A Coning Theory of Bullet Motions

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Each observable ballistic phenomenon of a spin-stabilized rifle bullet can be explained in terms of the acceleration of gravity and the total aerodynamic force acting on that bullet. In addition to the coning motion itself, Coning Theory explains the spinning bullet's aerodynamic jump and its steadily increasing yaw of repose together with its resulting spin-drift. The total aerodynamic force on the bullet comprises its drag and lift rectangular components and produces an associated overturning moment acting upon the rigid bullet. The coning motion of the bullet includes two distinct but synchronized aspects: 1) the well-known gyroscopic precession of the spin-axis of the bullet, and 2) the previously little-known orbiting of the center of gravity of the bullet around its mean trajectory with the nose of the bullet angled inward toward that trajectory. New equations are developed governing the orbital motion of the CG as a circular, isotropic harmonic oscillation driven by the lift and drag forces as they revolve together at the gyroscopic precession rate. Standard Tri-Cyclic Theory governs the uniform circular precession of the spin-axis driven by the overturning moment acting on the spinning bullet as a free-flying gyroscope. The synchronization of these two motions is the defining principle of Coning Theory.

Nomenclature

A. Forces and Moments

$\mathbf{F_D}$ = Drag force on the bullet (in pounds)

$\mathbf{F_L}$ = Lift force acting on the bullet (in pounds)

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F = Total aerodynamic force acting on the bullet (in pounds)

M = Overturning moment acting on the bullet (in pounds-feet)

α = Angle-of-attack of bullet spin-axis with respect to apparent wind and also the half-cone-angle of bullet's coning motion (always a non-negative value in radians)

q = $\boldsymbol{\rho V^2/2}$ **=** Dynamic pressure (in pounds per square foot)

ρ = Atmospheric density (in slugs per cubic foot)

V = Bullet velocity vector in earth-fixed coordinates tangent to the mean trajectory (in feet per second). [**V** is directed essentially horizontally in "flat firing," and its magnitude is essentially the supersonic airspeed of the bullet.]

S = $\boldsymbol{\pi d^2/4}$ **=** Reference (cross-sectional) area of the bullet (in square feet)

d = Barrel groove diameter = Caliber of the fired bullet (in feet)

CD = Coefficient of aerodynamic drag as a dimensionless function of Mach number and as an *even function* in **Sin(α)**

CLα = Coefficient of lift as a dimensionless function of Mach number used in determining the lift force on the bullet as an *odd function* in **Sin(α)**.

CMα = Coefficient for determining the overturning moment acting on the bullet as a dimensionless function of Mach number as an *odd function* in **Sin(α)**

B. Coning Motion Symbols

r = Orbital radius (in feet) of the CG of the coning bullet about a center moving along the mean trajectory with the bullet

$\mathbf{F_R}$ = Centripetal Hookean restoring force (in pounds) needed to maintain a circular harmonic orbit

$\mathbf{F_C}$ = Coning force component of the total aerodynamic force **F** perpendicular to the coning distance vector **D** (in pounds)

$\mathbf{k_R}$**,** $\mathbf{k_C}$ = Slowly varying "force constant" values of restoring force per unit of radial displacement away from a neutral point at the mean center of the coning motion (in pounds per foot)

m = Mass of the bullet in slugs (**0.000746** slugs for our example **168** grain bullet)

v = Circular orbital velocity of the CG of the coning bullet about its mean center (in feet per second)

$\mathbf{T_2}$ = $\mathbf{2\pi/\omega_2}$ = Period (in seconds) of a coning cycle at the slow-mode gyroscopic precession rate $\boldsymbol{\omega_2}$

$\boldsymbol{\omega_2}$ = $\mathbf{2\pi * f_2 = 2\pi/T_2}$ = Circular frequency of gyroscopic precession, slow-mode (in radians per second)

$\boldsymbol{\omega_1}$ = $\mathbf{2\pi * f_1}$ = Circular inertial frequency of gyroscopic nutation, fast-mode (in radians per second)

$\mathbf{f_2}$ = Frequency of slow-mode gyroscopic precession (**65** hertz initially in this example)

$\mathbf{f_1}$ = Inertial frequency of gyroscopic nutation, fast-mode (**311** hertz initially in this example)

D = Distance vector from cone apex to CG of the coning bullet (in feet)

$\mathbf{R_{CG}}$, $\mathbf{R_{Apex}}$ = Position vectors for CG of bullet and apex of cone, respectively, both in either earth-fixed or moving coordinates, as needed (in feet)

$\boldsymbol{\Gamma_C}$ = Rotating aerodynamic torque vector acting about the apex of the cone and driving the "torsional" coning motion (in pounds-feet)

$\boldsymbol{\beta}$ = Angle whose trigonometric tangent is the instantaneous "lift-to-drag ratio," $\mathbf{F_L/F_D}$ (in radians)

$\boldsymbol{\Delta}$ = Delta, the "small finite change in symbol following" operator. [$\boldsymbol{\Delta X}$ is a vector if the symbol **X** represents a vector quantity.]

$\mathbf{I_C}$ = $\mathbf{m * D^2}$ = *Coning moment of inertia* of coning bullet (as a *point mass*) about the cone apex (in slug-feet squared)

g = **32.174** feet per second squared = Nominal acceleration of gravity

Ix = Moment of inertia of the bullet about its spin-axis (in slug-feet squared) [**Ix = 0.000247** pound-inch^2 = $\mathbf{5.334x10^{-8}}$ sl-ft^2 for this bullet.]

Iy = Moment of inertia of the bullet about a cross-axis (in slug-feet squared) [**Iy = 0.001838** pound-inch^2 = $\mathbf{3.969x10^{-7}}$ sl-ft^2 for this bullet.]

L = $\mathbf{Ix * \omega}$ = Angular momentum of spinning bullet about its spin-axis (in slug-feet squared per second)

p = Spin-rate of the bullet about its axis of symmetry (in hertz) = **2800** revolutions/second (initially)

$\boldsymbol{\omega}$ = $\mathbf{2\pi * p}$ = Spin-rate of the bullet about its longitudinal principle axis of symmetry (in radians per second) [Note: $\boldsymbol{\omega}$, $\boldsymbol{\omega_1}$, and $\boldsymbol{\omega_2}$ are the three cyclic rates of the Tri-Cyclic Theory.]

$\boldsymbol{\alpha(t)}$ = Complex coning angle (in radians) as a function of time **t**

$\boldsymbol{\varphi(t)}$ = Pitch attitude of bullet spin-axis (in radians measured upward from **+V** direction), the real part of $\boldsymbol{\alpha(t)}$

$\theta(t)$ = Yaw attitude of bullet spin-axis (in radians measured rightward from **+V** direction), the imaginary part of **$\alpha(t)$**

i = Imaginary (**+yaw**) axis direction in the complex plane [$\mathbf{i^2 = -1}$]

K_0 = $\mathbf{[\varphi_0^{\ 2} + (\theta_0 + \gamma)^2]^{1/2}}$ **=** Initial magnitude of complex cone angle **α_0** (in radians)

ξ_0 = **ATAN2{$-\varphi_{0,}$ $-(\theta_0 + \gamma)$} $+\pi$ =** Initial "roll orientation" or phase angle of angle-of-attack **α_0**, measured positive clockwise from the real **$+\varphi$** axis (in radians from **zero** to **2π**) [$\mathbf{\alpha_0 = K_0 * Cos\xi_0 + i*(K_0*Sin\xi_0 + \gamma)}$]

C. Apparent Wind and Transient Aerodynamic Effects

W = Instantaneous true wind vector in earth-fixed coordinates [**W<<V**] (in feet per second)

W_A = Apparent wind in moving coordinate system (in feet per second)

γ = Angular offset of apparent wind direction **W_A** from **–V** direction due to crosswind **W** (a non-negative angular magnitude in radians) [The offset **γ** is in the **$-\varphi$** direction for the left-to-right crosswind in this example.]

δ = Angular magnitude of fast-mode nutating motion caused by sudden appearance of crosswind **W** (a non-negative angular value in radians)

T_N = Period of nutation with respect to the moving slow-mode arm **$= 2\pi/[(R - 1)*\omega_2] = 4.06$** milliseconds, initially in this example

ξ_1, ξ_2 = "Roll orientation" arguments as functions of time (**t**) of fast-mode and slow-mode arms, respectively, of epicyclic motion of bullet spin-axis about apparent wind direction, each measured clockwise (in radians) from the **$+\varphi$** (pitch axis) direction

f_N = **$1/T_N = 246$** hertz (initially) **=** Reduced nutation rate with respect to the moving slow-mode arm (in hertz) **[$f_2<f_N<f_1$]**

R = Ratio of epicyclic (inertial gyroscopic) rates (dimensionless) [$\mathbf{R = \omega_1/\omega_2 = f_1/f_2}$ = $\mathbf{\varphi'_1/\varphi'_2 = 4.79>>1}$, initially, in this example.]

