A Note on the Analyticity of Density of States

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Abstract

We consider the d-dimensional Anderson model, and we prove the density of states is locally analytic if the single site potential distribution is locally analytic and the disorder is large. We employ the random walk expansion of resolvents and a simple complex function theory trick. In particular, we discuss the uniform distribution case, and we obtain a sharper result using more precise computations. The method can be also applied to prove the analyticity of the correlation functions.

1 Introduction

We consider the Anderson tight binding model, i.e., a random Schrödinger operator

$$H^\omega = H_0 + V^\omega \quad \text{on } \mathcal{H} = \ell^2(\mathbb{Z}^d),$$

where $d \geq 1$, $V^\omega = \{V^\omega(n) \,|\, n \in \mathbb{Z}^d\}$ are i.i.d. random variables with the common distribution μ, and H_0 is given by

$$H_0 u(n) = h \sum_{|n-m|=1} u(m) \quad \text{for } u \in \mathcal{H}.$$

with a constant $h > 0$.

If Γ is a finite box of $\mathbb{Z}^d$, we will denote by $H^\omega_{\lambda,\Gamma}$ the operator H^ω_λ restricted to $\ell^2(\Gamma)$ with Dirichlet boundary conditions. The integrated density of states(IDS for short), $\mathcal{N}(E)$, is defined by

$$\mathcal{N}(E) = \lim_{\Gamma \to \mathbb{Z}^d} \frac{1}{\#\Gamma} \#\{\text{eigenvalues of } H^\omega_{\lambda,\Gamma} \leq E\}.$$

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Here we denote the cardinality of a set S by $\#S$. It is a consequence of ergodic theorem for almost every ω the limit exists for all $E \in \mathbb{R}$ and is independent of ω. Moreover supp $(d\mathcal{N}) = \sigma(H^\omega)$ a.e. ω. The basic facts about the density of states is found in any of the standard books in the area for example Cycon-Froese-Kirsch-Simon [5], Carmona-Lacroix [4] and Figotin-Pastur [8]. It is a result of Pastur [14] that $\mathcal{N}(E)$ is always continuous. The IDS $\mathcal{N}(E)$ is positive, non-decreasing and bounded (by 1) function satisfying $\mathcal{N}(\infty) = 1$. So it is the distribution function of a probability measure. In the case when this measure is absolutely continuous, the density $n(E)$ of this measure is called the "the density of states". One of the questions of interest is the degree of smoothness of the function $n(E)$, which is also often referred to as the smoothness of IDS, which we do in the following.

Then our main result is stated as follows:

Theorem 1. *Let $I \Subset I' \Subset \mathbb{R}$ be intervals, and suppose μ has an analytic density function $g(\lambda)$ on I'. Then there is $h_0 > 0$ such that $n(\lambda)$ is analytic on I if $0 < h < h_0$.*

Our argument is so simple that we have good control of the constant.

There are many results on the smoothness of IDS for one-dimensional case. For example, $\mathcal{N}(E)$ is differentiable, even infinitely differentiable under some regularity assumptions on μ (Companino-Klein [6] and Simon-Taylor [15]). Moreover the smoothness of IDS in the Anderson model on a strip are considered, for example, Klein-Speis[12], Klein-Lacroix-Speis[11], Glaffig[10] and Klein-Speis[13].

On the other hand, there are very few results on the smoothness of IDS for multi-dimensional case. Using Molchanov formula (of expressing the matrix elements of e^{-itH^ω} in terms of a random walk on the lattice), Carmona showed (see section VI.3 [4]) that for the Cauchy distribution the IDS is C^∞. Recently, Veselić [16] shows the Lipschitz-continuity of IDS for homogeneous Gaussian random potentials using a Wegner estimate.

