Covariant Mickelsson-Faddeev extensions of gauge and diffeomorphism algebras

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Abstract

We construct new extensions of current and diffeomorphism algebras in $N > 3$ dimensions, which are related to the Mickelsson-Faddeev algebra. The result is compatible with Dzhumadil'daev's classification of diffeomorphism cocycles. We also construct an extension of the current algebra in $N \geq 5$ dimensions which depends on the fourth Casimir operator.

1 Introduction

Higher-dimensional analogs of affine and Virasoro algebras have been known for a long time [7, 8, 14, 16, 18]. It is natural to ask whether the algebras that pertain to gauge and diff anomalies in three dimensions, in particular the Mickelsson-Faddeev (MF) algebra [6, 11, 12], can also be generalized to higher dimensions. The answer turns out to be affirmative, and is described in the present paper in a Fourier basis on the N-dimensional torus. The key step is to replace the three-dimensional delta function with an operator, which can be interpreted as an integral over a three-dimensional volume embedded in N-dimensional space. The resulting extensions are covariant in the sense that there is an intertwining action of N-dimensional diffeomorphisms.

Given a covariant extension of the current algebra for $\mathfrak{gl}(N)$, there is a standard procedure to construct an associated extension of the diffeomorphism algebra in N dimensions. We apply this construction to the covariant

MF extension, and relate the result to Dzhumadil'daev's classification of cocycles for the algebra of vector fields in N dimensions [5].

It is known that new gauge anomalies arise in all odd dimensions, which gives rise to a hierarchy of extensions of the current algebra. Well-defined extensions involving the n:th Casimir operator are expected to arise in all dimensions $N \geq 2n-3$. We explicitly describe the next element in this hierarchy, a covariant fourth-Casimir extension in $N \geq 5$ dimensions. This is presumably the simplest example of a non-abelian extension. There is some overlap between [4] and this part of the present work, but the covariantization which leads to well-defined extensions above the minimal dimension $N = 2n-3$ is new.

The final section contains a discussion on representations and possible relevance to physics.

2 Multi-dimensional affine algebra

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with generators J^a and totally antisymmetric structure constants f^{abc}. The brackets are given by $[J^a, J^b] = f^{abc}J^c$. Denote the symmetric Killing metric (proportional to the quadratic Casimir operator) by $\delta^{ab} = \mathrm{tr}\ J^a J^b$. It is not necessary to distinguish between upper and lower $\mathfrak{g}$ indices due to this metric; in contrast, the distinction is important for spacetime indices. The current algebra $\mathfrak{map}(M, \mathfrak{g})$ is the algebra of maps from a manifold M to $\mathfrak{g}$. We specialize to the N-torus $\mathbb{T}^N$ and expand all fields in a Fourier basis. The generators of $\mathfrak{map}(\mathbb{T}^N, \mathfrak{g})$ are $J^a(m) = \exp(im \cdot x)J^a$, where $m \equiv (m_0, m_1, ..., m_{N-1}) \in \mathbb{Z}^N$.

As is well known, $\mathfrak{map}(\mathbb{T}^1, \mathfrak{g})$ admits a central extension, the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$:

$$[J^a(m), J^b(n)] \quad = \quad if^{abc}J^c(m+n) + km\delta^{ab}\delta(m+n), \tag{1}$$

where $\delta(m)$ is the Kronecker delta. This extension is immediately generalized to $N > 1$ [7, 16]:

$$[J^a(m), J^b(n)] \quad = \quad if^{abc}J^c(m+n) + k^\mu m_\mu \delta^{ab}\delta(m+n), \tag{2}$$

where k^μ is a constant vector, e.g. $k^\mu = \delta^\mu_0$. This formulation is not covariant because there is a priviledged direction k^μ. To remedy this defect, note that the Kronecker delta in one dimension can be written as $\delta(m) = \frac{1}{2\pi}\int \exp(imx)dx$. The natural generalization to N dimensions is to replace the Kronecker delta with the curve operators

$$S^\mu(m) = \frac{1}{2\pi}\int \mathrm{e}^{im\cdot x} dx^\mu, \tag{3}$$

where the integral is taken over some curve embedded in $\mathbb{T}^N$. These operators satisfy the constraint

$$m_\mu S^\mu(m) = \frac{1}{2\pi i}\int d(\mathrm{e}^{im\cdot x}) \equiv 0. \tag{4}$$

The explicit form of $S^\mu(m)$ will not be important in the sequel.