Sg = **$(1 + R)^2/(4*R)$ =** Gyroscopic stability of the bullet (dimensionless)

J = Cross-track impulse due to transient aerodynamic force (in pound-seconds)

A_J = Aerodynamic jump angle (in radians)

$\mathbf{\Delta V_C}$ = Cross-track velocity "kick" (in feet per second)

$\mathbf{\Phi}$ = Flight path angle measured positive upward from local horizontal plane (in radians)

$\mathbf{\beta_R}$ = Yaw-of-repose angle of the bullet's coning-axis (in radians)

n = Twist rate of rifle barrel in calibers per turn

I. Introduction

This paper puts forward a comprehensive new *Coning Theory* explaining in detail the motions of spin-stabilized rifle bullets in flight. If not all completely new to the science of ballistics, at least these ideas are probably original in the aggregate. An early popular version of this theory was published in *Precision Shooting Magazine* **[1]**. Most basic is the concept that the center of gravity (CG) of the coning bullet always spirals around the mean trajectory at a larger radius than does the nose of the bullet. We will show that the spinning bullet "cones around" at its slow-mode, *gyroscopic precession* rate with its nose angled *inward* toward its *mean trajectory* as diagrammed in **Fig. 1** for a bullet fired from a right-hand twist barrel. The CG of the bullet revolves around the mean trajectory in the same rotational sense as the rifling twist, and so does the projected direction of the bullet's spin-axis seen in the ballistician's usual "wind axes" orthogonal pitch-versus-yaw coordinates plot as shown at the right side of the diagram in **Fig. 1**.

Currently accepted aeroballistic theory seems to hold, instead, that the bullet should cone around with its nose pointing *outward*, and that the CG of the bullet should move directly along the trajectory. For example, Harold R. Vaughn **[2]** has defined "coning motion" as:

> "*The motion a bullet makes with its nose traveling in a circle while the CG remains fixed on the flight path*."

This fallacy is cited only to illustrate the need for this new theory. While other working ballisticians might have disagreed with the good Mr. Vaughn, the situation has never been clarified. This paper is an attempt to do just that.

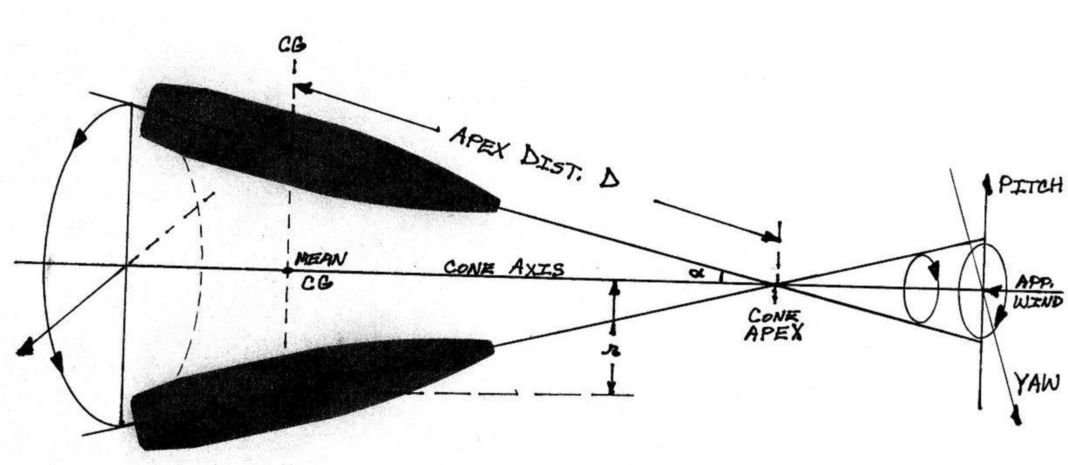

Fig. 1. Extreme Positions and Attitudes of Coning Bullet

The *coning motion* of a spin-stabilized rifle bullet in free flight is the result of a *gyroscopic precession* of the bullet's spin-axis driven by an aerodynamically produced overturning moment. As such, the pointing direction of the coning bullet's spin-axis follows a *circular* path in the "wind axes" coordinate system shown on the right-hand side in **Fig. 1**. If a *gyroscopic nutation* is superimposed on this precession the path of the bullet's spin-axis in the wind plot becomes the familiar epicyclic curve. Current Tri-Cyclic ballistics theory explains these gyroscopic phenomena perfectly well. According to this new Coning Theory, the motion of the CG of the free-flying bullet is a circular two-dimensional *isotropic harmonic oscillation* at the precession rate as shown on the left-hand side of **Fig. 1**. This orbiting motion of the CG at the bullet's gyroscopic precession rate is driven by the powerful primary aerodynamic forces of *lift* and *drag* as diagrammed in **Fig. 2**. The dual aspects of this coning motion are always perfectly synchronized with each other. We will develop this theory in detail.

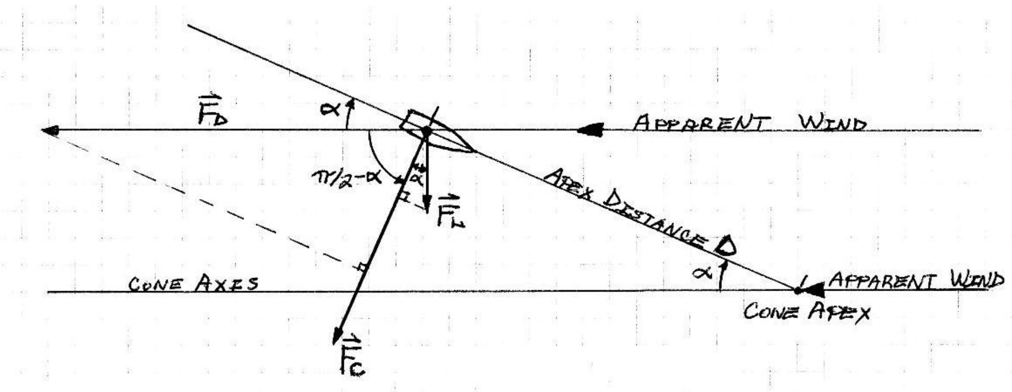

Fig. 2. Powering the Coning Motion of the Bullet's CG

We theorize here that both the lift and drag forces, $\mathbf{F_L}$ and $\mathbf{F_D}$, contribute to the coning force $\mathbf{F_C}$ driving the bullet's coning motion—as opposed to only the lift force $\mathbf{F_L}$ participating in this motion per the currently accepted analytical formulation of Robert L. McCoy **[3]** and others at the US Army's former Ballistics Research Laboratory (BRL) at Aberdeen, Maryland. This minor correction is necessary for self-consistency of the Coning Theory and produces better agreement with the empirical data. We frequently rely upon the primitive tools of vector differential geometry in this study of the physics behind the bullet's coning motions in addition to the elegant calculus favored at BRL. These rather crude tools are sufficient to the task of illuminating the details of the motions more clearly.

Another important new concept is that the *axis of the bullet's coning motion*, and *not* the *spin-axis of the bullet itself*, as seems commonly to be believed, always points directly into the "apparent wind" approaching the bullet. In fact, the direction of the apparent wind encountered by the bullet in flight *continually defines the axis of the coning motion*. The massless coning axis can incrementally change its direction even more readily than can the ficklest of real winds. The moving spin-axis of the bullet quickly accommodates each small change in the cone-axis direction into the coning motion—certainly within less than one fast-mode nutation cycle. We should point out one important exception to this "wind tracking" ability of a coning bullet, even though its occurrence is well outside the scope of this study: a spinning artillery projectile, nearing the apogee of a high-angle trajectory, might be unable to "arc over" quickly enough to continue its coning motion during the descending leg of its flight. We say such a projectile has “failed to trail” as it falls to earth sideways, or even backwards, badly missing its intended target **[4]**.

Whenever the coning axis has to move in order to remain aligned with a new apparent wind direction, the only mechanism available for adjusting the coning motion is for the *cone apex angle*, together with its corresponding *radius* of the coning motion, to *increase in magnitude* in order to accomplish this re-alignment. In this way, the coning bullet is able to align its cone axis, orienting it into a new apparent wind, even though the nose of the spinning bullet itself is being pushed away from that wind direction by the approaching wind. Only later in the flight, after the precession-rate coning motion due to the wind change has *damped out* as it does for dynamically stable bullets, will the spin-axis of the bullet be seen to have oriented itself into alignment with the apparent wind. The coning bullet cannot just magically “turns its nose into the wind.”

A one-time-per-disturbance, transient coning motion, commencing when the bullet first encounters a new *purely horizontal* crosswind, explains the small and recently documented vertical-direction “crosswind aerodynamic jump” in the flight path angle that we can observe reliably in precision shooting. This same trajectory-deflecting effect has been analytically formulated in calculus-based aeroballistics terms by Robert L. McCoy at BRL. Independent numerical calculations of this angular deflection of the trajectory are presented here based on using differential geometry and Coning Theory. These numerical results agree well with McCoy’s values for our example bullet after the BRL formulation is adjusted to incorporate the small contribution of the bullet’s drag force toward driving its coning motion. No “Magnus effect” of any type is involved in either formulation. A similar transient coning effect produces a similar type of angular deflection whenever the bullet enters the windstream with a non-zero aeroballistic yaw (or yaw rate).

A transient incremental deviation from the nominal coning motion that recurs *twice* per coning cycle as the bullet’s trajectory arcs downward due to gravity explains the slow increase of the “yaw-of-repose” and, thence, the long-known resulting “spin-drift” of the bullet in the same rightward horizontal direction as the sense of the rifling twist. The horizontal spin-drift of the bullet at long ranges is the accumulated effect of the aerodynamic lift force acting rightward on the bullet due to an angle-of-attack of the small, but steadily increasing, yaw-of-repose. The continually downward changing of the flight path angle due to gravity causes repeated rightward transient gyroscopic reactions centered about the extreme top and bottom dead center positions of the coning motion, which is how the double-rate yaw impulses come about. Calculations of the *rate of change* in the yaw-of-repose based on the differential geometry of Coning Theory have been fully reconciled with the analytical calculations of McCoy and others at BRL. The very small initial yaw-of-repose angle grows by about an order of magnitude over the maximum effective range for our example rifle bullet. Here, we have reformulated neither the yaw-of-repose, nor its resulting horizontal spin-drift.