Among the most important other results in the multi-dimensional case are Bovier-Campanino-Klein-Perez [1], Constantinescu-Fröhlich-Spencer[3] and Bellissard-Hislop [2] and all the available results require that h is small or the region of energy considered is away from the middle of the spectrum. We also consider the case with small h, which corresponds to the large disorder case. A typical result in Bovier-Companino-Klein-Perez [1] is that $\mathcal{N}(E)$ is $(n+1)$-times continuously differentiable under the condition that the Fourier transform $\phi(t)$ of $d\mu$ satisfies $(1+|t|)^{d+n}\phi(t) \in L^1$. On the other hand Constantinescu-Fröhlich-Spencer [3] show that $\mathcal{N}(E)$ is real analytic in E, for $|\mathrm{Re}E|$ large enough if the density of μ is analytic in the strip $\{V : |\mathrm{Im}V| < 2(d+\epsilon)\}$ for arbitrarily small, but positive ϵ. Bellissard-Hislop [2] proves that if the distribution $d\mu$ has a density analytic in a strip about the real axis, then these correlation functions are also analytic outside of the planes corresponding to coincident energies. In particluar, their result implies the analyticity of $n(E)$, and of current-current correlation function outside of the diagonal.

We prove Theorem 1 in the next section. In Section 3, we discuss an important example, i.e., the uniform distribution case, and present explicit constants. Section 4 is devoted to the discussion on correlation functions. We consider 2-correlation functions only, which is useful to study current-current correlation. The idea itself applies to higher correlation functions.

2 Proof of Theorem 1

2.1 The density of states

Let $\delta_m = (\delta_{nm})_{n\in\mathbb{Z}^d} \in \mathcal{H}$ for $m \in \mathbb{Z}^d$, where δ_{nm} is the Kronecker symbol. We denote the (n,m)-entry of an operator A on $\ell^2(\mathbb{Z}^d)$ by $A(n,m) = \langle \delta_n, A\delta_m\rangle$. The following formula of the density of states in terms of the resolvent is well-known:

$$n(\lambda) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\mathbb{E}(\mathrm{Im}(H^\omega - \lambda - i\varepsilon)^{-1}(0,0)) \tag{1}$$

where $\mathbb{E}(\cdot)$ denotes the expectation with respect to the randomness (see, e.g., [4], Remark VI.1.5). Hence, in order to prove the analyticity of $n(\lambda)$, it suffices to show that $\mathbb{E}((H^\omega - z)^{-1}(0,0))$ is analytic in a complex neighborhood of I. We use the *random walk expansion* of resolvent to analyze $\mathbb{E}((H^\omega - z)^{-1}(0,0))$. The random walk expansion is proposed by Fröhlich and Spencer[9] and used used by Constantinescu, Fröhlich and Spencer[3] for $n(E)$.

2.2 Random walk expansion of resolvents

We say $\gamma = (n_0, n_1, \dots, n_k) \in (\mathbb{Z}^d)^{k+1}$ is a *path of length* k if $|n_j - n_{j-1}| = 1$ for $j = 1, \dots, k$, and we write the initial point and the end point of γ as $i(\gamma) = n_0$ and $t(\gamma) = n_k$, respectively. We write the set of all paths of length k with the initial point n_0 and the end point n_k by $\Gamma_k(n_0, n_k)$. We note $\#\Gamma_k(n_0, n_k) \le (2d)^k$ for any $n_0, n_k \in \mathbb{Z}^d$. We use the Neumann series expansion of the resolvent:

$$(H^\omega - z)^{-1} = (V^\omega - z)^{-1}\sum_{k=0}^{\infty}\big(-H_0(V^\omega - z)^{-1}\big)^k,$$

which converges if $|\mathrm{Im} z| > \|H_0\| = 2dh$. It is easy to see

$$\big(H_0(V^\omega - z)^{-1}\big)^k(n,m) = \sum_{\gamma\in\Gamma_k(n,m)} h^k \prod_{j=1}^{k}(V^\omega(n_j) - z)^{-1},$$

where $\gamma = (n_0, \dots, n_k)$. Thus we learn

$$\mathbb{E}\big((H^\omega - z)^{-1}(n,m)\big) = \sum_{k=0}^{\infty} \sum_{\gamma \in \Gamma_k(n,m)} (-h)^k \mathbb{E}\bigg(\prod_{j=0}^{k} (V^\omega(n_j) - z)^{-1}\bigg). \quad (2)$$

