Airy etc., manifest as solutions of Schrödinger equation in quantum mechanics, in the study of waves, in optics and electromagnetism.

Apart from finding solutions of various differential equations, considerable effort has been put into understanding the symmetry properties of these equations, which throws light on the structure of the solution space [1]. Here, we illustrate a procedure of solving linear differential equations, which leads to an exact mapping between the desired polynomials and functions and the space of monomials [2, 3]. In this method, the Euler operator (EO) $D \equiv x\frac{d}{dx}$ plays a crucial role. It makes essential use of the two properties of the EO.

1. The monomials are the eigenfunctions of the EO:

$$x\frac{d}{dx}x^\lambda \equiv Dx^\lambda = \lambda x^\lambda. \tag{1}$$

2. The Euler operator 'measures' the degree of an operator, irrespective of its constituents,

$$[D, O^d] = dO^d. \tag{2}$$

---

[1]dr.n.gurappa@gmail.com
[2]abhijisen@yahoo.co.in
[3]darvish.rajneesh@gmail.com
[4]pprasanta@iiserkol.ac.in

In the above O^d is an operator of 'degree' d e.g., x^2 and $x^3 \frac{d}{dx}$ have degree 2, while $\frac{d^2}{dx^2}$ and $\frac{d^2}{dx^2} + \frac{\alpha}{x^2}$ have degree -2.

In the following, we start with the familiar Hermite, Laguerre and confluent hypergeometric equations and make use of their symmetry properties to obtain solutions in a form, which exhibits the mapping between the respective polynomials and the space of monomials. We then explicate a more general method, applicable to linear differential equations of arbitrary order and obtain the desired solutions making use of the above mentioned properties of the Euler operator. The procedure is explicated through solution of the oscillator problem. Subsequently, earlier solved examples are connected with the new approach, considering Hermite differential equation as the example. After illustrating the solution space of the hypergeometric equation, we tabulate the solutions and the structure of their respective differential equations for all the functions and polynomials, commonly encountered in physics literature. The case of the so called quasi exactly solvable Schrödinger equation is then studied, for which only a part of the spectrum is analytically obtainable. This example shows the use of the present approach for finding the exact solution, as well as in the development of approximation schemes. We then proceed to many-body interacting Calgero-Sutherland systems and show how our procedure can be profitably explored for finding out the eigenspectra and eigenfunctions. We then conclude with directions for future work.

2   Connecting Polynomials with Monomials

We start with the well studied Hermite differential equation:

$$[-\frac{1}{2}\frac{d^2}{dx^2} + (D - n)]H_n(x) \equiv \hat{O}H_n(x) = 0, \tag{3}$$

which has seen the use of traditional series solution method to the algebraic approach of raising and lowering operators, for finding the solution [4]. It is worthwhile to observe that the above equation has the Euler operator and a degree d=-2 operator $-\frac{1}{2}\frac{d^2}{dx^2} = P$, apart from a harmless constant. The fact that $[D, P] = -2P$ can be profitably used through the Baker-Campbell-Hausdorff (BCH) formula:

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2}[A, [A, B]] + ..., \tag{4}$$

in order to map the differential operator $\hat{O}$ onto (D-n).

For this purpose, it can be straightforwardly checked that

$$e^{\frac{1}{4}\frac{d^2}{dx^2}}[-\frac{1}{2}\frac{d^2}{dx^2} + D - n]e^{-\frac{1}{4}\frac{d^2}{dx^2}} = -\frac{1}{2}\frac{d^2}{dx^2} + D - n + \frac{1}{4}[\frac{d^2}{dx^2}, D] = (D - n). \tag{5}$$

This immidiately suggests to factorize the Hermite polynomial in the form

$$[-\frac{1}{2}\frac{d^2}{dx^2} + D - n]e^{-\frac{1}{4}\frac{d^2}{dx^2}}\varphi(x) = e^{-\frac{1}{4}\frac{d^2}{dx^2}}[D - n]\varphi(x) = 0, \tag{6}$$

which implies,

$$[D - n]\varphi(x) = 0. \tag{7}$$

The fact that the monomials are eigenfunctions of the Euler operator D leads to $\phi(x) = c_n x^n$ and hence the Hermite polynomial becomes, $H_n(x) = c_n e^{-\frac{1}{4}\frac{d^2}{dx^2}} x^n$ [5]. The above is the exact connection between the monomial x^n and the polynomial $H_n(x)$; the value of c_n is to be determined from normalization condition. The discerning reader can visualize that the structure of Laguerre differential equation,

$$[x\frac{d}{dx} - n - (\alpha + 1)\frac{d}{dx} - x\frac{d^2}{dx^2}]L_n^\alpha(x) = 0, \tag{8}$$

and confluent hypergeometric equation,

$$[x\frac{d}{dx} + \alpha - x\frac{d^2}{dx^2} - \gamma\frac{d}{dx}]\Phi(\alpha, \gamma, x) = 0, \tag{9}$$

are similar to the Hermite case. A similar mapping connects the above differential operators to $(D - n)$ and hence the solutions can be written in the form, $L_n^\alpha = c_n exp[-x\frac{d^2}{dx^2} - (\alpha + 1)\frac{d}{dx}]x^n$ and $\Phi(\alpha, \gamma, x) = c_{-\alpha}exp[-x\frac{d^2}{dx^2} - \gamma\frac{d}{dx}]x^{-\alpha}$.