The covariant form of the central extension (2) is defined by the relations

$$\begin{aligned} [J^a(m), J^b(n)] &= if^{abc}J^c(m+n) + k\delta^{ab}m_\rho S^\rho(m+n), \\ [J^a(m), S^\mu(n)] &= [S^\mu(m), S^\nu(n)] = 0, \\ m_\mu S^\mu(m) &\equiv 0. \end{aligned} \tag{5}$$

This algebra is covariant in the sense that it admits an intertwining action of general diffeomorphisms. Let $\mathfrak{vect}(\mathbb{T}^N)$ be the algebra of vector fields on $\mathbb{T}^N$, i.e. the algebra of infinitesimal diffeomorphisms. The generators $L_\mu(m) = -i\exp(im\cdot x)\partial/\partial x_\mu$ satisfy

$$[L_\mu(m), L_\nu(n)] = n_\mu L_\nu(m+n) - m_\nu L_\mu(m+n). \tag{6}$$

The following brackets complete the definition of the semi-direct product $\mathfrak{vect}(\mathbb{T}^N) \ltimes \mathfrak{map}(\mathbb{T}^N, \mathfrak{g})$:

$$\begin{aligned} [L_\mu(m), J^a(n)] &= n_\mu J^a(m+n), \\ [L_\mu(m), S^\nu(n)] &= n_\mu S^\nu(m+n) + k\delta^\nu_\mu m_\rho S^\rho(m+n). \end{aligned} \tag{7}$$

$J^a(m)$ transforms as a scalar density of weight +1 and $S^\mu(m)$ as a vector density.

3 Multi-dimensional Mickelsson-Faddeev algebra

The current algebra in three dimensions, $\mathfrak{map}(\mathbb{T}^3, \mathfrak{g})$, admits a different type of extension, the Mickelsson-Faddeev (MF) algebra [6, 11, 12]:

$$\begin{aligned} [J^a(\mathbf{m}), J^b(\mathbf{n})] &= if^{abc}J^c(\mathbf{m}+\mathbf{n}) + d^{abc}\epsilon^{ijk}m_i n_j A^c_k(\mathbf{m}+\mathbf{n}), \\ [J^a(\mathbf{m}), A^b_j(\mathbf{n})] &= if^{abc}A^c_j(\mathbf{m}+\mathbf{n}) + \delta^{ab}m_j\delta(\mathbf{m}+\mathbf{n}), \\ [A^a_i(\mathbf{m}), A^b_j(\mathbf{n})] &= 0, \end{aligned} \tag{8}$$

where Latin indices and boldface denote three-dimensional vectors: $\mathbf{m} = (m_1, m_2, m_3) \in \mathbb{Z}^3$. The totally anti-symmetric constant ϵ^{ijk} may be viewed as a tensor density of weight +1. Further, $A^a_\mu(\mathbf{m})$ are the Fourier components of the gauge connection, and $d^{abc} = \mathrm{tr}\ \{J^a, J^b\}J^c$ are the totally symmetric structure constants proportional to the third Casimir. Constancy of d^{bcd} implies the condition:

$$f^{abe}d^{ecd} + f^{ace}d^{bed} + f^{ade}d^{bce} = 0. \tag{9}$$

In analogy with the previous section, we observe that the three-dimensional Kronecker delta can be written as an integral over $\mathbb{T}^3$. In higher dimensions, we replace it by a volume operator, which is an integral over some 3-manifold embedded in $\mathbb{T}^N$:

$$S^{\mu\nu\rho}(m) = \frac{1}{V}\int \mathrm{e}^{im\cdot x}dx^\mu dx^\nu dx^\rho, \tag{10}$$

where V is the volume of the embedded manifold. $S^{\mu\nu\rho}(m)$ does not commute with diffeomorphisms when $N > 3$, but it does possess the crucial property $m_\rho S^{\mu\nu\rho}(m) \equiv 0$. Moreover, we replace

$$d^{abc}\epsilon^{ijk}A^c_k(\mathbf{m}) \Rightarrow A^{ab,\mu\nu}(m). \tag{11}$$