The agreement of these detailed bullet motions with the existing *equations of motion* for bullet flight is illustrated by the use of data outputs from existing six-degree-of-freedom (6-DOF) flight simulations (which numerically integrate these same *equations of motion*) to demonstrate how this new Coning Theory explains the motions of rifle bullets in flight. With proper initialization, these 6-DOF simulator outputs can agree very well with Doppler radar and instrumented range measurements of the flights of real bullets. This new Coning Theory generally coincides with and extends the bulk of the modern conventional analytic ballistic theory **[5]** for spin-stabilized projectiles. This new Coning Theory should be in complete agreement with the precepts of classical mechanics **[6]**, but the author must assume sole responsibility for any errors in its development.

This new *Coning Theory of Bullet Motions* does not rely upon any of the minor aeroballistic forces or moments (spin-damping, pitch-damping, Magnus force or moment, etc.) in analytical explanation of the basic observed motions of spin-stabilized rifle bullets. Consequently, no discussions of these non-relevant forces and moments are included. In this explanation of bullet motions, we need consider *only* the primary aerodynamic forces of *drag* and *lift* and the primary aerodynamic *overturning (or "pitching") moment* acting on the bullet, all of which combine at any given time during the bullet's flight into a single, instantaneous *total aerodynamic force*, acting at one particular point on the surface of the bullet and with its line-of-action passing through the instantaneous center of pressure (CP) on the axis of symmetry of the spinning bullet. Coning motion *always occurs* even if a perfectly made bullet could be perfectly launched into a completely wind-free atmosphere. The aerodynamic interaction of (1) a spin-stabilized bullet with (2) a sensible atmosphere in the presence of (3) a gravitational field having a cross-track component is sufficient to initiate a small precession-rate coning motion around a very small initial *yaw of repose* defining its center of rotation in a "wind axes" plot.

II. Assumptions and Limitations of Study

The modern Spitzer-style rifle bullet is a sharply-pointed, rigid, rotationally symmetric, statically unstable projectile of about 2.5 to 5.5 calibers in length. Spin-stabilization is applied to the bullet at launch to prevent its tumbling in flight. Throughout this study, the rotational sense of the bullet's spin is "right handed," i.e., the direction of rotation of a right-hand threaded screw advancing toward the target, or clockwise as seen from behind the bullet. For purposes of this discussion these rifle bullets are assumed to have been perfectly manufactured in balance, shape, and symmetry. The inertial spin-axis of the fired bullet is assumed to correspond exactly with the mechanical axis of symmetry of its outside profile. These rifle bullets are also assumed to have been perfectly launched so as to have an initial *aeroballistic yaw* of zero (and zero initial yaw-rate). Only our best target rifles can routinely approximate this level of perfection in bullet launching, and then only when using precision hand-loaded ammunition. [For our rotationally symmetric projectiles, the aeroballistic yaw can be thought of as the root-sum-square (RSS) of the small, non-Eulerian, orthogonal "aircraft type" *pitch* (up or down) and *yaw* (side-slip) attitude angles. This generalized aeroballistic yaw is also the "angle-of-attack" for rotationally symmetric projectiles as used herein.] The very real flight-disturbing effects of bullet imbalance, in-bore yaw, muzzle blast, or the motions of the muzzle of the recoiling rifle (for examples) are not discussed here.

Our subject rifle bullets are further assumed to be gyroscopically stable, but not over-stabilized, and also (usually) to be dynamically stable throughout their almost horizontal, supersonic flights in the flat-firing case being studied here. By saying that a rifle bullet is "statically unstable," we mean that the aerodynamic center of pressure (CP) for the modern rifle bullet flying normally at small angles-of-attack is *ahead* of its center of gravity (CG). Both centers are located on the axis of symmetry of the ideal rotationally symmetric bullet being considered here. Our selected example bullet, the well-studied 30-caliber 168-grain Sierra *MatchKing* (formerly *International*), is launched with a gyroscopic stability of 1.75 and is just slightly unstable dynamically. That is, the angular amplitude of the coning motion slowly increases with flight time rather than damping down as with most rifle bullets. This unusual flight behavior makes our selected example bullet particularly suitable for use in the study of coning motion.

The long axis of the rifle bullet is also a *principal axis of inertia*, producing an *extremum* (either a minimum or a maximum) in the second moment of the mass distribution of the bullet. In the case of a rifle bullet, its spin-axis is the axis having a *minimum* moment of inertia (i.e., the axial direction producing the smallest possible second moment of the mass distribution of the bullet). For our example match-type bullet, the moment of inertia about any transverse principal axis is **7.44 times** larger than that about its spin-axis. For a conventional gyroscope, the spin-axis has a *maximum* moment of inertia of just **twice** that of any transverse axis.

Of course, it is more convenient to simulate and to study these perfectly launched, ideal rifle bullets rather than dealing mathematically with the definition and physical effects of the many flaws that might occur with real bullets fired from actual rifles. However, we also make these simplifying assumptions here because: 1) they make our studies easier to perform and to understand; and, 2) the study of this idealized case is actually the stronger form of analysis in this instance. That is to say, this new Coning Theory does *not* rely upon the presence of small imperfections in the flight of the bullet as do, for examples, the "Tri-Cyclic Theory for Missiles Having Slight Configurational Asymmetries" of J. D. Nicolaides of BRL **[7]** and the x- and y-spirals theorized much earlier by Dr. Franklin W. Mann **[8]**.

III. Method of Studying the Coning Motion

To determine the character of the coning motion, one can examine streams of digital flight simulation data values calculated on small time intervals in a 6-DOF simulation of the flight of a perfectly launched, ideal example of our selected 30-caliber target-rifle bullet, the 168 grain Sierra *MatchKing*. We have the necessary aeroballistic coefficient data to perform these calculations for this bullet, at least in the guise of its substantially identical ancestor, the 168 grain Sierra *International* bullet **[9]**. These aeroballistic coefficients are tabulated as functions of the Mach number of the bullet, the ratio of the speed of the bullet through the air to the velocity of propagation of acoustic pressure waves through that atmosphere at the ambient temperature. We use a rather dense, dry sea-level ICAO standard atmosphere at 15 degrees Celsius (59 degrees Fahrenheit) throughout these studies in order to assure that reasonably large aerodynamic effects will be available for study. Our simulated rifle bullet is fired horizontally through a uniform 10 mile per hour crosswind approaching from 9:00 o'clock. The barrel of our assumed target rifle is chambered in 308 Winchester and is rifled at a right-hand twist rate of 12 inches per turn. The muzzle velocity used is 2800 feet per second.

Computed *bullet drift* and *bullet drop* data streams reported on 0.2 millisecond time centers were analyzed to determine the *horizontal* and *vertical* components, respectively, of the precession-rate *coning motion* of the bullet's CG as seen in an *earth-fixed coordinate system*. In addition to the coning and nutation motion of the CG, the *horizontal drift* (in inches) includes any crosswind-drift, horizontal Coriolis effect, horizontal aerodynamic jump components, and spin-drift. And the *vertical drop* (also in inches) below the projected axis of the bore, while mostly due to the acceleration of gravity, also includes the effects of any vertical crosswinds, the vertical component of the total aerodynamic force acting on the bullet, the vertical Coriolis effect, as well as any vertical direction aerodynamic jump (angular deflection) experienced by the bullet. Gyroscopic precession (coning) and nutation are the only possible periodic modulations of these data streams. Each of the others is a secular (non-periodic) effect. All of these analytic effects are boiled together in these two streams of *non-analytically produced* uniform-time-series data values as output from the simulator.

A time-symmetric, unit power, low-frequency-passing, *digital filtering technique* was employed in the temporal domain to remove all modulations from the two data streams, except for the extraneous low-rate secular variations noted above, without time-shifting or distorting the amplitudes of the remaining data. The low-pass filter was designed to have a sharp cut-off at a frequency matching the gyroscopic precession rate of the spinning bullet in one particular selected early portion of its flight so that all precession and nutation-rate modulation would be removed. The low-pass-filtered data arrays were then *subtracted*, point by point, from the original arrays of data samples calculated by the 6-DOF simulator. The data remaining after this procedure contained only the unmodified precession-rate (and higher-frequency) modulations of the path of the CG.

The period $\mathbf{T_2}$ (initially 15.4 milliseconds, here) of the coning motion of the spinning bullet (i.e., the reciprocal of $\mathbf{f_2}$, its slow-mode precession rate in hertz) was used in the design of a time-symmetric (non-causal), unit-power digital filter to extract the $\mathbf{f_2}$-rate (65 hertz) modulation from each of the two uniform-time-series data streams. An *equally weighted running mean* was selected for the type of *digital filter* to be used. It spanned a time interval of:

$$2*n*\Delta t \approx T_2 \geq 15.4 \text{ milliseconds} \qquad (1)$$

where **n** is a small positive integer (**n ≥ 38**) and **Δt** is the fixed data sample interval (0.2 msec). At each position of the moving filter, the low-pass-filtered average value was subtracted from the original data sample aligned with the center of the (**2*n+1**)-point running mean. The first and last **n** data points were not available as filtered values due to end-effects of the filter operator itself. The remaining data stream contained any modulating frequencies of $\mathbf{f_2}$ hertz and higher. The details of this procedure are included here to assist others in repeating this experiment using their own data sets.