This procedure can be used to identify the generating function and the algebraic structure of the solution space [6]. We refer the interested reader to Ref [2] for more details, as also to Ref [7] for construction of coherent states for the dynamical systems, associated with the harmonic oscillator and the Coulomb problem.

It is evident that, if the differential operator $\hat{O}$ contains an operator of definite degree, apart from the Euler operator and the constant term, the aforementioned approach can connect the solution space to monomials through an exponential mapping. It may not be the case in general. Keeping this in mind, in the next section we develop a more general approach connecting the solution space with monomials for general equations.

3 A simple approach to familiar differential equations

For simplicity, we will first consider the case of single variable linear differential equations (LDEs) and point out its multi-variate generalization later. A single variable LDE, as will become clear from the examples of later sections, can be cast in the form

$$[F(D) + P(x, d/dx)]\,y(x) = 0, \tag{10}$$

where, $F(D) \equiv \sum_{n=-\infty}^{n=\infty} a_n D^n$ and a_n's are some parameters; $P(x, d/dx)$ can be an arbitrary polynomial function of $x, d/dx$ and other operators. The solution to Eq. (10) can be written as [3],

$$y(x) = C_\lambda \left\{ \sum_{m=0}^{\infty} (-1)^m \left[\frac{1}{F(D)}P(x, d/dx) \right]^m \right\} x^\lambda, \tag{11}$$

provided, $F(D)x^\lambda = 0$; here C_λ is constant. It is easy to see that, the operator $1/F(D)$ is well defined in the above expression and will not lead to any singularity, if $P(x, d/dx)$ does not contain any degree zero operator. We note that D itself is an operator of degree zero.

The proof of Eq.(11) is straightforward and follows by direct substitution [3]. Alternatively, since $F(D)x^\lambda = 0$, equating $F(D)x^\lambda$ modulo C_λ and Eq. (10), one finds,

$$[F(D) + P(x, d/dx)]\, y(x) = C_\lambda F(D)x^\lambda \quad . \tag{12}$$

Rearranging the above equation in the form

$$F(D)\left[1 + \frac{1}{F(D)}P(x, d/dx)\right] y(x) = C_\lambda F(D)x^\lambda \quad , \tag{13}$$

and cancelling $F(D)$, we obtain

$$\left[1 + \frac{1}{F(D)}P(x, d/dx)\right] y(x) = C_\lambda x^\lambda \quad . \tag{14}$$

This yields

$$y(x) = C_\lambda \frac{1}{\left[1 + \frac{1}{F(D)}P(x, d/dx)\right]} x^\lambda \quad , \tag{15}$$

which can be cast in the desired series form:

$$y(x) = C_\lambda \sum_{m=0}^{\infty} (-1)^m \left[\frac{1}{F(D)}P(x, d/dx)\right]^m x^\lambda \quad .$$

It is explicit that, the above procedure connects the solution $y(x)$ to the space of the monomials x^λ. The generalization of this method to a wide class of many-variable problems is immediate. Using the fact that, $F(\bar{D})X^\lambda = 0$ has solutions, in the space of monomial symmetric functions [8], where $\bar{D} = \sum_i D_i \equiv \sum_i x_i \frac{d}{dx_i}$, the solutions of those multi-variate DEs, which can be separated into the form given in Eq. (10), can be solved like the single variable case. As will be seen later, this procedure enables one to solve a number of correlated many-body problems.

For illustration, we consider the harmonic oscillator problem. The Schrödinger eigenvalue equation (in the units, $\hbar=\omega=m=1$)

$$\left[\frac{d^2}{dx^2} + (2E_n - x^2)\right] \psi_n = 0 \quad , \tag{16}$$

can be written in the form given in Eq. (10), after multiplying it by x^2 :

$$\left[(D-1)D + x^2(2E_n - x^2)\right] \psi_n = 0 \quad . \tag{17}$$

Here, $F(D) = (D-1)D$ and the condition $F(D)x^\lambda = 0$ yields, $\lambda = 0$ or 1. Using Eq. (11), the solution for $\lambda = 0$ is,

$$
\begin{aligned}
\psi_0 &= C_0 \left\{ \sum_{m=0}^{\infty} (-1)^m \left[\frac{1}{(D-1)D}(x^2(2E_0 - x^2))\right]^m \right\} x^0 \\
&= C_0 \left[1 - \frac{[2E_0]}{2!}x^2 + \frac{(2! + [2E_0]^2)}{4!}x^4 - \frac{(4! + (2!)^2[2E_0] + (2!)[2E_0]^3)}{2!6!}x^6 + \cdots \right] \tag{18}
\end{aligned}
$$