The multi-dimensional MF algebra is defined by the brackets:

$$\begin{aligned}
[J^a(m), J^b(n)] &= if^{abc}J^c(m+n) + m_\mu n_\nu A^{ab,\mu\nu}(m+n), \\
[J^a(m), A^{bc,\mu\nu}(n)] &= if^{abd}A^{dc,\mu\nu}(m+n) + if^{acd}A^{bd,\mu\nu}(m+n) \\
&+ d^{abc}m_\rho S^{\mu\nu\rho}(m+n), \\
m_\rho S^{\mu\nu\rho}(m) &= 0, \qquad (12)\\
A^{ab,\mu\nu}(m) &= A^{ba,\mu\nu}(m) = -A^{ab,\nu\mu}(m), \\
S^{\mu\nu\rho}(m) &= -S^{\nu\mu\rho}(m) = S^{\rho\mu\nu}(m). \qquad (13)
\end{aligned}$$

All other brackets vanish: $[J, S] = [A, A] = [A, S] = [S, S] = 0$.

It is readily verified, using the relations (9), that (12) is a well-defined Lie algebra in any number of dimensions $N \geq 3$. Moreover, there is an intertwining action of $\mathfrak{vect}(\mathbb{T}^N)$, under which $J^a(m)$, $A^{ab,\mu\nu}(m)$ and $S^{\mu\nu\rho}(m)$ transform as densities of weight $+1$ and the appropriate tensor types. Explicity,

$$\begin{aligned}
[L_\mu(m), J^a(n)] &= n_\mu J^a(m+n), \\
[L_\mu(m), A^{ab,\nu\rho}(n)] &= n_\mu A^{ab,\nu\rho}(m+n) + \delta^\nu_\mu m_\sigma A^{ab,\sigma\rho}(m+n) \\
&+ \delta^\rho_\mu m_\sigma A^{ab,\nu\sigma}(m+n), \qquad (14)\\
[L_\mu(m), S^{\nu\rho\sigma}(n)] &= n_\mu S^{\nu\rho\sigma}(m+n) + \delta^\nu_\mu m_\tau S^{\tau\rho\sigma}(m+n) \\
&+ \delta^\rho_\mu m_\tau S^{\nu\tau\sigma}(m+n) + \delta^\sigma_\mu m_\tau S^{\nu\rho\tau}(m+n),
\end{aligned}$$

To my knowledge, the multi-dimensional MF algebra (12) is new.

The original MF algebra (8) is recovered as follows. In three dimensions, the closedness condition $m_\rho S^{\mu\nu\rho}(m) = 0$ has the unique solution

$$S^{\mu\nu\rho}(m) = \epsilon^{\mu\nu\rho}\delta(m), \tag{15}$$

and

$$A^a_\mu(m) = \epsilon_{\mu\nu\rho} d^{abc} A^{bc,\nu\rho}(m) \tag{16}$$

transforms as a connection.

4 Multi-dimensional Virasoro algebra

Consider the current algebra $\mathfrak{map}(\mathbb{T}^N, \mathfrak{gl}(N))$, with brackets

$$[T^\mu_\nu(m), T^\rho_\sigma(n)] \;=\; \delta^\rho_\nu T^\mu_\sigma(m+n) - \delta^\mu_\sigma T^\rho_\nu(m+n). \tag{17}$$

As usual, the current algebra generators transform as scalar densities of weight $+1$:

$$[L_\mu(m), T^\nu_\sigma(n)] \;=\; n_\mu T^\nu_\sigma(m+n). \tag{18}$$

There is an embedding $\mathfrak{vect}(\mathbb{T}^N) \hookrightarrow \mathfrak{vect}(\mathbb{T}^N) \ltimes \mathfrak{map}(\mathbb{T}^N, \mathfrak{gl}(N))$, defined by

$$L'_\mu(m) = L_\mu(m) + m_\nu T^\nu_\mu(m). \tag{19}$$

It is readily verified that the $L'_\mu(m)$ satisfy (6) if the unprimed $L_\mu(m)$ do so, and that the $T^\mu_\nu(m)$ transform as tensor densities:

$$[L'_\mu(m), T^\nu_\rho(n)] = n_\mu T^\nu_\rho(m+n) + \delta^\nu_\mu m_\sigma T^\sigma_\rho(m+n) - m_\rho T^\nu_\mu(m+n).$$