We now consider $\mathbb{E}\big(\prod_{j=0}^{k}(V^\omega(n_j) - z)^{-1}\big)$ for each $\gamma = (n_0, \dots, n_k) \in \Gamma_k(n,m)$. We denote $\#(\gamma,\alpha) = \#\big\{n_j \in \gamma \;\big|\; n_j = \alpha\big\}$ for $\alpha \in \mathbb{Z}^d$. By the independence of the site potentials, we can write

$$\begin{aligned}\mathbb{E}\bigg(\prod_{j=0}^{k} (V^\omega(n_j) - z)^{-1}\bigg) &= \prod_{\alpha \in \mathbb{Z}^d} \mathbb{E}\big((V^\omega(\alpha) - z)^{-\#(\gamma,\alpha)}\big) \\ &= \prod_{\alpha \in \mathbb{Z}^d} \int \frac{d\mu(\lambda)}{(\lambda - z)^{\#(\gamma,\alpha)}}. \quad (3)\end{aligned}$$

We note $\#(\gamma,\alpha) = 0$ except for finitely many α and hence the product is a finite product. We also note $\sum_{\alpha \in \mathbb{Z}^d} \#(\gamma,\alpha) = k$ for $\gamma \in \Gamma_k(n,m)$.

For $\ell \geq 0$, we set

$$B_\ell(z) = \int \frac{d\mu(\lambda)}{(\lambda - z)^\ell},$$

which is primarily defined for $z \in \mathbb{C}_+$, where $\mathbb{C}_\pm = \{z \in \mathbb{C} \mid \pm \mathrm{Im}(z) > 0\}$.

2.3 Analytic continuation of $B_\ell(z)$

In the following, we suppose $I = (a,b)$, $I' = (a - \delta, b + \delta)$ with some $\delta > 0$, and we write $\Omega_\delta = \big\{z \in \mathbb{C} \;\big|\; \mathrm{dist}(z, I) < \delta\big\}$. We suppose μ has a density function $g(\lambda)$ on I', and $g(\lambda)$ is extended to a complex function which is holomorphic in Ω_δ and continuous on $\overline{\Omega_\delta}$.

Lemma 2. *Under the assumptions above, $B_\ell(z)$ is extended to a holomorphic function in $\Omega_\delta \cup \mathbb{C}_+$. Moreover, there is $C > 1$ such that for any $0 < \delta' < \delta$,*

$$|B_\ell(z)| \leq C(\delta - \delta')^{-\ell}, \quad \text{for } z \in \Omega_{\delta'},\ \ell = 0, 1, \dots.$$

Proof. Let $\eta = \partial\Omega_\delta \cap \overline{\mathbb{C}_-}$ (see Figure 1).

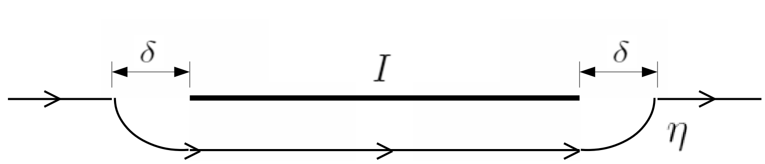

Figure 1: The integration path η

Then by the Cauchy theorem, we learn

$$B_\ell(z) = \int_{\mathbb{R}\setminus I'} \frac{d\mu(\lambda)}{(\lambda - z)^\ell} + \int_\eta \frac{g(w)dw}{(w-z)^\ell} \quad \text{for } z \in \mathbb{C}_+.$$

From this representation, it is clear that $B_\ell(z)$ is extended to $\Omega_\delta \cup \mathbb{C}_+$ as a holomorphic function. Also, by setting

$$C = 1 + (b - a + \pi\delta) \sup_{z\in\eta} |g(z)|$$

we have the inequality since $|z - w| \geq \delta - \delta'$ if $z \in \Omega_{\delta'}$ and $w \in \eta$ or $w \in \mathbb{R} \setminus I'$. □