Here ψ_0 is an expansion in powers of x, whose coefficients are polynomials in E_0. The above series can be written in a closed, square integrable form, $C_0 \exp(-x^2/2)$, only

when $E_0 = 1/2$. Analogously, $\lambda = 1$, yields the first excited state. To find the nth excited state, one has to differentiate the Schrödinger equation $(n-2)$ number of times and subsequently multiply it by x^n to produce a $F(D) = x^n \frac{d^n}{dx^n} = \prod_{l=0}^{n-1}(D - l)$, and proceed in a manner similar to the ground state case.

It is clear that, our procedure yields a series solution, where additional conditions like square integrability has to be imposed to obtain physical eigenfunctions and their corresponding eigenvalues. Once, the ground state has been identified and for those cases, where $\psi(x) = \psi_0 P(x)$, where $P(x)$ is a polynomial, one can effortlessly obtain the polynomial part, as will be shown below. Proceeding with the harmonic oscillator case and writing

$$\psi_\alpha(x) = \exp(-\frac{x^2}{2})H_\alpha(x) \quad ,$$

one can easily show that H_α satisfies

$$\left[D - \alpha - \frac{1}{2}\frac{d^2}{dx^2} \right] H_\alpha(x) = 0 \quad , \tag{19}$$

where $\alpha = E_n - 1/2$. The solution of the DE

$$H_\alpha(x) = C_\alpha \sum_{m=0}^{\infty}(-1)^m \left[-\frac{1}{(D - \alpha)}\frac{1}{2}\frac{d^2}{dx^2} \right]^m x^\alpha \quad . \tag{20}$$

yields a polynomial only when α is an integer, since the operator d^2/dx^2 reduces the degree of x^α by two, in each step. Setting $\alpha = n$ in Eq. (19), we obtain the Hermite DE and $E_n = (n+1/2)$ as the energy eigenvalue. Below, we give the algebraic manipulations required to cast the series solution of Eq. (20) into a form, not very familiar in the literature. For the Hermite DE, $F(D) = D-n$ and $P(x, d/dx) = -\frac{1}{2}\frac{d^2}{dx^2}$, the condition $F(D)x^\lambda = 0$ yields $\lambda = n$, hence,

$$H_n(x) = C_n \sum_{m=0}^{\infty}(-1)^m \left[-\frac{1}{(D - n)}\frac{1}{2}\frac{d^2}{dx^2} \right]^m x^n \quad . \tag{21}$$

Using, $[D, (d^2/dx^2)] = -2(d^2/dx^2)$ and making use of the fact

$$\frac{1}{(D - n)} = \int_0^{\infty} ds\, e^{-s(D-n)} \tag{22}$$

we can write,

$$\left[-\frac{1}{2}\frac{1}{(D - n)}\frac{d^2}{dx^2} \right] = -\frac{1}{2}\frac{d^2}{dx^2}\frac{1}{(D - n - 2)} \quad ,$$

$$\left[-\frac{1}{2}\frac{1}{(D - n)}\frac{d^2}{dx^2} \right]\left[-\frac{1}{2}\frac{1}{(D - n)}\frac{d^2}{dx^2} \right] = \left[-\frac{1}{2}\frac{d^2}{dx^2} \right]^2 \frac{1}{(D - n - 4)}\frac{1}{(D - n - 2)} \tag{23}$$

Hence in general,

$$\left[-\frac{1}{2}\frac{1}{(D - n)}\frac{d^2}{dx^2} \right]^m x^n = \left(-\frac{1}{2}\frac{d^2}{dx^2} \right)^m \prod_{l=1}^{m}\frac{1}{(-2l)}x^n \quad ,$$

$$= \frac{1}{m!}\left(\frac{1}{4}\frac{d^2}{dx^2} \right)^m x^n \quad . \tag{24}$$

Substituting Eq. (24) in Eq. (21), we obtain,

$$
\begin{aligned}
H_n(x) &= C_n \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!} \left(\frac{1}{4} \frac{d^2}{dx^2} \right)^m x^n \quad , \\
&= C_n \exp \left(-\frac{1}{4} \frac{d^2}{dx^2} \right) x^n \quad ,
\end{aligned}
\tag{25}
$$

a result, not commonly found in the literature. The arbitrary constant C_n is chosen to be 2^n, so that the polynomials obtained can match with the standard definition [9].