The embedding (19) is useful to create diffeomorphism analogs of current algebra concepts. E.g., given the $\mathfrak{vect}(N)$ module of scalar fields and a $\mathfrak{gl}(N)$ representation R, it produces the module of tensor densities of type R.

We will use the embedding to construct $\mathfrak{vect}(\mathbb{T}^N)$ extensions from $\mathfrak{map}(\mathbb{T}^N, \mathfrak{gl}(N))$ extensions. To construct the diffeomorphism analog of the affine cocycle in section 2, we need the Killing metric for $\mathfrak{gl}(N)$. For a generic representation, the second Casimir must be of the form

$$\delta^{\mu\rho}_{\nu\sigma} = \mathrm{tr}\; T^\mu_\nu T^\rho_\sigma = c_1 \delta^\mu_\sigma \delta^\rho_\nu + c_2 \delta^\mu_\nu \delta^\rho_\sigma, \tag{20}$$

for some constants c_1 and c_2. Clearly, $\delta^{\rho\mu}_{\sigma\nu} = \delta^{\mu\rho}_{\nu\sigma}$. In particular, the $\mathfrak{sl}(N)$ subalgebra of $\mathfrak{gl}(N)$ is characterized by the condition $T^\mu_\mu = 0$, which leads to the relation $c_1 + Nc_2 = 0$.

Now consider the multi-dimensional affine algebra (5) in the particular case $\mathfrak{g} = \mathfrak{gl}(N)$, and make use of the embedding (19). The result is the multi-dimensional Virasoro algebra:

$$\begin{aligned} [L_\mu(m), L_\nu(n)] &= n_\mu L_\nu(m+n) - m_\nu L_\mu(m+n) \\ &+ (c_1 m_\nu n_\mu + c_2 m_\mu n_\nu) m_\rho S^\rho(m+n), \\ [L_\mu(m), S^\nu(n)] &= n_\mu S^\nu(m+n) + \delta^\nu_\mu m_\rho S^\rho(m+n), \\ [S^\mu(m), S^\nu(n)] &= 0, \\ m_\mu S^\mu(m) &= 0. \end{aligned} \tag{21}$$

To see that this algebra indeed reduces to the usual Virasoro algebra when $N = 1$, we notice that the condition $m_0 S^0(m_0)$ implies that $S^0(m_0)$ is proportional to the Kronecker delta, which indeed commutes with diffeomorphisms. So the Virasoro extensions is central when $N = 1$ but not otherwise. Nevertheless, (21) defines a well-defined and non-trivial Lie algebra extension of $\mathfrak{vect}(N)$ for every N.

The cocycle proportional to c_1 was discovered by Rao and Moody [18], and the one proportional to c_2 by myself [8].

5 Multi-dimensional MF-diffeomorphism algebra

In this section we intend to use the embedding (19) to construct an analog of the MF extension for $\mathfrak{vect}(\mathbb{T}^N)$. To this end, we first need to specialize the multi-dimensional MF algebra (12) to $\mathfrak{g} = \mathfrak{gl}(N)$:

$$\begin{aligned} [T^\mu_\nu(m), T^\rho_\sigma(n)] &= \delta^\rho_\nu T^\mu_\sigma(m+n) - \delta^\mu_\sigma T^\rho_\nu(m+n) \\ &+ m_\kappa n_\lambda R^{\mu\rho,\kappa\lambda}_{\nu\sigma}(m+n), \\ [T^\mu_\nu(m), R^{\rho\tau,\kappa\lambda}_{\sigma\omega}(n)] &= \delta^\rho_\nu R^{\mu\tau,\kappa\lambda}_{\sigma\omega}(m+n) - \delta^\mu_\sigma R^{\rho\tau,\kappa\lambda}_{\nu\omega}(m+n) \\ &+ \delta^\tau_\nu R^{\rho\mu,\kappa\lambda}_{\sigma\omega}(m+n) - \delta^\mu_\omega R^{\rho\tau,\kappa\lambda}_{\sigma\nu}(m+n) \\ &+ \delta^\kappa_\nu R^{\rho\tau,\mu\lambda}_{\sigma\omega}(m+n) + \delta^\lambda_\nu R^{\rho\tau,\kappa\mu}_{\sigma\omega}(m+n) \\ &+ d^{\mu\rho\tau}_{\nu\sigma\omega} m_\pi S^{\kappa\lambda\pi}(m+n), \end{aligned} \tag{22}$$

$$R^{\mu\rho,\kappa\lambda}_{\nu\sigma}(m) = R^{\rho\mu,\kappa\lambda}_{\sigma\nu}(m) = -R^{\mu\rho,\lambda\kappa}_{\nu\sigma}(m), \tag{23}$$

where $d^{\mu\rho\tau}_{\nu\sigma\omega}$ is the third Casimir for $\mathfrak{gl}(N)$. For symmetry reasons it must be of the form

$$
\begin{aligned}
d^{\mu\rho\tau}_{\nu\sigma\omega} &= a_1(\delta^\mu_\sigma\delta^\rho_\omega\delta^\tau_\nu + \delta^\mu_\omega\delta^\rho_\nu\delta^\tau_\sigma) \\
&+ a_2(\delta^\mu_\nu\delta^\rho_\omega\delta^\tau_\sigma + \delta^\mu_\omega\delta^\rho_\sigma\delta^\tau_\nu + \delta^\mu_\sigma\delta^\rho_\nu\delta^\tau_\omega) \\
&+ a_3\delta^\mu_\nu\delta^\rho_\sigma\delta^\tau_\omega.
\end{aligned} \tag{24}
$$

In particular for $\mathfrak{sl}(N) \subset \mathfrak{gl}(N)$, the condition $d^{\mu\rho\tau}_{\nu\sigma\tau} = 0$ leads to $a_1 = N^2 a$, $a_2 = -2Na$, $a_3 = 4a$. We check that the third Casimir vanishes for the trivial algebra $\mathfrak{sl}(1)$.

The MF extension of $\mathfrak{vect}(\mathbb{T}^N)$ becomes:

$$
\begin{aligned}
[L_\mu(m), L_\nu(n)] &= n_\mu L_\nu(m+n) - m_\nu L_\mu(m+n) \\
&+ m_\rho m_\kappa n_\sigma n_\lambda R^{\rho\sigma,\kappa\lambda}_{\mu\nu}(m+n), \\
[L_\mu(m), R^{\rho\tau,\kappa\lambda}_{\sigma\omega}(n)] &= n_\mu R^{\rho\tau,\kappa\lambda}_{\sigma\omega}(m+n) \\
&+ \delta^\rho_\mu m_\nu R^{\nu\tau,\kappa\lambda}_{\sigma\omega}(m+n) - m_\sigma R^{\rho\tau,\kappa\lambda}_{\mu\omega}(m+n) \\
&+ \delta^\tau_\mu m_\nu R^{\rho\nu,\kappa\lambda}_{\sigma\omega}(m+n) - m_\omega R^{\rho\tau,\kappa\lambda}_{\sigma\mu}(m+n) \\
&+ \delta^\kappa_\mu m_\nu R^{\rho\tau,\nu\lambda}_{\sigma\omega}(m+n) + \delta^\lambda_\mu m_\nu R^{\rho\tau,\kappa\nu}_{\sigma\omega}(m+n) \\
&+ d^{\nu\rho\tau}_{\mu\sigma\omega} m_\nu m_\pi S^{\kappa\lambda\pi}(m+n), \\
[L_\mu(m), S^{\nu\rho\sigma}(n)] &= n_\mu S^{\nu\rho\sigma}(m+n) + \delta^\nu_\mu m_\tau S^{\tau\rho\sigma}(m+n) \\
&+ \delta^\sigma_\mu m_\tau S^{\nu\tau\sigma}(m+n) + \delta^\rho_\mu m_\tau S^{\nu\rho\tau}(m+n), \\
m_\rho S^{\mu\nu\rho}(m) &= 0,
\end{aligned} \tag{25}
$$