For convenience of analysis, the pointing directions of the right-hand spinning bullet's spin-axis, as tabulated in orthogonal pitch and yaw angles for each sample time in the data streams, are converted into *polar coordinates* centered on the *apparent wind direction*. Examination of these four resulting tabulated data streams yields the following observations:

> *The residual precession-rate modulation of these two drift and drop data streams shows that the periodic motion of the bullet's CG matches the expected amplitude (**r**) of the coning motion, but that the CG of the bullet rotates 180 degrees out of phase with the motion of the bullet's spin-axis as seen in "wind axes" plots*.

This relative phasing of the synchronous coning motions of the CG and of the spin-axis can only be consistent with an *inwardly angled* orientation of the bullet's nose that, in turn, could occur only if the CG of the bullet were located behind a "crossing point" (i.e., a cone apex) moving along the mean trajectory ahead of the bullet, as shown earlier in **Fig. 1**. In this simulated flight, the coning motion is initiated by having the perfectly launched, non-coning, simulated bullet encounter a constant left-to-right 10 mile-per-hour horizontal crosswind immediately after launch. A nutation-rate wobbling motion due to suddenly hitting the crosswind is superimposed upon the precession-rate (65 hertz) coning motion. This high-rate (311 hertz) nutating motion damps to imperceptibility after 5 or 6 additional coning cycles for our example bullet and does not produce any noticeable additional CG motion. We designed a similar digital filter to isolate any modulation at the nutation rate, but none could be detected.

Wind-axes plots of all precession-rate coning motions from 6-DOF simulator runs show circular, or at most slowly inward or outward spiraling, centered coning motions completely lacking any hint of ellipticity. The centers of the circular or spiraling pitch- and yaw-coordinate values in the observed wind-axes plots consistently indicate that the coning bullet's spin-axis always revolves about the *instantaneous apparent wind direction*, which further indicates that the axis of the bullet's coning motion always points *directly into the apparent wind*.

IV. Aerodynamic Forces Acting on the Bullet

For certain bullets **[10]**, we have tables of aeroballistic coefficients as functions of the Mach number of the bullet's airspeed in flight that allow us to calculate the total aerodynamic force **F** experienced by the bullet at any point in its flight in terms of its rectangular components, the drag force $\mathbf{F_D}$ and lift force $\mathbf{F_L}$, as functions of the airspeed **V** of the bullet, the density **ρ** of the atmosphere, and the bullet's angle-of-attack **α**:

$$\mathbf{F_D = q*S*CD} \qquad \mathbf{(2)}$$

$$\mathbf{F_L = q*S*Sin(\alpha)*CL\alpha} \qquad \mathbf{(3)}$$

then as a rectangular vector-summing relationship:

$$\mathbf{F = F_D + F_L} \qquad \mathbf{(4)}$$

Variations in the density **ρ** of the local atmosphere are handled by incorporating that variable directly into the formulation for the dynamic pressure **q**. Variation in the elasticity of the local atmosphere with ambient temperature, and hence variation in the "speed of sound," is handled by tabulating the aeroballistic coefficients as functions of the bullet's Mach number instead of its airspeed **V**. Our example Spitzer-style rifle bullet is typical in that its *transonic instability* limits our study to airspeeds above about Mach 1.2 where steady flight can be maintained. In the bullet speed range of interest here (above Mach 1.2), the supersonic drag is not really quite proportional to the square of velocity, as incorporated into the formulation of the (subsonic type) dynamic pressure **q**. Instead, the bullet's supersonic drag is proportional to the 3/2-power of its airspeed, and this difference explains much of the variance in the tabulated drag coefficients as functions of Mach number.

The drag force $\mathbf{F_D}$ acts in a *downwind direction* along the direction of relative motion of the undisturbed local air-mass as seen from the moving bullet; i.e., in the direction of movement of the apparent wind $\mathbf{W_A}$ approaching the bullet. The *direction* of the drag force $\mathbf{F_D}$ is *independent* of the *orientation* of the bullet in flight. The *magnitude* of the bullet's drag coefficient **CD** is found from the sum of a tabulated primary "zero yaw" drag $\mathbf{CD_0}$ function of Mach number and an additive adjustment table of coefficients $\mathbf{CD_{\delta 2}}$ to be multiplied by $\boldsymbol{\delta}^2$ (the square of the sine of $\boldsymbol{\alpha}$, the angle-of-attack of the bullet) before summing. The lift force $\mathbf{F_L}$ is defined to act *perpendicularly* to the drag force $\mathbf{F_D}$ and is directed, in this case, toward the axis of the coning motion of the bullet. The coefficient of lift $\mathbf{CL\alpha}$ is tabulated as a function of Mach number, but is itself independent of the angle-of-attack $\boldsymbol{\alpha}$ for our particular example rifle bullet. The "roll orientation" of the lift force vector $\mathbf{F_L}$ for some non-zero angle-of-attack $\boldsymbol{\alpha}$ is completely determined by the instantaneous orientation of the *plane* that contains the spin-axis of the bullet and the "eye" of the apparent wind. The "eye of the wind" is an old nautical expression for the exact *direction* from which the apparent wind is blowing at any instant.

From the study of the *statics of rigid bodies*, we can state the following *three theorems* about the complete system of aerodynamic forces acting on the free-flying bullet at any instant:

> **A**. Disregarding any possible aerodynamic effect of the spinning of the bullet and treating the bullet as a solid rigid body, we can sum the entire system of aerodynamic pressures and friction forces acting over the whole surface of the bullet into a *single*, total aerodynamic force **F** acting at the one point on the surface of the bullet that uniquely produces the exact instantaneous overturning moment **M** that is also being experienced by the bullet. [There are apparently no aerodynamic spin effects that apply for a bullet flying with a small angle-of-attack $\boldsymbol{\alpha}$.]

B. Because 1) the shape of the bullet is effectively a closed-ended, axisymmetric "solid of revolution," 2) the aerodynamic force **F** is a line-vector that produces a torque, and 3) the bullet is a rigid body; we can translate this total aerodynamic force vector **F** from the surface of the bullet, near its nose, along the line-of-action of the line-vector **F**, that must intersect the bullet's spin-axis by symmetry, to a center-of-pressure CP lying on that spin-axis. [The force **F** still produces the same moment **M** when it is applied at the axial CP of the bullet.]

C. We can once again translate the force vector **F** rearward from the CP along the spin-axis to the center-of-gravity CG of the bullet if we also separately consider the overturning moment **M** as being produced by the resulting *force couple* (**F**, **-F**) acting on the whole bullet as a rigid body, but considered as a torque vector acting about the CG of the bullet. [The original force **F** of the couple always remains acting through the CP of the bullet, while the added force **–F**, completing the couple, acts at the CG of the bullet exactly compensating the newly-added *translated* force **F** now also acting at the CG.]

By this procedure, we "de-couple" the translational and overturning aerodynamic effects so that these can be analyzed separately. The location of the CP might well migrate along the spin-axis of our example bullet as the flight progresses and the cone angle gradually increases with ongoing time-of-flight. In addition, we should remember that changes over time in the total aerodynamic force **F**, together with its lift $\mathbf{F_L}$ and drag $\mathbf{F_D}$ component forces (which drive the orbital coning motion of the CG) and its associated overturning moment **M** (which drives the Tri-Cyclic precession and nutation motions of the spin axis), always remain *perfectly synchronized*. Of course, this is because they are always just different manifestations of the common source vector **F**. [We will consider the overturning moment **M** in later sections.]

V. Forces Driving the Coning Motion

If the CG of the bullet is to orbit about the mean trajectory at its gyroscopic precession-rate $\boldsymbol{\omega_2}$ as an *isotropic harmonic oscillation*, it must move as if the bullet were subject to some hypothetical, radially symmetric, centripetal, linearly proportional restoring force $\mathbf{F_R}$ of the form given in Hooke's Law as:

$$\mathbf{F_R = -k_R * r} \qquad \textbf{(5)}$$

Furthermore, from Newton's *Second Law of Motion*, for a body moving in a *circular orbit* at this *precession rate* $\boldsymbol{\omega_2}$ in such a force field, this unspecified centripetal restoring force $\mathbf{F_R}$, whatever its source, must also equal:

$$\mathbf{F_R = (Bullet\ Mass)*(Centripetal\ Acceleration\ for\ a\ Circular\ Orbit)}$$

$$\mathbf{F_R = m*(-v^2/r) = -m*(\boldsymbol{\omega_2}*r)^2/r = -m*\boldsymbol{\omega_2}^2*r = -k_R*r} \qquad \textbf{(6)}$$

More precisely, the isotropic coning motion of the rifle bullet is a type of "torsional" harmonic oscillation in cone angle $\boldsymbol{\alpha}$ about a cone apex that remains at a relatively fixed distance **D** ahead of the CG of the bullet in flight. One could envision the CG of the bullet being affixed as a "point mass" at the tip of a "massless" fly-rod of length **D**. Earlier analysis **[3]** had the CG of the bullet moving in response to a simple radial restoring force, the aerodynamic lift force $\mathbf{F_L}$ alone, perpendicular to the apparent wind, with the drag force $\mathbf{F_D}$ *contributing nothing* toward driving the coning motion.