2.4 Proof of the main theorem

By Lemma 2 and (3), we now learn that $\mathbb{E}\Big(\prod_{j=0}^k (V^\omega(n_j) - z)^{-1}\Big)$ is extended to a holomorphic function in $\mathbb{C}_+ \cup \Omega_\delta$, and it is bounded by $C^k(\delta - \delta')^{-k}$ on $\Omega_{\delta'}$. We recall $\#\Gamma_k(n,m) \leq (2d)^k$, and hence

$$\begin{aligned} &\sum_{k=0}^\infty \sum_{\gamma\in\Gamma_k(n,m)} h^k \left| \mathbb{E}\Big(\prod_{j=0}^k (V^\omega(n_j) - z)^{-1}\Big)\right| \\ &\qquad \leq \sum_{k=0}^\infty h^k (2d)^k C^k (\delta - \delta')^{-k} = \sum_{k=0}^\infty \left(\frac{2dCh}{\delta - \delta'}\right)^k < \infty \end{aligned}$$

for $z \in \Omega_{\delta'}$ if $h < (\delta - \delta')/(2dC)$. Thus by (2), under this condition, $\mathbb{E}\big((H^\omega - z)^{-1}(n,m)\big)$ is holomorphic in $z \in \Omega_{\delta'}$. In particular, we learn $\mathbb{E}\big((H^\omega - z)^{-1}(0,0)\big)$ is holomorphic in $z \in \Omega_{\delta'}$, and we conclude Theorem 1 thanks to (1). □

3 An example

Here we consider a typical Anderson model, that is, the case when μ has the uniform distribution on $[-a, a]$ with $a > 0$. In this case, it is well-known $\sigma(H^\omega) = [-a - 2dh, a + 2dh]$ almost surely.

Since μ has the density $1/2a$ on $(-a, a)$, we can apply Theorem 1 to conclude that for any $b < a$, $n(\lambda)$ is analytic on $(-b, b)$ if h is sufficiently small. However, in this case, we can explicitly compute $B_\ell(z)$ to obtain a sharper result. We have

$$B_1(z) = \frac{1}{2a}\int_{-a}^a \frac{d\lambda}{\lambda - z} = \frac{1}{2a}(\log(a - z) - \log(-a - z)),$$

and for $\ell \geq 2$,

$$B_\ell(z) = \frac{1}{2a}\int_{-a}^a \frac{d\lambda}{(\lambda - z)^\ell} = \frac{1}{2a(\ell - 1)}\left(\frac{1}{(a-z)^{\ell-1}} - \frac{1}{(-a-z)^{\ell-1}}\right).$$

Now we consider the case $h = 1$ and change a, which is equivalent by a simple scaling. . If $\delta(\log\delta + \pi) \le a$ and $1 \le \delta \le a$, then we have

$$|B_\ell(z)| \le \delta^{-\ell} \quad \text{for } z \text{ such that } \mathrm{dist}(z, \{\pm a\}) \ge \delta, \mathrm{Re} z \in (-a, a),$$

for all $\ell \in \mathbb{N}$. If $a > 2d(\log(2d) + \pi)$, then we can choose $\delta > 2d$ arbitrarily close to $2d$ to satisfy the above conditions. Then by modifying the above argument, we learn that $\mathbb{E}((H^\omega - z)^{-1}(0, 0))$ is analytic on $\{z \,|\, \mathrm{dist}(z, \{\pm a\}) > \delta, \mathrm{Re} z \in (-a, a)\}$. Thus we have the following theorem.

Theorem 3. *Let $h = 1$, and suppose μ has the uniform distribution on $[-a, a]$ with $a > 2d(\log(2d)+\pi)$. Then $n(\lambda)$ is analytic on $(-a+2d, a-2d)$.*

4 Correlation functions

Here we extend our method to show the analyticity of correlation functions (see Bellissard-Hislop[2]). For simplicity, we consider 2-correlation functions only, which applies to current-currect correlation functions.

We denote

$$\delta(H^\omega - E) = \frac{1}{2\pi i} \lim_{\varepsilon\to+0} \mathrm{Im}(H^\omega - E - i\varepsilon)^{-1}$$

when the limit is well-defined. Let A_1, A_2 be bounded operators, which is local in the following sense: there are $R, M > 0$ such that

$$A_j(k, \ell) = 0 \text{ if } |k - \ell| > R, \quad |A_j(k, \ell)| \le M \text{ for } \forall k, \ell,$$

with $j = 1, 2$. The 2-correlation function for H^ω, A_1 and A_2 is defined by

$$K(e_1, e_2) = \mathbb{E}\big[(\delta(H^\omega - e_1)A_1\delta(H^\omega - e_2)A_2)(0, 0)\big]$$

for $e_1, e_2 \in \mathbb{R}$.