All other brackets vanish; $[R,R] = [R,S] = [S,S] = 0$. The R and S symmetry properties were written down in (23) and (13), respectively. Using the expression (24) for the $\mathfrak{gl}(N)$ third Casimir, the final term in the LR bracket can be written more explicitly as

$$
\begin{aligned}
&\Big(a_1(m_\sigma\delta^\rho_\omega\delta^\tau_\mu + m_\omega\delta^\rho_\mu\delta^\tau_\sigma) + a_2(m_\mu\delta^\rho_\omega\delta^\tau_\sigma + m_\omega\delta^\rho_\sigma\delta^\tau_\mu + m_\sigma\delta^\rho_\mu\delta^\tau_\omega) \\
&\quad + a_3 m_\mu\delta^\rho_\sigma\delta^\tau_\omega\Big) \times\ m_\nu S^{\kappa\lambda\nu}(m+n).
\end{aligned} \tag{26}
$$

Dzhumadil'daev has classified extensions of the algebra of polynomial vector fields by modules of tensor densities [5]. His classification applies morally to $\mathfrak{vect}(M)$ in general, and in particular to $\mathfrak{vect}(\mathbb{T}^N)$ which contains polynomial vector fields as a proper subalgebra. The MF-diffeomorphism algebra (25) is closely related to cocycles $\psi^W_3 - \psi^W_{10}$ in his classification. Namely, these cocycles are recovered if we set $S^{\mu\nu\rho}(m) \equiv 0$ (which can be consistently done because the bracket of S with anything is proportional to S) and decompose $R^{\mu\rho,\kappa\lambda}_{\nu\sigma}$ into irreducible $\mathfrak{gl}(N)$ representations. These cocycles are also described in subsection 3.3 of the review [10].

6 A fourth Casimir extension of current algebras

The descent equations suggest that there is an entire hierarchy of extensions of $\mathfrak{map}(\mathbb{T}^N, \mathfrak{g})$. In this section we explicitly describe the next element in this hierarchy, associated with the fourth order Casimir operator. Let

$$d^{abcd} \equiv \mathrm{tr}(J^a J^b J^c J^d + \text{permutations}) \tag{27}$$

be the structure constants of the fourth Casimir operator of $\mathfrak{g}$. They satisfy the condition

$$f^{abf} d^{fcde} + f^{acf} d^{bfde} + f^{adf} d^{bcfe} + f^{aef} d^{bcdf} = 0, \tag{28}$$

which follows from the fact that d^{abcd} commutes with $\mathfrak{g}$.

The fourth Casimir extension of $\mathfrak{map}(\mathbb{T}^N, \mathfrak{g})$ is defined by the relations

$$\begin{aligned}
[J^a(m), J^b(n)] &= i f^{abc} J^c(m+n) + m_\mu n_\nu A^{ab,\mu\nu}(m+n), \\
[J^a(m), A^{bc,\mu\nu}(n)] &= i f^{abd} A^{dc,\mu\nu}(m+n) + i f^{acd} A^{bc,\mu\nu}(m+n) \\
&+ m_\rho n_\sigma B^{abc,\mu\nu\rho\sigma}(m+n), \\
[J^a(m), B^{bcd,\mu\nu\rho\sigma}(n)] &= i f^{abe} B^{ecd,\mu\nu\rho\sigma}(m+n) + i f^{ace} B^{bed,\mu\nu\rho\sigma}(m+n) \\
&+ i f^{ade} B^{bce,\mu\nu\rho\sigma}(m+n) + k d^{abcd} m_\tau S^{\mu\nu\rho\sigma\tau}(m+n), \\
[A^{ab,\mu\nu}(m), A^{cd,\rho\sigma}(n)] &= i f^{ace} B^{bde,\mu\nu\rho\sigma}(m+n) + i f^{bce} B^{ade,\mu\nu\rho\sigma}(m+n) \\
&+ i f^{ade} B^{bce,\mu\nu\rho\sigma}(m+n) + i f^{bde} B^{ace,\mu\nu\rho\sigma}(m+n) \\
&+ k d^{abcd} m_\tau S^{\mu\nu\rho\sigma\tau}(m+n), \\
m_\tau S^{\mu\nu\rho\sigma\tau}(m) &= 0.
\end{aligned} \tag{29}$$