We can define a vector **D**, giving the position of the CG of the bullet relative to the apex of the cone in any suitable coordinate system by the vector relationship:

$$\mathbf{D = R_{CG} - R_{Apex}} \qquad \textbf{(7)}$$

Then, as diagrammed in **Fig. 2** above, we can formulate an aerodynamic torque vector $\boldsymbol{\Gamma_C}$, driving the torsional coning oscillation about the apex of the cone, as the vector cross-product:

$$\boldsymbol{\Gamma_C} \mathbf{= D\ x\ F = D\ x\ F_D + D\ x\ F_L}$$

or, in *magnitudes*

$$\mathbf{\Gamma_C = D{*}F{*}Sin(\alpha{+}\beta) = D{*}F_C} \qquad \mathbf{(8)}$$

where the magnitude of the *coning force* $\mathbf{F_C}$ is the size of the aerodynamic force component *perpendicular* to **D** and thus directly available to drive the coning motion. The angle between the vectors **D** and **F** is $\boldsymbol{\alpha+\beta}$, where $\boldsymbol{\beta}$ represents the small angle whose tangent is the lift-to-drag ratio $\mathbf{F_L/F_D}$ of the bullet flying through the specified atmosphere at airspeed **V** and with angle-of-attack $\boldsymbol{\alpha}$:

$$\mathbf{\beta = Tan^{-1}[F_L/F_D] = Tan^{-1}[(CL\alpha/CD){*}Sin(\alpha)]} \qquad \mathbf{(9)}$$

Thus, the angle $\boldsymbol{\beta}$ will normally exceed the cone angle $\boldsymbol{\alpha}$ by a significant amount.

If the aerodynamic driving force $\mathbf{F_C}$ is perpendicular to the direction of **D**, then from trigonometry its component forces project in this perpendicular direction as:

$$\mathbf{F_C = F{*}Sin(\alpha{+}\beta) = [F{*}Sin(\beta)]{*}Cos(\alpha) + [F{*}Cos(\beta)]{*}Sin(\alpha) = F_L{*}Cos(\alpha) + F_D{*}Sin(\alpha)} \qquad \mathbf{(10)}$$

For these small coning angles ($\boldsymbol{\alpha < 0.10}$ radians = 5.7 degrees), we can approximate:

$$\mathbf{Cos(\alpha) \approx 1.00}$$

Then, after this simplification, and substitution of the aeroballistic expressions from **Eq. 2** and **Eq. 3** for the components of the force **F** into the expression for $\mathbf{F_C}$ in **Eq. 10** above, we see an interesting and fundamental relationship defining the *magnitude* $\mathbf{\{F_C\}}$ of the coning force $\mathbf{F_C}$ available to drive the coning motion:

$$\mathbf{\{F_C\} = q{*}S{*}Sin(\alpha){*}[CL\alpha{+}CD]} \qquad \mathbf{(11)}$$

So, the drag force $\mathbf{F_D}$ *does* contribute somewhat (about 10 to 20 percent for small coning angles in most cases) toward driving the coning motion of the rifle bullet along with the more direct contribution from the smaller lift force $\mathbf{F_L}$.

Since **Sin(α)**, the trigonometric sine of the half-cone-angle **α**, can also be expressed geometrically as the ratio **r/D**, and adopting our negative sign convention for a centripetal force, we can put this expression into the form of a Hookean "restoring force" by those changes:

$$F_C = -q*S*Sin(\alpha)*[CL\alpha+CD] = -[q*S*(CL\alpha+CD)/D]*r = -k_C*r \qquad (12)$$

Furthermore, if the aerodynamic coning force F_C on the bullet is actually to provide the hypothetical centripetal force F_R necessary to maintain this circular harmonic orbit, then at any given time these two force constants k_R (from **Eq. 6**) and k_C above, must be equal to each other (at least for **α < 5.7** degrees), so that, solving for the cone apex distance **D**, we have another important flat-firing relationship:

$$D = q*S*(CL\alpha+CD)/(m*\omega_2^2) \qquad (13)$$

and, the coning radius **r** can then be computed as:

$$r = D*Sin(\alpha) = q*S*Sin(\alpha)*(CL\alpha+CD)/(m*\omega_2^2) \qquad (14)$$

Thus, we have expressions for the cone apex distance **D** and the coning radius **r** as functions of several slowly varying aeroballistic parameters. The cone apex distance **D** and radius **r** are fundamental parameters describing the bullet's coning motion. The coning bullet adjusts its apex position, and hence its apex distance **D**, as its aeroballistic parameters change slowly during the flight. The cone apex distance **D** starts out at about **1.25 inches** (or four calibers) at launch for our example 30-caliber rifle bullet and gradually increases to about **3 inches** (or ten calibers) at 900 yards downrange. The coning radius **r** is **0.10*D** for **α** = 0.10 radians (5.7 degrees), so **r** is seldom much larger than about **0.3 inches** (or one caliber). The distance **D**, from the apex of the cone to the CG of the bullet, serves effectively as a lever arm, converting the forces driving the coning motion into a net torque Γ_C about the cone apex driving the "torsional" harmonic oscillation of the bullet. The cone angle **α** itself, and its corresponding coning radius **r**, are *unconstrained* in the precession-rate oscillation and are thus available to vary in accommodating any changes in flight conditions that may be encountered by the bullet.

As the bullet slightly increases its coning angle **α** upon encountering a change in the apparent wind direction, it can only selectively *increase* its orbital radius **r** in in accordance with **Eq. 14**, above, to accomplish the necessary cone-axis re-orientation. Unusually in physics, the period of oscillation ($\mathbf{T_2 = 1/f_2 = 2\pi/\omega_2}$) for a harmonic oscillator is *completely independent* of variations in the amplitude of its oscillation (i.e., variations in **α** and in **r** in this case). This amplitude independence would seem rarer still, except that upon analysis many different types of mechanical vibrations turn out to be either true or slightly non-linear versions of harmonic oscillation. The coning rate $\boldsymbol{\omega_2}$ and cone apex distance **D** are fixed independently by bullet spin-rate and other aeroballistic conditions (but *not* including the cone angle **α**). Thus, only the independently determined amplitude of the oscillation in **α** (or, equivalently, in **r**) is available for variation in response to changes in crosswinds.

VI. Mathematics of the Coning Motion

The oscillation in cone angle **α** about the cone apex can properly be described mathematically in terms of a *complex cone angle* **α(t)**, having *real* pitch **φ(t)** and *imaginary* yaw **θ(t)** orthogonal attitude angle components as functions of ongoing time **t**:

$$\boldsymbol{\alpha(t) = \varphi(t) + i*[\theta(t) - \gamma]} \qquad \textbf{(15)}$$

Here, **γ** is a non-negative, leftward, angular yaw offset of the incoming direction of the apparent wind $\mathbf{W_A}$ from the origin while the bullet is experiencing a left-to-right crosswind. No similar pitch offset is shown because of the definition of the origin direction of the standard "wind axes" plots (being always in the **+V** direction) and because we are not studying the effects of vertical crosswinds here. To the extent that the "pitching over" of the bullet follows the change in flight path angle **Φ** (the tangent to the trajectory) during the flight, this change in pitch attitude is "invisible" in wind-axes plots.

Making use of a torsional version of Newton's *Second Law of Motion* and **Eq. 8** above, we can write:

$$\mathbf{\Gamma_C = I_C*d^2\alpha/dt^2 = D*F_C} \qquad \textbf{(16)}$$

Substituting our previous expressions for the magnitudes of $\mathbf{I_C}$ (formulated as equal to $\mathbf{m*D^2}$, considering the bullet as a point mass), **D** (from **Eq. 13**), and $\mathbf{F_C}$ (from **Eq. 12**), and invoking the same small-angle approximation that we use in reducing a similar expression to the form of Hooke's law when treating the small-amplitude motion of a pendulum as simple harmonic motion, we find that:

$$\mathbf{d^2\alpha/dt^2 = -(\omega_2)^2 * \alpha(t)} \qquad \mathbf{(17)}$$

A typical solution in real and imaginary parts for this second-order differential equation in complex $\boldsymbol{\alpha}\mathbf{(t)}$ can be expressed as an orthogonal pair of the well-known relationships for harmonic oscillation at the precession rate $\boldsymbol{\omega_2}$:

$$\mathbf{\varphi(t) = K_0{*}Cos(\omega_2{*}t + \xi_0)}$$

$$\mathbf{\theta(t) = K_0{*}Sin(\omega_2{*}t + \xi_0)} \qquad \mathbf{(18)}$$

with $\mathbf{(K_0, \xi_0)}$ and $\mathbf{(K_0, \xi_0\text{-}\pi/2)}$ taken as arbitrary constants of the four integrations.

These two parametric equations, with time **t** as the independent variable, mathematically describe an isotropic *circular clockwise coning motion* of the spin-axis of the bullet at the slow-mode gyroscopic precession rate $\boldsymbol{\omega_2}$ and with an initial angular radius $\mathbf{K_0}$, centered about the apparent wind direction (**0**, **-i***γ), as seen in the traditional "wind axes" plot for this example and shown in **Fig. 3**, below. This exercise shows how the observed isotropic coning motion of the CG of the bullet can be derived from the driving forces as these have been formulated above. This coning motion can be described as the CG of the bullet orbiting in a clockwise circular path about its mean trajectory at a cyclic rate determined by the rate of gyroscopic precession $\boldsymbol{\omega_2}$ of the bullet's spin-axis.