Likewise, the solution to the hypergeometric DE

$$
\left[x^2 \frac{d^2}{dx^2} + (\alpha + \beta + 1) x \frac{d}{dx} + \alpha\beta - x \frac{d^2}{dx^2} - \gamma \frac{d}{dx} \right] F(\alpha, \beta; \gamma; x) = 0
\tag{26}
$$

can be written as,

$$
F(\alpha, \beta; \gamma; x) = (-1)^{-\beta} \frac{\Gamma(\alpha - \beta)\Gamma(\gamma)}{\Gamma(\gamma - \beta)\Gamma(\alpha)} \exp \left[\frac{-1}{(D + \alpha)} \left(x \frac{d^2}{dx^2} + \gamma \frac{d}{dx} \right) \right] . x^{-\beta} \quad .
\tag{27}
$$

For convenience a table has been provided at the end, which lists a number of commonly encountered DEs and the novel exponential forms of their solutions. It is worth pointing out that, the solution of hypergeometric DEs is a polynomial solution, provided either α or β are negative integers. It should be noticed that, unlike the conventional expressions, the monomials in the above solutions are arranged in decreasing powers of x.

Multiplying the hypergeometric DE with x yields two roots, $\lambda = 0, 1 - \gamma$ and $\lambda = 0$ solution gives rise to the well-known, Gauss hypergeometric series. Since a number of quantum mechanical problems can be related to confluent and hypergeometric DEs [10], we hope that the novel expressions given above and in the table will find physical applications.

4   Quasi-exactly solvable problems

This section is devoted to the study of QES problems [11]. These problems are intermediate to exactly and non-exactly solvable quantum potentials, in the sense that, only a part of the spectrum can be determined analytically. These potentials have attracted considerable attention in recent times, because of their connection to various physical problems [12].

We illustrate our procedure, through the sextic oscillator in the units $\hbar = 2m = \omega = 1$, for the purpose of comparison with the standard literature. The corresponding eigenvalue equation is given by:

$$
\left[-\frac{d^2}{dx^2} + \alpha x^2 + \gamma x^6 \right] \psi(x) = E\psi(x) \quad .
\tag{28}
$$

Asymptotic analysis suggests a trial wave function of the form,

$$
\psi(x) = \exp(-bx^4)\tilde{\psi}(x) \quad ,
\tag{29}
$$

6

which leads to,

$$\left[-\frac{d^2}{dx^2} + 2\sqrt{\gamma}x^3\frac{d}{dx} + (\alpha + 3\sqrt{\gamma})x^2 \right] \tilde{\psi}(x) = E\tilde{\psi}(x) \quad , \tag{30}$$

where x^6 term has been removed by the condition $16b^2 = \gamma$. One notices that, the operator $\tilde{O} = (\alpha + 3\sqrt{\gamma})x^2 + 2\sqrt{\gamma}x^3 d/dx$ increases the degree of $\tilde{\psi}(x)$ by two, if $\tilde{\psi}(x)$ is a polynomial. Confining ourselves to polynomial solutions and assuming that the highest power of the monomial in $\tilde{\psi}(x)$ is n, one obtains,

$$-\frac{\alpha}{\sqrt{\gamma}} = 2n + 3 \quad , \tag{31}$$

after imposing the condition that $\tilde{O}$ does not increase the degree of the polynomial. This is the well-known relationship between the coupling parameters of the quasi-exactly solvable sextic oscillator [12]. Taking $n = 4$ and $\gamma = 1$ for simplicity, and after multiplying the above equation with x^2 :

$$\left[D(D-1) + Ex^2 + 8x^4 - 2x^5\frac{d}{dx} \right] \tilde{\psi}(x) = 0 \quad , \tag{32}$$

we get,

$$\tilde{\psi}_0(x) = C_0 \left\{ \sum_{m=0}^{\infty} (-1)^m \left[\frac{1}{D(D-1)} \left(x^2 E_0 + 8x^4 - 2x^5\frac{d}{dx} \right) \right]^m \right\} 1 \quad . \tag{33}$$

Modulo C_0, the above series can be expanded as,

$$\tilde{\psi}_0(x) = 1 - E_0\frac{x^2}{2!} + (E_0^2 - 16)\frac{x^4}{4!} + (64E_0 - E_0^3)\frac{x^6}{6!} + \cdots \quad . \tag{34}$$

The monomials having degree greater than four vanish provided, $E_0 = 0, \pm 8$. It can be explicitly checked that, for these values of E_0, Eq. (32) is satisfied. The eigenfunctions corresponding to these three values are given by,

$$\psi_{-8}(x) = \exp(-\frac{x^4}{4})[1 + 4x^2 + 2x^4] \tag{35}$$

$$\psi_0(x) = \exp(-\frac{x^4}{4})[1 - \frac{2}{3}x^4] \quad , \tag{36}$$

$$\text{and} \quad \psi_{+8}(x) = \exp(-\frac{x^4}{4})[1 - 4x^2 + 2x^4] \quad . \tag{37}$$

This procedure generalizes to a wide class of QES problems [13].