Note that the constants k in the third and fourth equations are equal. The symmetry properties can be summarized as

$$\begin{aligned}
A^{ab,\mu\nu}(m) &= A^{(ab),[\mu\nu]}(m), \\
B^{adc,\mu\nu\rho\sigma}(m) &= B^{(adc),[\mu\nu\rho\sigma]}(m), \\
S^{\mu\nu\rho\sigma\tau}(m) &= S^{[\mu\nu\rho\sigma\tau]}(m),
\end{aligned} \tag{30}$$

where (...) denotes symmetrization and [...] denotes anti-symmetrization of indices. Finally, the structure constants satify the conditions (9) and (28).

To verify the Jacobi identities is straightforward albeit tedious.

In the special case of five dimensions, the hyper-surface operator is proportional to the Kronecker delta:

$$S^{\mu\nu\rho\sigma\tau}(m) = \epsilon^{\mu\nu\rho\sigma\tau}\delta(m), \tag{31}$$

and

$$B^a_\mu(m) = \epsilon_{\mu\nu\rho\sigma\tau} d^{abc} B^{bc,\nu\rho\sigma\tau}(m) \tag{32}$$

transforms as a connection.

The algebra (29) is an example of a non-abelian extension, because the $[A, A]$ bracket is nonzero. There are some similarities to the algebra described in equation (1.2) of [4]. However, their algebra is only defined in five dimensions, whereas (29) is well defined in all dimensions $N \geq 5$, because the five-dimensional Kronecker delta has been replaced by the covariant five-volume operator $S^{\mu\nu\rho\sigma\tau}(m)$. Moreover, their algebra only involves the third Casimir d^{abc}, so (29) may be new even in $N = 5$ dimensions.

It is clear from the existence of the embedding (19) that $\mathfrak{vect}(N)$ possesses an analog extension for $N \geq 5$. The explicit expression will be quite cumbersome and not very illuminating, and we have not written it down.

7 Discussion

In this paper we have indicated how the n:th Casimir extensions of the current and diffeomorphism algebras in $N = 2n - 3$ dimensions give rise to extensions also in higher dimensions. The construction includes the multi-dimensional affine algebras [7, 16], Virasoro algebras [8, 18], as well as the presumably new multi-dimensional MF and fourth Casimir algebras. The MF extension of the current algebra in four dimensions could possibly be useful to study gauge anomalies in a covariant formalism.

The price to pay is that we must introduce hyper-surface operators to replace the delta functions. In the minimal dimension, the hyper-surface is essentially unique (a circle can be embedded into the circle in one way only), and the surface operator commutes with diffeomorphisms. Above the minimal dimension $N = 2n - 3$, this is no longer true.

It is difficult to imagine that such a surface operator could have any physical meaning. Hence the algebras described in this paper do probably not occur in nature. This is further corroborated by that fact that the MF algebra apparently lacks good quantum representations; more precisely, it has no unitary lowest-weight representation acting on a separable Hilbert space [15]. Mickelsson has constructed a different type of representations [13], but since these involve a classical background gauge field they can not arise in a fundamental, fully quantum theory. Nature abhors algebras without unitary quantum representations, which is confirmed by the fact that gauge anomalies associated with the MF algebra cancel in the standard model.

The situation is different for the multi-dimensional affine and Virasoro algebras. There is a natural occurring one-dimensional curve – the observer's spacetime trajectory. Moreover, these algebras do possess unitary lowest-weight representations. In fact, a classification of such representations for the multi-dimensional affine algebra appeared already in chapter 4 of [16], and they have been further studied in e.g. [1, 2, 9, 14]. Instead of working with quantum fields, one must consider spacetime histories in the space of p-jets, which locally can be identified with Taylor series truncated at order p. The privileged one-dimensional curve is identified as the time evolution of the expansion point, i.e. the observer's trajectory in spacetime. Because the space of p-jets is finite-dimensional, the space of p-jet trajectories is spanned by finitely many functions of a single variable. In this situation we can normal order operators without producing infinities.