As long as the spin-rate of a gyroscope remains nearly the same, and the overturning torque remains nearly constant, the rate $\boldsymbol{\omega_2}$ of the stable slow-mode precession of the gyroscope will also remain nearly constant (once established, and absent any $\boldsymbol{\omega_1}$-rate undamped fast-mode nutating motion) and the spin-axis of the gyroscope will maintain a nearly constant angle $\boldsymbol{\alpha}$ with its neutral axis of precession (i.e., the direction of the approaching apparent wind). The vector rate of gyroscopic precession $\boldsymbol{\omega_2}$ is related to the angular momentum vector **L** of the spinning bullet and to the overturning moment vector **M** acting on the bullet by the *magnitude* of the *vector cross-product* **[6]**:

$$\boldsymbol{\omega_2 \text{ x } L = M = q*S*d*Sin(\alpha)*CM\alpha} \qquad \textbf{(19)}$$

where the angle between the vectors $\boldsymbol{\omega_2}$ (the approaching wind direction) and **L** (the bullet's spin-axis direction) is just the angle-of-attack **α** (and also the half-cone-angle **α**). The overturning moment coefficient **CMα** itself comprises a tabulated primary Mach-dependent function $\mathbf{CM_0}$ plus a tabular *negative* corrective coefficient $\mathbf{CM_{\delta 2}}$ function multiplied by $\boldsymbol{\delta^2 = Sin^2(\alpha)}$ and algebraically summed into the coefficient **CMα** as a function of Mach number and angle-of-attack.

Note that **Sin(α)** appears as a factor in the *magnitudes* on both the left and right sides of **Eq. 19** and, thus, divides out *for non-zero angles-of-attack* so that the magnitude of the precession rate $\boldsymbol{\omega_2}$ (in radians per second) can be calculated from:

$$\boldsymbol{\omega_2 * L * Sin(\alpha) = q*S*d*Sin(\alpha)*CM\alpha}$$

$$\boldsymbol{\omega_2 = q*S*d*CM\alpha/L} \qquad \textbf{(20)}$$

Also notice that the pseudo-regular precession rate $\boldsymbol{\omega_2}$ is *not directly dependent* on the amplitude of the cone angle **α**. The variation of the overturning moment coefficient **CMα** with angle-of-attack **α** is quite small for the small **α**-angles considered here. The angular momentum **L** of the spinning bullet is the product of its moment of inertia **Ix** about its spin-axis and its spin-rate **ω (ω = 2*π*p)**. The spin-rate of the bullet **p** slows *only very gradually* in flight in accordance with an aeroballistic spin-damping coefficient.

VII. Wind Shift Effects

Whenever the coning bullet encounters a new wind **W**, its massless cone axis *can* and necessarily *does* instantly move so as to point directly into the new apparent wind $\mathbf{W_A}$, controlled by the three-dimensional vector relationship:

$$\mathbf{W_A = W - V} \tag{21}$$

The apparent wind $\mathbf{W_A}$ is just the true wind vector **W** translated into a coordinate system moving at velocity **V** along with the bullet.

Envision for a moment a horizontally fired, perfectly launched, spin-stabilized, ideal rifle bullet that has just emerged from the muzzle blast cloud and has not yet begun any actual coning motion. If a steady crosswind **W**, of much slower speed than **V**, is blowing horizontally from 9:00 o'clock (i.e., from left-to-right) across the trajectory, this non-coning bullet will experience an immediate small change in the direction of the approaching apparent wind vector, from straight ahead over to just leftward of straight ahead, by a small, *inherently non-negative* angle-of-attack $\boldsymbol{\gamma}$, given in radians *for this limited special case* by:

$$\boldsymbol{\gamma = \mathrm{Tan}^{-1}[W/V] \approx W/V} \tag{22}$$

This angular difference $\boldsymbol{\gamma}$ between the **–V** direction and the apparent wind direction $\mathbf{W_A}$ creates a small cross-track component of the aerodynamic drag force $\mathbf{F_D}$, of magnitude $\boldsymbol{\gamma}*\mathbf{F_D}$, which in turn causes the familiar horizontal drift of the rifle bullet fired through a crosswind as first formulated by Didion in 1859.

As a result of this left-to-right crosswind **W**, the horizontally fired rifle bullet, with its CP ahead of its CG, immediately begins experiencing a *nose-rightward* aerodynamic overturning moment **M**, a torque vector pointing *vertically downward* in this case, and of magnitude given by:

$$\mathbf{M = q*S*d*Sin(\boldsymbol{\gamma})*CM\boldsymbol{\alpha}} \tag{23}$$

As a *gyroscopic reaction* to the application of this moment **M** to the spinning bullet, the forward-pointing angular momentum vector **L** of the right-hand-spinning bullet will be just as strongly forced downward, *in the direction of the moment vector* **M**, and at a rate proportional to the magnitude of **M**, according to the vector relationship:

$$\mathbf{M} = \mathbf{dL/dt} \qquad \textbf{(24)}$$

This *gyroscopic relationship* is just the rotational analogue of Newton's *Second Law of Motion*. Of course, the nose of the bullet is pulled downward along with the bullet's angular momentum vector.

VIII. The Crosswind Aerodynamic Jump

As the coning motion is becoming established for the originally non-coning, horizontally fired bullet that is just encountering a purely horizontal left-to-right crosswind, the spin-axis direction will initially accelerate *rightward* and then *predominately downward* from its original orientation in the **+V**-direction. This is clearly shown in **Fig. 3**, the computer-generated wind axes plot from 6-DOF data provided by Bryan Litz.

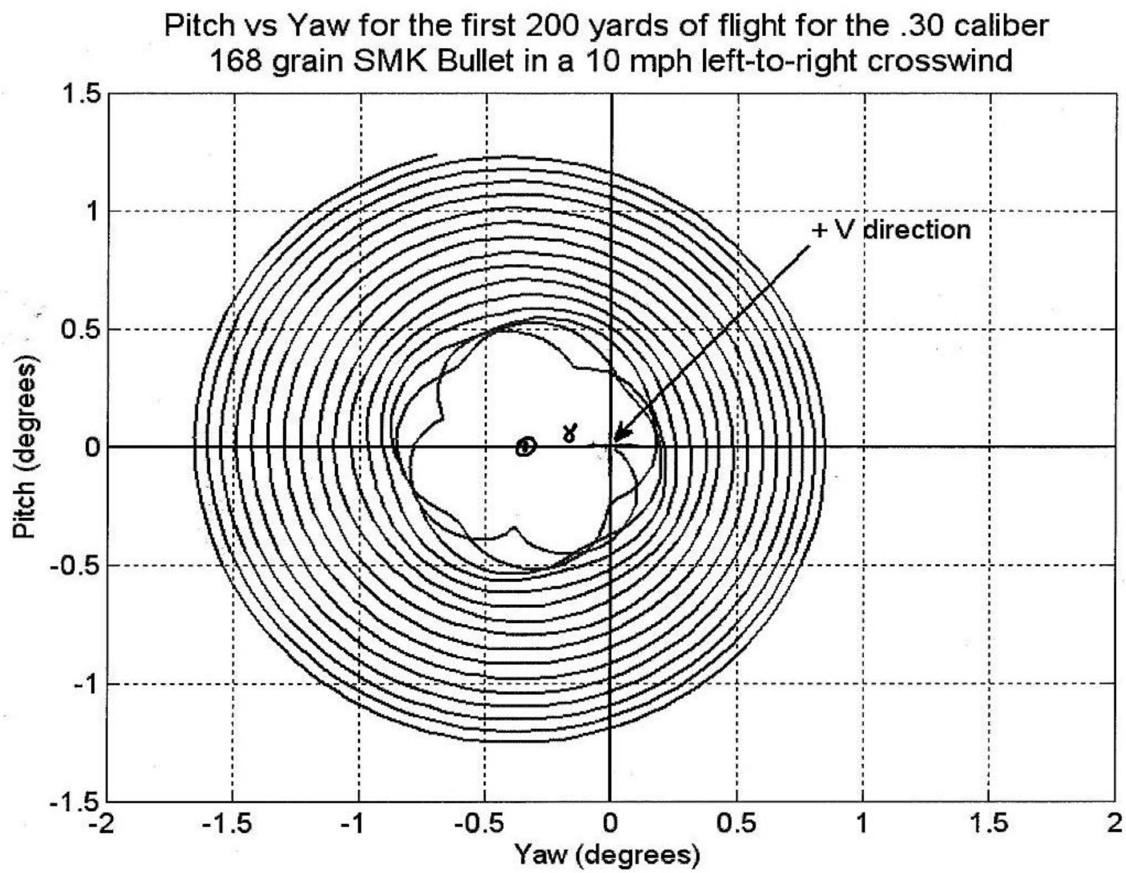

Fig. 3. Epicyclic Motion of Spin-Axis Direction (Provided by Bryan Litz)

*During the coning start-up period, with the initial movement of the spin-axis being rightward in the downwind direction of the crosswind **W**, the bullet's trajectory becomes permanently deflected slightly downward by the transient, vertically downward aerodynamic impulse **J** due to its momentarily nose-down attitude relative to the approaching windstream.*

After completion of the first *fast-mode* nutation cycle, the *uniformly rotating* total aerodynamic force vector **F** produces no further *net* deflection of the bullet's path; i.e., the precessing lift vector sums to zero over each successive cycle of nutation (or coning) motion.