Below, we demonstrate the method of finding approximate [13] eigenvalues and eigenfunctions for non-exactly solvable problems, using the well studied anharmonic oscillator as the example :

$$\left[-\frac{d^2}{dx^2} + \alpha x^2 + \beta x^4 - E_n \right] \psi_n(x) = 0 \quad . \tag{38}$$

Proceeding as before, $\psi_0(x)$ can be written as,

$$\psi_0(x) = C_0 \left\{ \sum_{m=0}^{\infty} (-1)^m \left[\frac{1}{(D-1)D} \left(E_0 x^2 - \alpha x^4 - \beta x^6 \right) \right]^m \right\} .1 \quad , \tag{39}$$

which can be expanded as,

$$\psi_0(x) = 1 - \frac{E_0}{2!}x^2 + \frac{1}{4!}\left(2\alpha + E_0^2\right)x^4 - \frac{1}{6!}\left(24\beta - (14\alpha E_0 + E_0^3)\right)x^6 + \cdots \qquad (40)$$

Although a number of schemes can be devised for the purpose of approximation, we consider the simplest one of starting with a trial function $\tilde{\psi}_0(x) = \exp(-\mu x^2 - \nu x^4)$ and matching it with $\psi_0(x)$. Comparison of the first three terms yields,

$$
\begin{aligned}
\mu &= \frac{E_0}{2!} \quad, \\
\frac{\mu^2}{2!} - \nu &= \frac{2\alpha}{4!} + \frac{E_0^2}{4!} \quad, \\
\text{and} \quad \mu\nu - \frac{\mu^3}{3!} &= \frac{\beta}{30} - \frac{(14E_0 + E_0^3)}{6!} \quad.
\end{aligned} \qquad (41)
$$

The resulting cubic equation in energy, $E_0^3 - E_0\alpha = 3\beta/2$, leads to one real root and two complex roots. Choosing the real root on physical grounds, one obtains,

$$E_0 = \frac{2^{1/3}\alpha}{A} + \frac{A}{3 \cdot 2^{1/3}} \quad, \qquad (42)$$

where $A = \left[40.5\beta + (1640.25\beta^2 - 108\alpha^3)^{1/2}\right]^{1/3}$. The value of E_0, obtained in the weak coupling regime, matches reasonably well with the earlier obtained results [14]. An approximate ψ_0 can be obtained from Eq. (40). One can easily improve upon the above scheme by taking better trial wave functions. Similar analysis can be carried out for the excited states. The above expansion of the wave function may be better amenable for a numerical treatment. For example, an accurate numerically determined energy value can lead to a good approximate wave function.

5 Many-body interacting systems

In this section, we will be dealing with correlated many-body systems, particularly of the Calogero-Sutherland [15] and Sutherland type [16]. These models have found application in diverse branches of physics like fluid flow, random matrix theory, novel statistics, quantum Hall effect and others [2].

We start with the relatively difficult Sutherland model, where the particles are confined to a circle of circumference L. The two-body problem treated explicitly below, straightforwardly generalizes to N particles. The Schrödinger equation is given by (in the units $\hbar = m = 1$)

$$\left[-\frac{1}{2}\sum_{i=1}^{2}\frac{\partial^2}{\partial x_i^2} + \beta(\beta - 1)\frac{\pi^2}{L^2}\frac{1}{\sin^2[\pi(x_1 - x_2)/L]} - E_\lambda\right]\psi_\lambda(\{x_i\}) = 0 \qquad . \qquad (43)$$

Taking, $z_j = e^{2\pi i x_j/L}$ and writing $\psi_\lambda(\{z_i\}) = \prod_{i,i\neq j} z_i^{-\beta/2}(z_i - z_j)^\beta J_\lambda(\{z_i\})$, the above equation becomes,

$$\left[\sum_{i=1}^{2}D_i^2 + \beta\frac{z_1 + z_2}{z_1 - z_2}(D_1 - D_2) + \tilde{E}_0 - \tilde{E}_\lambda\right]J_\lambda(\{z_i\}) = 0 \qquad , \qquad (44)$$