Since the multi-dimensional affine and Virasoro algebras do possess unitary lowest-weight representations, nothing prevents them from appearing in nature. Such an extension is a gauge or diff anomaly, but that is by itself not a sign of inconsistency[1]. A gauge anomaly simply means that the classical and quantum theories have different symmetries. It is of course inconsistent to treat an anomalous gauge symmetry as a redundancy; a gauge anomaly converts a classical gauge symmetry into a quantum global symmetry, which acts on the Hilbert space rather than reducing it. The crucial consistency condition is that the symmetry algebra is represented unitarily.

However, these extensions can not arise within QFT. In four dimensions, QFT gauge anomalies are proportional to the third Casimir d^{abc}, and are hence described by the MF algebra rather than the multi-dimensional affine algebra. Moreover, there are no QFT diff anomalies at all in four dimensions [3]. The reason is that these extensions involve the curve operator $S^\mu(m)$, which requires that the observer's trajectory has been introduced. This suggests that in order to fully understand gauge and general-covariant theories, QFT must be amended with a physical observer with a quantized

[1] Counterexample: the free subcritical string, which according to the no-ghost theorem can be quantized with a ghost-free spectrum despite its conformal gauge anomaly.

position operator.

References

[1] S. Berman and Y. Billig, *Irreducible representations for toroidal Lie algebras*, J. Algebra **221** (1999) 188–231.

[2] Y. Billig, *Principal vertex operator representations for toroidal Lie algebras*, J. Math. Phys. **7** (1998) 3844–3864.

[3] L. Bonora, P. Pasti and M. Tonin, *The anomaly structure of theories with external gravity*, J. Math. Phys. **27** (1986) 2259–2270.

[4] Cederwall, M., G. Ferretti, B.E.W. Nilsson and A. Westerberg, *Higher-dimensional loop algebras, non-abelian extensions and p-branes*, Nucl. Phys. **B 424** (1994).

[5] Dzhumadil'daev A., *Virasoro type Lie algebras and deformations*, Z. Phys. C **72**, 509–517 (1996).

[6] L. D. Faddeev, Phys. Lett. **145B** (1984) 81.

[7] C. Kassel, *Kahler differentials and coverings of complex simple Lie algebras extended over a commutative algebra*, J. Pure and Appl. Algebra **34** (1985) 256–275.

[8] T.A. Larsson, *Central and non-central extensions of multi-graded Lie algebras*, J. Phys. A. **25** (1992) 1177–1184.

[9] T.A. Larsson, *Extended diffeomorphism algebras and trajectories in jet space*, Comm. Math. Phys. **214** (2000) 469–491, `math-ph/9810003`

[10] T.A. Larsson, *Extensions of diffeomorphism and current algebras*, `math-ph/0002016`

[11] J. Mickelsson, Comm. Math. Phys. **97** (1985) 361.

[12] J. Mickelsson, *Current algebras and groups*, Plenum Monographs in Nonlinear Physics, London: Plenum Press, 1989.

[13] J. Mickelsson, *Hilbert space cocycles as representations of* $(3+1)-D$ *current algebras*, Lett. Math. Phys. **28** (1993) 97–106.

[14] R.V. Moody, S.E. Rao, and T. Yokonoma, *Toroidal Lie algebras and vertex representations*, Geom. Ded. **35** (1990) 283–307.

[15] D. Pickrell, *On the Mickelsson-Faddeev extensions and unitary representations*, Comm. Math. Phys. **123** (1989) 617.

[16] A. Pressley, and G. Segal, *Loop groups*, Clarendon Press, 1988.

[17] S.E. Rao, R.V. Moody, and T. Yokonuma, *Lie algebras and Weyl groups arising from vertex operator representations*, Nova J. of Algebra and Geometry **1** (1992) 15–57.

[18] S.E. Rao and R.V. Moody, *Vertex representations for N-toroidal Lie algebras and a generalization of the Virasoro algebra*, Comm. Math. Phys. **159** (1994) 239–264.