Theorem 4. *Let I, I' be intervals as in Theorem 1. Then there is $\gamma > 0$ such that $K(e_1, e_2)$ is analytic on $I \times I \setminus D(\gamma h)$, where $D(\beta) = \big\{(e_1, e_2) \in \mathbb{R}^2 \,\big|\, |e_1 - e_2| \le \beta\big\}$.*

Namely, the 2-correlation function $K(e_1, e_2)$ is analytic away from a small neighborhood of the diagonal, and the width of the exceptional set is $O(h)$ as $h \to 0$. We note that for a fixed $h > 0$, we do not have the analyticity in (e_1, e_2) very close to the diagonal set.

Proof. Let $I = [a, b]$, and let $a \le E_1 < E_2 \le b$. We show

$$F(z_1, z_2) = \mathbb{E}\big[((H^\omega - z_1)^{-1}A_1(H^\omega - z_2)^{-1}A_2)(0, 0)\big]$$

is analytically extended to a complex neighborhood of $(z_1, z_2) = (E_1, E_2)$ from $(z_1, z_2) \in \mathbb{C}_\pm \times \mathbb{C}_\mp$, or $(z_1, z_2) \in \mathbb{C}_\pm \times \mathbb{C}_\mp$. We consider the case:

$(z_1, z_2) \in \mathbb{C}_+ \times \mathbb{C}_-$ only. The other cases can be handled similarly. We choose $\delta > 0$ so that

$$E_2 \geq E_1 + 2\delta, \quad [a - \delta, b + \delta] \Subset I'. \tag{4}$$

By direct computations as in previous sections, we have

$$(H^\omega - z_1)^{-1} A_1 (H^\omega - z_2)^{-1} A_2 = \sum_{k,\ell=0}^{\infty} (V^\omega - z_1)^{-1} \big[(-H_0)(V^\omega - z_1)^{-1}\big]^k \times \\ \times A_1 (V^\omega - z_2)^{-1} \big[(-H_0)(V^\omega - z_2)^{-1}\big]^\ell A_2.$$

We denote the set of the paths satisfying the following conditions by $\Gamma_{k,\ell}(n,m)$: $\gamma = (n_0, n_1, \dots, n_k, m_0, \dots, m_\ell, m_{\ell+1}) \in (\mathbb{Z}^d)^{k+\ell+3}$ such that $n_0 = n$, $m_{\ell+1} = m$, $|n_i - n_{i-1}| = 1$ for $i = 1, \dots, k$, $|m_j - m_{j-1}| = 1$ for $j = 1, \dots, \ell$, $|n_k - m_0| \leq R$ and $|m_\ell - m_{\ell+1}| \leq R$. We note $\#\Gamma_{k,\ell}(n,m) \leq (2R)^{2d}(2d)^{k+\ell}$. Then we have

$$\mathbb{E}\big[((H^\omega - z_1)^{-1} A_1 (H^\omega - z_2)^{-1} A_2)(n,m)\big] \\ = \sum_{k,\ell=0}^{\infty} \sum_{\gamma \in \Gamma_{k,\ell}(n,m)} (-h)^{k+\ell} A_1(n_k, m_0) A_2(m_\ell, m_{\ell+1}) \times \\ \times \mathbb{E}\Big[\prod_{i=1}^{k} (V^\omega(n_i) - z_1)^{-1} \prod_{j=1}^{\ell} (V^\omega(m_j) - z_2)^{-1}\Big].$$