where, $D_i \equiv z_i \frac{\partial}{\partial z_i}$, $\tilde{E}_\lambda \equiv 2(\frac{L}{2\pi})^2 E_\lambda$, $\tilde{E}_0 \equiv 2(\frac{L}{2\pi})^2 E_0$ and $E_0 = (\frac{\pi}{L})^2 \beta^2$, is the ground-state energy. Here, $J_\lambda(\{z_i\})$ is the polynomial part, which in the multivariate case is the well known Jack polynomial [8]. Here, λ is the degree of the symmetric function and $\{\lambda\}$ refers to different partitions of λ. $\sum_i D_i^2$ is a diagonal operator in the space spanned by the monomial symmetric functions, $m_{\{\lambda\}}$, with eigenvalues $\sum_{i=1}^2 \lambda_i^2$. A monomial symmetric function is a symmetrized combination of monomials of definite degree. For example for two particle case, there are two monomial symmetric functions having degree two. These are $m_{2,0} = x_1^2 + x_2^2$ and $m_{1,1} = x_1 x_2$. Readers are referred to Ref. [9] for more details about various symmetric functions and their properties. Rewriting Eq. (44) in the form,

$$\left[\sum_i (D_i^2 - \lambda_i^2) + \beta \frac{z_1 + z_2}{z_1 - z_2}(D_1 - D_2) + \tilde{E}_0 + \sum_i \lambda_i^2 - \tilde{E}_\lambda \right] J_\lambda(\{z_i\}) = 0 \quad , \quad (45)$$

one can immediately show that,

$$J_\lambda(\{z_i\}) = C_\lambda \left\{ \sum_{n=0}^\infty (-1)^n \left[\frac{1}{\sum_i (D_i^2 - \lambda_i^2)} \left(\beta \frac{z_1 + z_2}{z_1 - z_2}(D_1 - D_2) + \tilde{E}_0 + \sum_i \lambda_i^2 - \tilde{E}_\lambda \right) \right]^n \right\}$$
$$\times m_\lambda(\{z_i\}) \quad . \quad (46)$$

For the sake of convenience, we define

$$\hat{S} \equiv \left[\frac{1}{\sum_i (D_i^2 - \lambda_i^2)} \hat{Z} \right] \quad ,$$

$$\text{and} \quad \hat{Z} \equiv \beta \frac{z_1 + z_2}{z_1 - z_2}(D_1 - D_2) + \tilde{E}_0 + \sum_i \lambda_i^2 - \tilde{E}_\lambda \quad . \quad (47)$$

The action of $\hat{S}$ on $m_\lambda(\{z_i\})$ yields singularities, unless one chooses the coefficient of m_λ in $\hat{Z} m_\lambda(\{z_i\})$ to be zero; this condition yields the eigenvalue equation

$$\tilde{E}_\lambda = \tilde{E}_0 + \sum_i (\lambda_i^2 + \beta[3 - 2i]\lambda_i) \quad .$$

Using the above, one can write down the two particle Jack polynomial as,

$$J_\lambda(\{z_i\}) = \sum_{n=0}^\infty (-\beta)^n \left[\frac{1}{\sum_i (D_i^2 - \lambda_i^2)} (\frac{z_1 + z_2}{z_1 - z_2}(D_1 - D_2) - \sum_i (3 - 2i)\lambda_i) \right]^n$$
$$\times m_\lambda(\{x_i\}) \quad . \quad (48)$$

Starting from $m_{2,0} = z_1^2 + z_2^2$, it is straightforward to check that

$$\hat{Z} m_{2,0} = 4\beta(z_1 + z_2)^2 = 4\beta m_{1,0}^2$$

$$\hat{S} m_{2,0} = \frac{1}{\sum_i (D_i^2 - 4)}(4\beta m_{1,0}^2) = -2\beta m_{1,0}^2 \quad ,$$

$$\hat{S}^n m_{2,0} = -2(\beta)^n m_{1,0}^2 \quad \text{for} \quad n \geq 1 \quad .$$

Substituting the above result in Eq. (46), apart from C_2, one obtains

$$J_2 = m_{2,0} + \left(\sum_{n=1}^\infty (-1)^n (-2)(\beta)^n \right) m_{1,0}^2$$

9

$$= m_{2,0} + 2\beta \left(\sum_{n=0}^{\infty} (-\beta)^n \right) m_{1,0}^2$$

$$= m_{2,0} + \frac{2\beta}{1+\beta} m_{1,0}^2 \quad ; \tag{49}$$

which is the desired result. The above approach can be easily generalized to the N-particle case.

Another class of many-body problems, which can be solved by the present approach is the Calogero-Sutherland model (CSM) and its generalizations. Proceeding along the line, as for the Sutherland model, one finds the eigenvalues and eigenfunctions for the CSM.