Actually, the suddenly encountered crosswind **W** creates a non-zero fast-mode (nutation) arm of initial angular magnitude $\boldsymbol{\delta_0}$, as well as a slow-mode (precession) arm of magnitude $\boldsymbol{\gamma_0+\delta_0}$. Each arm rotates clockwise for our right-hand-spinning rifle bullet, but the initial "roll-orientations" (ξ_1 and ξ_2) of the two epicyclic arms are *oppositely directed*, so that the slow-mode precession arm ($\boldsymbol{\gamma_0+\delta_0}$) initially points rightward from the apparent wind direction ($\mathbf{0, -i}*\gamma_0$) along the **+yaw** axis, and the fast-mode nutation arm ($\boldsymbol{\delta_0}$) initially points back to the left in the **-yaw** direction. Their combined *epicyclic sum* must initially equal $\boldsymbol{\gamma_0}$, the offset angle due to the initial apparent wind at time **t = 0** when the bullet has just exited the muzzle, because the spin-axis of this perfectly launched bullet starts out pointing in the **+V** direction (toward the origin of the wind axes plot).

As shown in **Fig. 3** for our example bullet, the fast-mode nutations damp fairly rapidly, to insignificance after 5 or 6 slow-mode coning cycles. After completion of the first full, *relative* nutation cycle at $\mathbf{1/T_N = f_N}$ hertz, the steadily rotating aerodynamic forces acting on the coning bullet subsequently integrate out to zero net deflection of the trajectory. Certainly the rotating lift force attributable to the fast-mode ($\mathbf{f_N}$) nutation itself averages to a net of zero rather quickly.

The epicyclic motion of the bullet's spin-axis about the apparent wind direction in "wind axes" coordinates can be defined in terms of the complex cone angle function $\boldsymbol{\alpha}(\mathbf{t})$ as:

$$\boldsymbol{\alpha(t) = -i*\gamma_0 + (\gamma_0+\delta_0)*(Cos\xi_2 + i*Sin\xi_2) + \delta_0*(Cos\xi_1 + i*Sin\xi_1)} \qquad \mathbf{(25)}$$

These three vector terms are, respectively; **§1** the initial and nearly constant apparent wind offset angle $\boldsymbol{\gamma_0}$ for this example; **§2** the initial slow-mode, clockwise rotating coning motion arm of length $\boldsymbol{\gamma_0+\delta_0}$; and, **§3** the initial fast-mode, clockwise rotating nutation arm of length $\boldsymbol{\delta_0}$. These vectors are best envisioned as being summed "head-to-tail" in this sequential order. The apparent wind vector **§1** is always defined with respect to the **–V** direction. This **–V** vector points toward the origin of the "wind axes" coordinate system, but from behind the plot. ξ_1 and ξ_2 are the phase angles of the clockwise rotations of the fast and slow epicyclic arms, respectively. These angles are measured positive clockwise from the **+φ** (pitch) axis direction. The fast arm **§3** rotates clockwise at the nutation rate ω_1, while the slow arm **§2** rotates clockwise at the coning (or precession) rate ω_2. The "yaw of repose" β_R is too small to be seen at the scale of this plot.

The angular arguments $\xi_1(t)$ and $\xi_2(t)$ start out just as explained above and grow with ongoing time **t** at their respective rotation rates, so that:

$$\xi_1(t) = \omega_1 * t - \pi/2$$

$$\xi_2(t) = \omega_2 * t + \pi/2 \qquad (26)$$

The attitude angles **γ+δ** and **δ** also vary slowly with time **t** according to their respective slow-mode and fast-mode *damping factors*. For our somewhat dynamically unstable example bullet, the slow-mode coning angle slowly *increases* with ongoing time **t**.

The *vertical component* of this epicyclic motion at any time **t** during the initial relative nutation cycle, is given by the *real part* of the expression (**Eq. 23**) for **α(t)**:

$$Re\{\alpha(t)\} = (\gamma_0+\delta_0)*[\mathrm{Cos}(\omega_2*t + \pi/2)] + \delta_0*[\mathrm{Cos}(\omega_1*t - \pi/2)]$$

$$Re\{\alpha(t)\} = -(\gamma_0+\delta_0)*[\mathrm{Sin}(\omega_2*t)] + \delta_0*[\mathrm{Sin}(R*\omega_2*t)] \qquad (27)$$

From the initial condition of *rightward first motion*, we can set the time-derivative of this vertical component expression (**Eq. 27**) *initially equal to zero*, so that:

$$\gamma_0 + \delta_0 = R*\ \delta_0 = 4.79*\ \delta_0$$

$$\boldsymbol{\delta_0 = \gamma_0/(R - 1) = \gamma_0/3.79} \qquad \textbf{(28)}$$

Thus, the initial condition of rightward first motion of the spin-axis from the **+V** direction determines the initial ratio **1/(R – 1)** that fixes the size of the nutation $\boldsymbol{\delta_0}$ relative to the size of the apparent wind offset $\boldsymbol{\gamma_0}$. From the conditions established for this flight simulation run, we know that initially:

$$\boldsymbol{\gamma_0 = (14.67 \text{ fps})/(2800 \text{ fps}) = 5.24 \text{ milliradians}} \qquad \textbf{(29)}$$

$$\boldsymbol{\delta_0 = \gamma_0/3.79 = 1.38 \text{ milliradians}} \qquad \textbf{(30)}$$

$$\boldsymbol{\gamma_0+\delta_0 = 6.62 \text{ milliradians}} \text{ (in this example)} \qquad \textbf{(31)}$$

The vertical-direction aerodynamic impulse **J** is the time integral of the vertical component of the perpendicular aerodynamic force $\boldsymbol{F_C(\alpha,t)}$ over this first full fast-mode nutation cycle relative to the moving slow-mode arm, which spans the time required to establish a steady coning motion at the precession rate $\boldsymbol{\omega_2}$. The time $\boldsymbol{T_N}$ of completion of the first full *relative* nutation cycle is found (with **R = 4.79**, initially, in this example) from:

$$\boldsymbol{T_N = 2\pi/[(R - 1)*\omega_2] = T_2/3.79 = 4.06 \text{ milliseconds}} \qquad \textbf{(32)}$$

While the coning motion is establishing itself, the vertical (pitch) component of the total epicyclic angle-of-attack $\boldsymbol{\alpha(t)}$ is increasing in magnitude from zero to a maximum amplitude of **-1.35*γ** (pitch downward).

An impulse **J** (in pound-seconds) directly causes a vertically downward *change* in the linear momentum of the bullet, which can be well-approximated by use of a "linearizing" technique:

$$\boldsymbol{J = \int F_C(\alpha,t)dt = AVE\{F_C(\alpha,t)\}*T_N \approx F_C[AVE\{\alpha\}]*T_N} \qquad \textbf{(33)}$$

$$\boldsymbol{J = \Delta(mV_C) = m*\Delta V_C} \qquad \textbf{(34)}$$

where $\mathbf{\Delta V_C}$ is a negative, or downward, "kick" velocity in this example.

The time-average of the vertical component of the angle-of-attack **AVE{α}** can be evaluated by the definite integration of ***Re*{α(t)}** from zero to $\mathbf{T_N}$, i.e., over the first complete relative nutation cycle:

$$\mathbf{AVE\{\alpha\} = (1/T_N)\int\{[-(\gamma_0+\delta_0)/\omega_2]*Sin(\omega_2*t)d(\omega_2*t) + [\delta_0/(4.79*\omega_2)]*Sin(4.79*\omega_2*t)d(4.79*\omega_2*t)\}}$$

$$\mathbf{AVE\{\alpha\} = -(\gamma_0+\delta_0)*(3.79/2\pi)*[1 - Cos(2\pi/3.79)] + (3.79/4.79)*(\delta_0/2\pi)*\{1 - Cos[(4.79/3.79)*2\,\pi]\}}$$

$$\mathbf{AVE\{\alpha\} = -4.340\ mrad + 0.189\ mrad = -4.151\ milliradians} \qquad \mathbf{(35)}$$

Note from the slow-mode angular argument (**2π/3.79**) that the coning motion has progressed through **95 degrees** to the first inward-pointing epicyclic cusp at time $\mathbf{T_N}$ while the fast-mode argument $\mathbf{\xi_1(T_N)}$ has progressed through **455 degrees**:

$$\mathbf{\xi_1(T_N) = 4.79*95\ degrees = 360 + 95\ degrees = 455\ degrees} \qquad \mathbf{(36)}$$

Then

$$\mathbf{J \approx T_N*F_C(-4.15\ mrad) = T_N*q*S*Sin(-4.15\ mrad)*[CL\alpha + CD]}$$

$$\mathbf{J \approx (4.06\ msec)*(4.82\ lbs.)*(-.00415)*[2.85 + 0.32] \approx -2.58x10^{-4}\ pound\text{-}seconds} \qquad \mathbf{(37)}$$

The permanent downward angular deflection $\mathbf{A_J}$ of the trajectory of the bullet resulting from this downward cross-track "kick velocity" $\mathbf{\Delta V_C}$ (from **Eq. 34**) is given by:

$$\mathbf{A_J = Tan^{-1}[\Delta V_C/V] \approx \Delta V_C/V = J/(m*V)} \text{ (in radians)}$$

$$\mathbf{A_J \approx -2.58x10^{-4} \text{ pound-seconds}/2.09 \text{ pound-seconds} \approx -0.123 \text{ milliradians}} \qquad \mathbf{(38)}$$

Note that **m*V = 2.09 pound-seconds** is just the linear momentum of the bullet itself at the time it encountered the crosswind **W**.