Now we write

$$\nu_1(\gamma, \alpha) = \#\{n_i \mid n_i = \alpha\}, \quad \nu_2(\gamma, \alpha) = \#\{m_j \mid m_j = \alpha\}$$

for $\gamma = (n_0, n_1, \dots, n_k, m_0, \dots, m_\ell, m_{\ell+1}) \in \Gamma_{k,\ell}(n,m)$. Then by the independence, it is easy to observe

$$\mathbb{E}\Big[\prod_{i=1}^{k} (V^\omega(n_i) - z_1)^{-1} \prod_{j=1}^{\ell} (V^\omega(m_j) - z_2)^{-1}\Big] \\ = \prod_{\alpha \in \mathbb{Z}^d} \mathbb{E}\big[(V^\omega(\alpha) - z_1)^{-\nu_1(\gamma,\alpha)} (V^\omega(\alpha) - z_2)^{-\nu_2(\gamma,\alpha)}\big] \\ = \prod_{\alpha \in \mathbb{Z}^d} \int \frac{d\mu(\lambda)}{(\lambda - z_1)^{\nu_1(\gamma,\alpha)} (\lambda - z_2)^{\nu_2(\gamma,\alpha)}}.$$

We also note $\sum_\alpha \nu_1(\gamma, \alpha) = k$ and $\sum_\alpha \nu_2(\gamma, \alpha) = \ell$ for $\gamma \in \Gamma_{k,\ell}(n,m)$. We denote

$$B_{k,\ell}(z_1, z_2) = \int \frac{d\mu(\lambda)}{(\lambda - z_1)^k (\lambda - z_2)^\ell}, \quad (z_1, z_2) \in \mathbb{C}_+ \times \mathbb{C}_-,$$

and $\Omega_\delta(E) = \big\{z \in \mathbb{C} \bigm| |z - E| < \delta\big\}$. Then, as well as Lemma 2, we have the following lemma:

Lemma 5. *$B_{k,\ell}(z_1, z_2)$ is extended to a holomorphic function in $\Omega_\delta(E_1) \times \Omega_\delta(E_2)$. Moreover, there is $C > 1$ such that for any $0 < \delta' < \delta$, k and ℓ,*

$$|B_{k,\ell}(z_1, z_2)| \leq C(\delta - \delta')^{-k-\ell} \quad \textit{for } (z_1, z_2) \in \Omega_\delta(E_1) \times \Omega_{\delta'}(E_2).$$

We note the constant C is also independent of E_1, E_2 and $\delta, \delta' > 0$ as long as they satisfy (4). The proof of Lemma 5 is similar to that of Lemma 2, but we use the contour:

$$\eta = \partial(\mathbb{C}_+ \cup \Omega_\delta(E_1) \setminus \Omega_\delta(E_2))$$

to represent $B_{k,\ell}(z_1, z_2)$ by a contour integral (see Figure 2). We omit the detail.

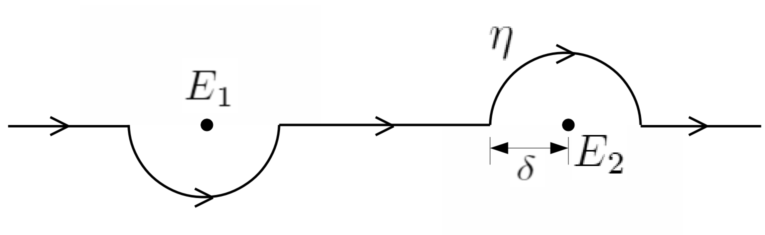

Figure 2: The integration path

Using the lemma, we learn

$$|F(z_1, z_2)| \leq \sum_{k,\ell=0}^{\infty} (2R)^{2d} (2d)^{k+\ell} h^{k+\ell} M^2 C^{k+\ell} (\delta - \delta')^{-k-\ell}$$

for $(z_1, z_2) \in \Omega_{\delta'}(E_1) \times \Omega_{\delta'}(E_2)$. If

$$2dhC(\delta - \delta')^{-1} < 1, \text{ i.e., } \delta - \delta' > 2dCh,$$

then the random walk expansion converges uniformly in $\Omega_{\delta'}(E_1) \times \Omega_{\delta'}(E_2)$, and inparticular, $F(z_1, z_2)$ is analytic. We may choose $\delta' = \delta/2$, and we conclude $F(e_1, e_2)$ is analytic if $|e_1 - e_2| > 8dCh$. This, combined with other cases, implies the assertion. □

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