The Schrödinger equation for the CSM in the previous units, is given by,

$$\left[-\frac{1}{2} \sum_{i=1}^{N} \frac{\partial}{\partial x_i^2} + \frac{1}{2} \sum_{i=1}^{N} x_i^2 + \frac{g^2}{2} \sum_{\substack{i,j \\ i \neq j}}^{N} \frac{1}{(x_i - x_j)^2} - E_n \right] \psi_n(\{x_i\}) = 0 \quad , \tag{50}$$

where the wave function is of the form [15],

$$\psi_n(x) = \psi_0 P_n(\{x_i\}) = ZG P_n(\{x_i\}) \quad . \tag{51}$$

Here $Z \equiv \prod_{i<j} (x_i - x_j)^\beta$, $G \equiv \exp\left\{ -\frac{1}{2} \sum_i x_i^2 \right\}$, $g^2 = \beta(\beta - 1)$ and $P_n(\{x_i\})$ is a polynomial. After removing the ground state, the polynomial $P_n(\{x_i\})$ satisfies,

$$\left[\sum_i x_i \frac{\partial}{\partial x_i} + E_0 - E_n - \frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} - \beta \sum_{i \neq j} \frac{1}{(x_i - x_j)} \frac{\partial}{\partial x_i} \right] P_n(\{x_i\}) = 0 \quad , \tag{52}$$

where $E_0 = \frac{1}{2}N + \frac{1}{2}\beta N(N-1)$. Defining

$$\hat{A}(\beta) \equiv \frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} + \beta \sum_{i \neq j} \frac{1}{(x_i - x_j)} \frac{\partial}{\partial x_i} \quad ,$$

one can easily see, following the procedure adopted for the Hermite DE that,

$$P_n(\{x_i\}) = C_n e^{-\hat{A}(\beta)} m_{\{n\}}(\{x_i\}) \quad , \tag{53}$$

where $m_{\{n\}}(\{x_i\})$ is a monomial symmetric function of degree n. The corresponding energy is given by

$$E_n = E_0 + n \quad , \tag{54}$$

One can use the above procedure to solve many other interacting systems.

6 Conclusions

In conclusion, we have presented a novel scheme to treat exactly, quasi-exactly and non-exactly solvable problems, which also extends to a wide class of many-body interacting systems. The procedure can be used for the construction of the ladder operators for various orthogonal polynomials and the quantum systems associated with them.

The approximation scheme presented needs further refinement. It should be analyzed in conjunction with computational tools for finding its efficacy as compared to other methods. The many-body problems presented here have deep connection with diverse branches of physics and mathematics. The fact that the procedure employed for solving them, connects the solution space of the problem under study to the space of monomials, will make it useful for constructing ladder operators for the many-variable case. This will throw light on the structure of the Hilbert space of these correlated systems. Some of these questions are currently under study and will be reported elsewhere.

References

[1] W. Miller Jr., *Symmetry and Separation of Variables (Encyclopedia of Mathematics and its Applications Volume 4)*, Cambridge University Press, 1984 (ISBN: 978-0521302241).

[2] N. Gurappa and P. K. Panigrahi, *Phys. Rev.* **B 59**, R2490-R2493 (1999) and references therein.

[3] N. Gurappa and P. K. Panigrahi, *Phys. Rev.* **B 62**, 1943-1949 (2000); N. Gurappa, P. K. Panigrahi, T. Shreecharan and S. Sree Ranjani in *Frontiers of Fundamental Physics 4*, Eds: B. G. Sidharth and M. V. Altaisky, Kluwer Academic Plenum Publishers, New York, 269-277 (2001), N. Gurappa, P. K. Panigrahi and T. Shreecharan, Phys. News **34** and **35**, 4-11 (2004).

[4] J. Mathews and R. L. Walker, *Mathematical methods of physics (2nd Edition)*, Addison-Wesley, 1970; P. M. Morse and H. Feschbach, *Methods of Theoretical Physics*, **Vol 1**, McGraw-Hill, New-York, 1953.

[5] F. M. Fernández, *Phys. Lett.* **A 237**, 189-191 (1998).

[6] N. Gurappa, P. K. Panigrahi and T. Shreecharan, *J. Comp. Appl. Math*, **160** 103 (2003).

[7] T. Shreecharan, P. K. Panigrahi and J. Banerji, *Phys. Rev.* **A 69** 012102 (2004).

[8] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford, 1979.

[9] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products*, Academic Press Inc., 1965.

[10] L. Schiff, *Quantum Mechanics*, McGraw-Hill, New-York, 1968;
L. D. Landau and E. Lifshitz, *Quantum Mechanics*, Pergamon, New-York, 1977.

[11] M. A. Shifman, *Int. J. Mod. Phys.* **A 4**, 2827-2952 (1989).

[12] A. G. Ushveridze, *Quasi-Exactly Solvable Models in Quantum Mechanics*, Inst. of Physics Publishing, Bristol, 1994.