The permanent, one-time-per-disturbance, downward angular deflection $\mathbf{A_J}$ of the entire remaining trajectory produces a downward displacement on the target that varies *linearly* with range to the target (minus about 2 yards). If the horizontal wind-drift due to our assumed constant 10 MPH crosswind from 9:00 o'clock in this example is the expected 3.0 inches *to the right* on a 200-yard target, and the bullet also strikes 0.887 inches *below* the center of the bullseye, the combined downward-sloping skew angle is 16.5 degrees below the horizontal across the face of the target. Since the horizontal wind deflection itself varies approximately with the *square* of the range to the target when firing through a constant crosswind, the downward left-to-right skew angle of the combined windage effects on the target due to this purely horizontal crosswind must "flatten out" *significantly* at longer ranges (and *steepen* at shorter ranges). In addition, since this vertical deflection effect is not quite as directly proportional to the strength of the horizontal crosswind as is the basic horizontal windage effect, this skew angle across the face of the target must also "flatten out" *very slightly* with increasing crosswind speeds.

This vertically downward (negative), angular deflection $\mathbf{A_J}$ of the bullet's trajectory, is one type of *aerodynamic jump* caused by encountering a purely horizontal crosswind, as it has been termed by Bob McCoy of BRL. McCoy **[11]** formulates this jump $\mathbf{J_A}$ as:

$$\mathbf{J_A = -[Ix/(m*d^2)]*(CL\alpha/CM\alpha)*(2*\pi/n)*(W/V)} \qquad \mathbf{(Eq.\ 12.98,\ McCoy)}$$

$$\mathbf{J_A = -[0.10854]*(2.85/2.56)*(2*\pi/38.96)*(14.67/2800) = -0.102 \text{ milliradians}} \qquad \mathbf{(39)}$$

$$\mathbf{J_A = 0.829*A_J} \text{ (our coning value)} \qquad \mathbf{(40)}$$

Had McCoy used **$(CL\alpha + CD)/CM\alpha$** as proposed herein, instead of just **$(CL\alpha/CM\alpha)$** in his formulation for $\mathbf{J_A}$, he would have found:

$$\mathbf{J_A = -\ 0.114\ milliradians = 0.922*A_J}\ \text{(the coning value)} \qquad \mathbf{(41)}$$

or a value 7.8 percent smaller than our calculated $\mathbf{A_J}$. This jump angle $\mathbf{J_A}$ corresponds to a skew angle of 15.3 degrees across the target. Modern benchrest competition rifles and across-the-course target rifles achieve the level of repeatability necessary to demonstrate quite clearly that this small vertical displacement *does occur* while flat-firing through purely horizontal near-surface crosswinds. Several available PRODAS 6-DOF runs for the quite similar 175-grain Sierra *MatchKing* bullet fired in military M118LR ammunition show a 200-yard skew angle of 16.9 degrees in these same conditions.

If the initial crosswind **W** had been blowing from right-to-left instead, the angular deflection of the trajectory would have been positive (upward) for our example right-hand spinning bullet. Additionally, similar one-time transient aerodynamic jumps will occur whenever an already-coning bullet encounters any significant change in the direction $\boldsymbol{\gamma}$ of the approaching apparent wind. We analyze a case here starting with a non-coning bullet simply for convenience.

If a bullet is launched with some non-zero initial pitch and (or) yaw attitude, a similar aerodynamic jump will occur very early in its flight, as well. Unfortunately, real rifle bullets routinely suffer "in-bore yaw" during firing and enter the airstream with a greatly enlarged first-maximum yaw angle after emerging into the atmosphere as given by Kent's Equation (Eq. 12.92, McCoy) **[12]**. A bullet's having a non-zero initial pitch rate and (or) yaw rate will also produce a similar type of one-time aerodynamic jump. In any event, the rotational orientation of the one-time *jump angle deflection* of the bullet's path will always be *rotated 90 degrees clockwise* (as seen from behind the bullet) from the roll-orientation of the *initial movement* of the spin-axis of the bullet. This 90-degree clockwise rotation is indicated in the complex plane by the initial factor of **i** (where $\mathbf{i^2 = -1}$) in the BRL formulation (Eq. 12.83, McCoy) **[13]** for this type of aerodynamic jump (or **–i** for our complex plane definition). The angular deflection of the trajectory caused by any of these types of initial aerodynamic jump effects gets established during the first fast-mode nutation cycle that takes place during the first 2 or 3 yards of travel for most rifle bullets. Perhaps this is one of the reasons why experienced benchrest competitors pay particular attention to any wind blowing across directly in front of their firing benches.

IX. Conclusions

We have shown that the bullet cones around at its gyroscopic precession rate in accordance with the Tri-Cyclic Theory, but with its nose angled *inward*, toward the mean trajectory. We posit that the incoming direction of the apparent wind seen by the moving bullet continually defines the orientation of the axis of this coning motion so that the spin-axis of the coning bullet always precesses around the instantaneous "eye of the apparent wind." We theorize that the spin-stabilized bullet in free flight acts exactly as would a gimbal-mounted gyroscope under similar conditions. We have shown that the coning motion of the spin-axis of the bullet is a pseudo-regular gyroscopic precession and that the CG of the bullet orbits in a circle around the mean trajectory "in sync" with this same precession of its axis. We found that higher-rate gyroscopic nutation does not produce significant bullet motion. The observed vertical deflection of rifle bullets fired through purely horizontal crosswinds has been shown numerically to be a one-time transient effect occurring during start-up of the coning motion upon first encountering that crosswind.

Mathematically, we can describe the coning motion of the spin-axis direction of a rifle bullet using complex notation as the vector sum:

$$\boldsymbol{\alpha(t) = (\beta_R + \gamma)*i + (\beta_R + \gamma + \delta)*\exp[i*\xi_2(t)] + (\beta_R + \delta)*\exp[i*\xi_1(t)]} \qquad \textbf{(42)}$$

This vector relationship defining the complex coning angle $\boldsymbol{\alpha(t)}$ is shown in **Fig. 4**, below.

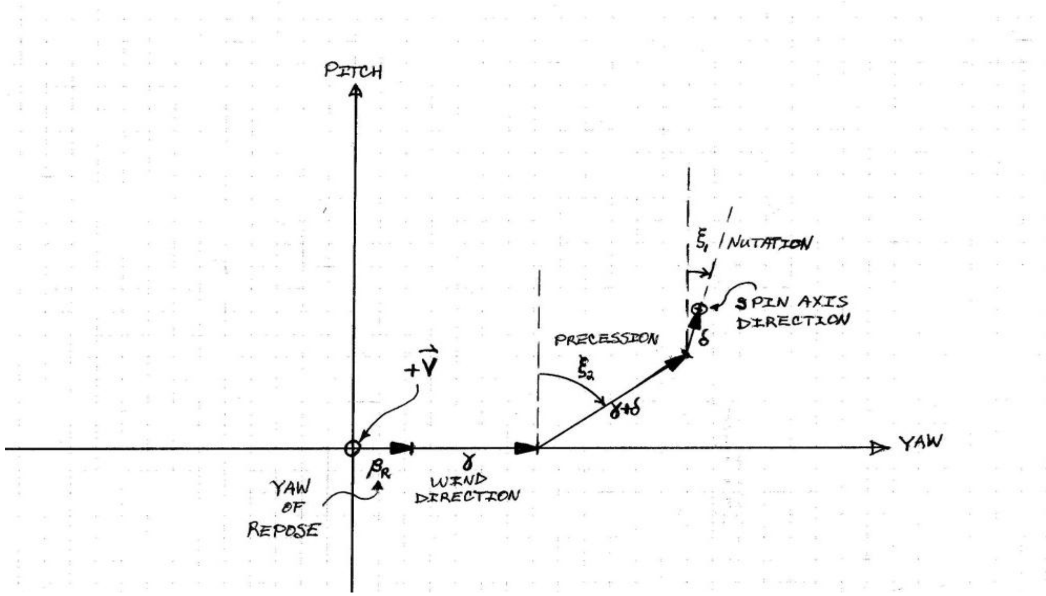

Fig. 4. Epicyclic Motions of Spin-Axis Direction for a Coning Bullet

Summary

This detailed new analysis of the physics behind the equations of motion describing the flight of a rifle bullet is entirely consistent with BRL's standard calculus-based formulation of these equations and with the mathematics currently implemented in existing 6-DOF flight simulation software. The major accomplishment of this effort is illuminating the dual nature of the bullet's coning motion—the circular orbiting of the CG of the bullet synchronized with the gyroscopic precession of its spin-axis. Only the primary aeroballistic forces, drag and lift, and the overturning moment are used in this new analysis of bullet motions. The minor aeroballistic forces and moments (e.g., Magnus force and moment, pitch-damping force and moment, and spin-damping moment, etc.) do not seem to be needed directly in the analytical explanation of all observable phenomena of modern rifle bullets flying essentially horizontally. The expected advantages of adopting this new *Coning Theory of Bullet Motions* are perceived to lie primarily in the teaching of exterior ballistics. Demystifying a subject always improves its pedagogy.

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