[13] R. Atre and P. K. Panigrahi, *Phys. Lett.* **A 317**, 46-53 (2003).

[14] M. H. Macfarlane, *Ann. Phys.* **271**, 159-202 (1999) and references therein.

[15] F. Calogero, *J. Math. Phys.* **12**, 419-436 (1971);
B. Sutherland, *ibid* **12**, 246-250 (1971); **12**, 251-256 (1971).

[16] B. Sutherland, *Phys. Rev.* **A, 4**, 2019-2021 (1971); **5**, 1372-1376 (1972).

TABLE I. Some frequently encountered DEs and their novel solutions.

Below, the differential equations from top to bottom, respectively, are Hermite, Laguerre, Legendre, Gegenbauer, Chebyshev Type I, Chebyshev Type II, Bessel, confluent hypergeometric and hypergeometric.

Differential Equation	$\mathbf{F(D)}$, $D \equiv x(d/dx)$	Solution
$\left[x\frac{d}{dx} - n - \frac{1}{2}\frac{d^2}{dx^2}\right] H_n(x) = 0$	$(D - n)$	$H_n(x) = C_n \exp\left[-\frac{1}{4}\frac{d^2}{dx^2}\right].x^n$
$\left[x\frac{d}{dx} - n - (\alpha+1)\frac{d}{dx} - x\frac{d^2}{dx^2}\right] L_n^\alpha = 0$	$(D - n)$	$L_n^\alpha(x) = C_n \exp\left[-x\frac{d^2}{dx^2} - (\alpha+1)\frac{d}{dx}\right].x^n$
$\left[x^2\frac{d^2}{dx^2} + 2x\frac{d}{dx} - n(n+1) - \frac{d^2}{dx^2}\right] P_n(x) = 0$	$(D+n+1)(D-n)$	$P_n(x) = C_n \exp\left[-\frac{1}{2(D+n+1)}\frac{d^2}{dx^2}\right].x^n$
$\left[x^2\frac{d^2}{dx^2} + (2\lambda+1)x\frac{d}{dx} - n(2\lambda+n) - \frac{d^2}{dx^2}\right] \times$ $\times C_n^\lambda(x) = 0$	$(D+n+2\lambda)(D-n)$	$C_n^\lambda(x) = C_n \exp\left[-\frac{1}{2(D+n+2\lambda)}\frac{d^2}{dx^2}\right].x^n$
$\left[x^2\frac{d^2}{dx^2} + x\frac{d}{dx} - n^2 - \frac{d^2}{dx^2}\right] T_n(x) = 0$	$(D+n)(D-n)$	$T_n(x) = C_n \exp\left[-\frac{1}{2(D+n)}\frac{d^2}{dx^2}\right].x^n$
$\left[x^2\frac{d^2}{dx^2} + 3x\frac{d}{dx} - n(n+2) - \frac{d^2}{dx^2}\right] U_n(x) = 0$	$(D+n+2)(D-n)$	$U_n(x) = C_n \exp\left[-\frac{1}{2(D+n+2)}\frac{d^2}{dx^2}\right].x^n$
$\left[x^2\frac{d^2}{dx^2} + x\frac{d}{dx} - \nu^2 + x^2\right] J_{\pm\nu}(x) = 0$	$(D+\nu)(D-\nu)$	$J_{\pm\nu}(x) = C_{\pm\nu} \exp\left[-\frac{1}{2(D\pm\nu)}x^2\right].x^{\mp\nu}$
$\left[x\frac{d}{dx} + \alpha - x\frac{d^2}{dx^2} - \gamma\frac{d}{dx}\right] \Phi(\alpha, \gamma, x) = 0$	$(D+\alpha)$	$\Phi(\alpha, \gamma, x) = C_{-\alpha} \exp\left[-x\frac{d^2}{dx^2} - \gamma\frac{d}{dx}\right].x^{-\alpha}$
$\left[x(1-x)\frac{d^2}{dx^2} + (\gamma - [\alpha+\beta+1]x)\frac{d}{dx} - \alpha\beta\right] \times$ $\times F(\alpha, \beta, \gamma, x) = 0$	$(D+\alpha)(D+\beta)$	$F(\alpha, \beta, \gamma, x) = C_{-(\alpha,\beta)} \times$ $\times \exp\left[-\frac{1}{[D+(\alpha,\beta)]}(x\frac{d^2}{dx^2} + \gamma\frac{d}{dx})\right].x^{-(\beta,\alpha)}$

The solution to the DE $[F(D) + P(x, d/dx)] y(x) = 0$, is,

$$y(x) = C_\lambda \left\{ \sum_{m=0}^\infty (-1)^m \left[\frac{1}{F(D)} P(x, d/dx)\right]^m \right\} x^\lambda, \text{ provided } F(D)x^\lambda = 0.$$